Infinitely many associated primes of Frobenius powers and local cohomology

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Abstract. A modification of Katzman's example is given to produce a two-generated ideal in a two-dimensional Noetherian integral domain for which the set of associated primes of all the Frobenius powers is infinite. A further modification yields a four-dimensional Noetherian integral domain and a five-dimensional Noetherian local integral domain for which an explicit second local cohomology module has infinitely many associated primes.

Katzman gave an example in [K1] of an ideal I in a two-dimensional ring for which the set of associated primes of all the Frobenius powers of I is infinite. The ring in Katzman's example was not an integral domain. In this paper it is shown that the infinite cardinality of the set of associated primes of all the Frobenius powers of an ideal can happen even in a two-dimensional integral domain.

An application is another example of a local cohomology module with infinitely many associated primes. Singh in [Si] found the first example of such a module. His example was a non-local six-dimensional integral domain R for which $H^3_I(R)$ has infinitely many associated primes for some ideal I. Katzman in [K2] revisited his own example from [K1] to construct a five-dimensional local integral domain R for which $H^2_I(R)$ has infinitely many associated primes for some ideal I. Similarly also the ideal in this paper yields a five-dimensional local integral domain R for which $H^2_I(R)$ has infinitely many associated primes for some ideal I. Theorem 8 gives a general method for constructing local cohomology modules with infinitely many associated primes from certain families of matrices. Both Katzman's ideal and the ideal in this paper yield such families of matrices.

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Katzman's example in [K1] was motivated by the theory of tight closure, in particular by the question whether tight closure commutes with localization. For most of this paper, no knowledge of tight closure is needed, but a good reference for tight closure is Hochster-Huneke [HH]. Having an infinite set of associated primes of all the Frobenius powers of an ideal showed that the localization of tight closure is a non-trivial problem, and came as a surprise. However, the localization question can always be reduced to integral domains by passing to all the quotients by the minimal prime ideals: Katzman's ideal is principal modulo each minimal prime, whence its tight closure equals its integral closure, so localization and tight closure commute on Katzman's ideal. Furthermore, modulo each minimal prime ideal, the Frobenius powers of Katzman's ideal equal the ordinary powers, thus the set of associated primes of all the Frobenius powers there is a finite set (see Ratliff [R]). Thus the question remained whether there exists an ideal in a low-dimensional integral domain for which the set of associated primes of all the Frobenius powers is an infinite set. This paper provides just such an example: a two-dimensional hypersurface domain.

The following is the ring: $R = \frac{k[t,x,y]}{(x^4+x^3y+x^2y^2+tx^2y^2+txy^3+t^2y^4)}$, where k is a field of characteristic 2, and x, y and t are variables over k. See Lemma 1 for a proof of when R is an integral domain. By Theorem 5 the set of associated primes of all the Frobenius powers of $(x, y)R$ is infinite.

This example is a modification of Katzman's example: Katzman's ring was $k[t, x, y]$ modulo $(x^3y + x^2y^2 + tx^2y^2 + txy^3) = xy(x + y)(x + ty)$, where k is a field k of arbitrary positive characteristic p. Here, the characteristic is assumed to be 2, and Katzman's ideal is modified into an irreducible one by adding $x^4 + t^2y^4$.

The eth Frobenius power of an ideal I in a ring of positive prime characteristic p is defined as the ideal $(i^{p^e} | i \in I)$. Thus if $i_1, \ldots, i_n$ generate I, then the $i_j^{p^e}$ generate the eth Frobenius power of I. In the sequel some proofs will be easier if for a given set of generators x and y of I we admit the generalized "Frobenius powers" (x^n, y^n) of (x, y), as n varies over all positive integers. An attempt is also made throughout the paper to not be restricted to characteristic 2. However, characteristic 2 is needed for the main results.

The integral domain property of the ring can be analyzed for all characteristics:

Lemma 1: *Let k be a field of arbitrary characteristic, x, y and t variables over k. Consider the polynomial $g = x^4 + x^3y + x^2y^2 + tx^2y^2 + txy^3 + t^2y^4$. Then g is irreducible in $k[t, x, y]$ if and only if one of the following conditions hold:*
(1) the characteristic of k is different from 2,
(2) the characteristic of k is 2, and k contains no root of the polynomial $U^2 + U + 1 = 0$.

In particular, g is irreducible over $\mathbb{Z}/2\mathbb{Z}$.

Proof: Note that g is homogeneous in (x, y) of degree 4.

Suppose that g has a linear factor. Up to a unit scalar multiple, this factor is $x - ut^i y$ for some $u \in k^*$ and some integer $i \in \{0, 1, 2\}$. By rewriting x as $ut^i y$ in g and then dividing by y^4 one gets $u^4t^{4i} + u^3t^{3i} + (1 + t)u^2t^{2i} + ut^{1+i} + t^2 = 0$. Thus by the degree count necessarily $i = 0$. But then the homogeneous part of degree 2 of the equation forces $1 = 0$, which is a contradiction.

Now assume that g factors into a product of two quadratics. Then up to a unit scalar multiple,

$$g = (x^2 + axy + ut^i y^2)(x^2 + (1 - a)xy + u^{-1}t^{2-i}y^2)$$

for some $a \in k[t]$, some unit u in k, and some $i \in \{0, 1, 2\}$. The coefficients of xy^3 yield $t = au^{-1}t^{2-i} + (1 - a)ut^i$, or $t - ut^i = a(u^{-1}t^{2-i} - ut^i)$. By degree count, the case $i = 0$ is impossible. Also the case $i = 2$ is impossible for then $t - ut^2 = a(u^{-1} - ut^2)$ forces $a = a't$ for some $a' \in k[t]$, so that $1 - ut = a'(u^{-1} - ut^2)$, which is also impossible by the degree count. In the remaining case $i = 1$ the coefficient of xy^3 yields $1 - u = a(u^{-1} - u)$, so that either $u = 1$ or else that $a \in k^*$ with $1 = a(u^{-1} + 1)$. The coefficient of x^2y^2 then results in $1+t = u^{-1}t+ut+a(1-a)$. The case $u = 1$ gives $1-t = a(1-a)$, which has no solution for a in $k[t]$, so that necessarily $u \neq 1$. In that case $a \in k^*$, so that $1+t = u^{-1}t+ut+a(1-a)$ splits into equations

$$1 = u^{-1} + u, 1 = a(1 - a).$$

Thus both a and u are roots of $U^2-U+1 = 0$, so that $a = u^{\pm 1}$. The equation $1 = a(u^{-1}+1)$ rules out the case $a = u$, so that necessarily $a = u^{-1}$. But then a is a root of $U^2-U+1 = 0$ and of $U^2 + U - 1 = 0$, which is impossible in characteristic other than 2.

The rest of the proposition now follows easily. ■

In order to construct associated primes, one needs to find zerodivisors. A clear pattern for the zerodivisors (elements τ_e below) modulo (generalized) Frobenius powers holds in characteristic 2:

Proposition 2: *Let k be a field of characteristic 2, x, y and t variables over k, and $A = k[t, x, y]$. Let $g = x^4 + x^3y + x^2y^2 + tx^2y^2 + txy^3 + t^2y^4$. For each positive integer e define*

$$I_e = (x^{2^{2e}}, y^{2^{2e}}, g)A, \qquad \tau_e = 1 + \sum_{i=1}^{2e} t^{2^{i-1}}, \qquad c_e = x^{2^{2e}-1}y^2.$$

Then $\tau_e c_e \in I_e$.

Proof: Define

$$A_e = \tau_e c_e + (y + x\sum_{i=1}^{2e} t^{2^{i-1}-1})x^{2^{2e}} + (x+y)t^{2^{2e}-1}y^{2^{2e}}.$$

It suffices to prove that each A_e is a multiple of g. We will proceed by induction on e. When $e = 1$,

$$A_1 = (1+t+t^2)x^3y^2 + (y + x(1+t))x^4 + (x+y)t^3y^4.$$

Modulo g, x^4 can be written as $x^3y + x^2y^2 + tx^2y^2 + txy^3 + t^2y^4$, so that A_1 is congruent to

$$\begin{aligned}
&\equiv (1+t+t^2)x^3y^2 + (y+x(1+t))(x^3y + x^2y^2 + tx^2y^2 + txy^3 + t^2y^4) + (x+y)t^3y^4 \\
&= (t+t^2)x^3y^2 + (y+x(1+t))(x^2y^2 + tx^2y^2 + txy^3 + t^2y^4) \\
&\qquad + (1+t)x^4y + (x+y)t^3y^4 \\
&\equiv (t+t^2)x^3y^2 + (y+x(1+t))(x^2y^2 + tx^2y^2 + txy^3 + t^2y^4) \\
&\qquad + (1+t)(x^3y + x^2y^2 + tx^2y^2 + txy^3 + t^2y^4)y + (x+y)t^3y^4.
\end{aligned}$$

It is easy to check that the coefficients of x^3y^2, x^2y^3, xy^4 and y^5 in the last expression are all 0. Thus A_1 is a multiple of g.

Now suppose that $e > 1$ and that by induction for all $a < e$, A_a is a multiple of g. We will prove that A_e is a multiple of g. We rewrite A_e as follows:

$$A_e = x^{2^{2e}-1}\beta_e + (x+y)t^{2^{2e}-1}y^{2^{2e}},$$

where $\beta_e = xy + y^2 + (x^2+ty^2)\sum_{i=1}^{2e} t^{2^{i-1}-1}$. As β_{e-1} is not a factor of the irreducible polynomial g, it suffices to prove that $A_e\beta_{e-1}$ is a multiple of g. Also, as A_{e-1} is a multiple of g, it suffices to prove that $A_{e-1}\beta_e x^{3\cdot 2^{2e-2}} + A_e\beta_{e-1}$ is a multiple of g:

$$\begin{aligned}
A_{e-1}\beta_e x^{3\cdot 2^{2e-2}} + A_e\beta_{e-1} &= (x+y)t^{2^{2e-2}-1}y^{2^{2e-2}}\beta_e x^{3\cdot 2^{2e-2}} + (x+y)t^{2^{2e}-1}y^{2^{2e}}\beta_{e-1} \\
&= (x+y)t^{2^{2e-2}-1}y^{2^{2e-2}}\left(\beta_e x^{3\cdot 2^{2e-2}} + t^{3\cdot 2^{2e-2}}y^{3\cdot 2^{2e-2}}\beta_{e-1}\right),
\end{aligned}$$

so that it suffices to show that

$$B = \beta_e x^{3\cdot 2^{2e-2}} + t^{3\cdot 2^{2e-2}}y^{3\cdot 2^{2e-2}}\beta_{e-1}$$

is a multiple of g. Then again as β_{e-1} is not a zero-divisor modulo g it suffices to prove that $B\beta_{e-1}^3$ is a multiple of g. But modulo A_{e-1}, which is a multiple of g,

$$\begin{aligned}
B\beta_{e-1}^3 &= \beta_e x^{3\cdot 2^{2e-2}}\beta_{e-1}^3 + t^{3\cdot 2^{2e-2}}y^{3\cdot 2^{2e-2}}\beta_{e-1}^4 \\
&\equiv \beta_e x^3(x+y)^3 t^{3\cdot 2^{2e-2}-3}y^{3\cdot 2^{2e-2}} + t^{3\cdot 2^{2e-2}}y^{3\cdot 2^{2e-2}}\beta_{e-1}^4 \\
&= t^{3\cdot 2^{2e-2}-3}y^{3\cdot 2^{2e-2}}\left(\beta_e x^3(x+y)^3 + t^3\beta_{e-1}^4\right)
\end{aligned}$$

so that it suffices to prove that

$$C = \beta_e x^3(x+y)^3 + t^3\beta_{e-1}^4$$

is a multiple of g. As modulo g, $x^4 + x^3y + x^2y^2 + tx^2y^2 + txy^3 + t^2y^4 \equiv 0$, then also

$$\begin{aligned}
x^3(x+y)^3 &= x^6 + x^5y + x^4y^2 + x^3y^3 \equiv tx^4y^2 + (1+t)x^3y^3 + t^2x^2y^4 \\
&\equiv x^3y^3 + tx^2y^4 + t^2xy^5 + t^3y^6, \\
x^4 &\equiv x^3y + x^2y^2 + tx^2y^2 + txy^3 + t^2y^4, \\
x^5 &\equiv x^4y + x^3y^2 + tx^3y^2 + tx^2y^3 + t^2xy^4 \equiv tx^3y^2 + x^2y^3 + (t+t^2)xy^4 + t^2y^5, \\
x^6 &\equiv tx^4y^2 + x^3y^3 + (t+t^2)x^2y^4 + t^2xy^5 \equiv (1+t)x^3y^3 + t^3y^6, \\
x^8 &\equiv (1+t)x^5y^3 + t^3x^2y^6 \equiv (1+t)(tx^3y^2 + x^2y^3 + (t+t^2)xy^4 + t^2y^5)y^3 + t^3x^2y^6 \\
&= (t+t^2)x^3y^5 + (1+t+t^3)x^2y^6 + (t+t^3)xy^7 + (t^2+t^3)y^8,
\end{aligned}$$

so that

$$\begin{aligned}
C &\equiv \left(xy + y^2 + (x^2+ty^2)\sum_{i=1}^{2e} t^{2^{i-1}-1}\right)(x^3y^3 + tx^2y^4 + t^2xy^5 + t^3y^6) \\
&\quad + t^3\left(x^4y^4 + y^8 + (x^8 + t^4y^8)\sum_{i=1}^{2(e-1)} t^{4(2^{i-1}-1)}\right) \\
&= x^4y^4 + tx^3y^5 + t^2x^2y^6 + t^3xy^7 + (x^3y^3 + tx^2y^4 + t^2xy^5 + t^3y^6)y^2\left(1 + \sum_{i=1}^{2e} t^{2^{i-1}}\right) \\
&\quad + (x^5y^3 + tx^4y^4 + t^2x^3y^5 + t^3x^2y^6)\sum_{i=1}^{2e} t^{2^{i-1}-1} \\
&\quad + t^3x^4y^4 + x^8\sum_{i=3}^{2e} t^{2^{i-1}-1} + t^3y^8\left(1 + \sum_{i=3}^{2e} t^{2^{i-1}}\right) \\
&= x^4y^4 + tx^3y^5 + t^2x^2y^6 + t^3xy^7 + x^3y^5 + tx^2y^6 + t^2xy^7 \\
&\quad + (tx^3y^5 + t^2x^2y^6 + t^3xy^7 + x^5y^3 + tx^4y^4 + t^2x^3y^5 + t^3x^2y^6)\sum_{i=1}^{2e} t^{2^{i-1}-1} \\
&\quad + t^3x^4y^4 + x^8\sum_{i=3}^{2e} t^{2^{i-1}-1} + t^3y^8(t+t^2) \\
&\equiv tx^3y^5 + t^2x^2y^6 + t^3xy^7 + t^2xy^7 + x^2y^6 + txy^7 + t^2y^8 + t^3y^8(t+t^2) \\
&\quad + ((t+t^2)x^3y^5 + (1+t+t^3)x^2y^6 + (t+t^3)xy^7 + (t^2+t^3)y^8)\sum_{i=1}^{2e} t^{2^{i-1}-1}
\end{aligned}$$

$$+ t^3x^3y^5 + t^3x^2y^6 + t^4x^2y^6 + t^4xy^7 + t^5y^8$$
$$+ ((t+t^2)x^3y^5 + (1+t+t^3)x^2y^6 + (t+t^3)xy^7 + (t^2+t^3)y^8) \sum_{i=3}^{2e} t^{2^{i-1}-1}$$

and now it is easy to verify that the coefficients of x^3y^5, x^2y^6, xy^7 and y^8 are all 0. Thus C is a multiple of g, which proves the proposition. ■

With this we proceed as follows: we prove first that the elements τ_e obtained in the proposition above are not contained in a finite set of prime ideals, and after that we prove that the τ_e are indeed zero-divisors modulo the – generalized and genuine – Frobenius powers.

Proposition 3: *Let k be a field of characteristic 2, t a variable over k. For every $e \geq 1$, set $\tau_e = \sum_{i=1}^{2e} t^{2^{i-1}} \in k[t]$. Then every set of prime ideals in $k[t]$ containing all the τ_e is infinite. Specifically, the polynomials τ_{2^e} have no common factors.*

Proof: Consider the set E of all positive integers e for which all the τ_e have a common irreducible factor f. Let e_0 be the smallest element of E. For all $e, e' \in E$ also $\tau_e - \tau_{e'}$ has f as a factor. But if $e > e'$,

$$\tau_e - \tau_{e'} = \sum_{i=2e'+1}^{2e} t^{2^{i-1}} = \left(\sum_{i=1}^{2e-2e'} t^{2^{i-1}} \right)^{2^{2e}},$$

so that f divides all the $\sigma_{e-e'} = \sum_{i=1}^{2(e-e')} t^{2^{i-1}}$. Then also all the diferences $\sigma_{e-e'} - \sigma_{e'-e'}$ have f as a factor, and $\sigma_{e-e'} - \sigma_{e'-e'}$ is a power of some $\sigma_{e'}$. Keep taking the differences of all the σ_e obtained in this process. Note that the smallest possible subscript on σ obtained by repeating this procedure is the greatest common divisor of the differences $e - e'$ of $e, e' \in E$. Let d be the greatest common divisor of these differences $e - e'$. As $\sigma_1 = t + t^2$ and f is not t or $t+1$, it follows that $d \geq 2$.

Suppose that $e_0 \geq d$. Then f divides

$$\tau_{e_0} - \sigma_d = 1 + \sum_{i=2d+1}^{2e_0} t^{2^{i-1}} = \left(1 + \sum_{i=1}^{2e_0-2d} t^{2^{i-1}} \right)^{2^{2d}} = (\tau_{e_0-d})^{2^{2d}},$$

which contradicts the minimality of e_0. So necessarily $e_0 < d$. Then $\sigma_d - \tau_{e_0} = (\tau_{d-e_0})^{2^{2e_0}}$, so that $d - e_0 \in E$, and so necessarily $d - e_0 \geq e_0$. Then by the definition of d, d divides $d - 2e_0$, so that d divides $2e_0$. So necessarily $d = 2e_0$. Thus E is a subset of the set of all odd multiples of e_0.

For each $a \geq 0$, set F_a to be the set of all odd multiples of 2^a. Clearly the sets F_a partition the set of all positive integers.

Suppose that all the τ_e are contained in a finite set of prime ideals $P_1, \ldots, P_s$. For each $i = 1, \ldots, s$, set E_i to be the set of all positive integers e such that $\tau_e \in P_i$. We have proved that if e_i denotes the smallest element of E_i, then every element of E_i is an odd multiple of e_i. But as $e_i \in F_{j(i)}$ for some $j(i)$, it follows that $E_i \subseteq F_{j(i)}$, which proves that there cannot be only finitely many P_i. ■

The elements τ_e can of course be defined over a field of arbitrary characteristic. The next result proves that each τ_e is indeed a zerodivisor modulo $(x^{2^{2e}}, y^{2^{2e}})R$ in characteristic 2:

Proposition 4: *Let k be an arbitrary field, x, y and t variables over k, and $A = k[t, x, y]$. Let $g = x^4 + x^3y + x^2y^2 + tx^2y^2 + txy^3 + t^2y^4$ and for each positive integer $n \geq 4$ let J_n be the ideal $(x^n, y^n, g)A$. If n is congruent to 2, 3, or 4 modulo 6, then $x^{n-1}y^2 \notin J_n$.*

In particular, if n is of the form 2^e or 3^e, then $x^{n-1}y^2 \notin J_n$.

Proof: Suppose for contradiction that

$$x^{n-1}y^2 = qx^n + ry^n + sg,$$

for some $q, r, s \in A$. As $x^{n-1}y^2, x^n, y^n$ and g are homogeneous in x and y, without loss of generality $q = q_1x + q_2y$, $r = r_1x + r_2y$, $s = \sum_{i=0}^{n-3} s_i x^{n-3-i} y^i$ for some $q_i, r_i, s_i \in k[t]$. The equation above then expands to

$$\begin{aligned} x^{n-1}y^2 = q_1x^{n+1} + q_2x^ny + r_1xy^n + r_2y^{n+1} + \sum_{i=0}^{n-3} s_i x^{n+1-i}y^i + \sum_{i=0}^{n-3} s_i x^{n-i} y^{i+1} \\ + \sum_{i=0}^{n-3} s_i x^{n-1-i} y^{i+2}(1+t) + \sum_{i=0}^{n-3} s_i x^{n-2-i}y^{i+3}t + \sum_{i=0}^{n-3} s_i x^{n-3-i} y^{i+4} t^2. \end{aligned}$$

The coefficients of the various monomials of degree $n+1$ in x and y in the equation above then satisfy the following equations:

$$\begin{aligned} &x^{n+1}: && 0 = q_1 + s_0, \\ &x^n y: && 0 = q_2 + s_1 + s_0, \\ &x^{n-1}y^2: && 1 = s_2 + s_1 + s_0(1+t), && (A) \\ &x^{n-2}y^3: && 0 = s_3 + s_2 + s_1(1+t) + s_0 t, && (B) \end{aligned}$$

$$
\begin{array}{lll}
x^{n+1-i}y^i: & 0 = s_i + s_{i-1} + s_{i-2}(1+t) + s_{i-3}t + s_{i-4}t^2, i = 4, \ldots, n-3, & (C_i) \\
x^3y^{n+1}: & 0 = s_{n-3} + s_{n-4}(1+t) + s_{n-5}t + s_{n-6}t^2, & (D) \\
x^2y^{n-1}: & 0 = s_{n-3}(1+t) + s_{n-4}t + s_{n-5}t^2, & (E) \\
xy^n: & 0 = r_1 + s_{n-3}t + s_{n-4}t^2, & \\
y^{n+1}: & 0 = r_2 + s_{n-3}t^2, &
\end{array}
$$

The only equations that need to be worked on are the $n-2$ equations $(A)-(E)$ in the s_i. Let M_n be the corresponding $(n-2)\times(n-2)$ matrix of coefficients:

$$
M_n = \begin{bmatrix}
1+t & 1 & 1 & & & & & & & \\
t & 1+t & 1 & 1 & & & & & & \\
t^2 & t & 1+t & 1 & 1 & & & & & \\
0 & t^2 & t & 1+t & 1 & 1 & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & t^2 & t & 1+t & 1 & 1 & 0 \\
 & & & & t^2 & t & 1+t & 1 & 1 \\
 & & & & & t^2 & t & 1+t & 1 \\
 & & & & & & t^2 & t & 1+t
\end{bmatrix}.
$$

Then $M_n(s_0, \ldots, s_{n-3}) = (1, 0, \ldots, 0)$. Note that M_n has non-zero determinant (plug in $t = 0$ to get a clearly invertible upper-triangular matrix with non-zero entries on the diagonal). Thus Cramer's rule applies: $s_0 = \frac{\det M_{n-1}}{\det M_n}$. We will prove that under the assumptions on n, s_0 is not an element of the polynomial ring $k[t]$, which will prove the proposition.

The first few of these matrices are as follows:

$$
M_4 = \begin{bmatrix} 1+t & 1 \\ t & 1+t \end{bmatrix}, M_5 = \begin{bmatrix} 1+t & 1 & 1 \\ t & 1+t & 1 \\ t^2 & t & 1+t \end{bmatrix}, M_6 = \begin{bmatrix} 1+t & 1 & 1 & 0 \\ t & 1+t & 1 & 1 \\ t^2 & t & 1+t & 1 \\ 0 & t^2 & t & 1+t \end{bmatrix},
$$

so that $\det M_4 = t^2 + t + 1$, $\det M_5 = 2t^2 + t + 1$, $\det M_6 = -t^3 + 3t^2 + t + 1$. Also, $\det M_7 = -2t^3 + 4t^2 + t + 1$. In the sequel let $n \geq 8$.

For any matrix M, set M_{ij} to be the submatrix of M obtained from M by deleting the ith column and the jth row. Then for example $\det(M_4)_{21} = 1$, $\det(M_5)_{21} = 1$, $\det(M_6)_{21} = 1 + t^2$.

By expanding the determinant of M_n down the first column, we get

$$
\det M_n = (1+t)\det M_{n-1} - t\det(M_n)_{21} + t^2\det(M_n)_{22}.
$$

Consider $(M_n)_{21}$:

$$(M_n)_{21} = \begin{bmatrix} 1 & 1 & & & & & \\ t & 1+t & 1 & 1 & & & \\ t^2 & t & 1+t & 1 & 1 & & \\ & t^2 & t & 1+t & 1 & 1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & t^2 & t & 1+t & 1 \\ & & & & t^2 & t & 1+t \end{bmatrix}.$$

The determinant of this can be expanded along the first row:

$$\det(M_n)_{21} = \det M_{n-2} - \det((M_n)_{21})_{12}.$$

Next,

$$((M_n)_{21})_{12} = \begin{bmatrix} t & 1 & 1 & & & & \\ t^2 & 1+t & 1 & 1 & & & \\ 0 & t & 1+t & 1 & 1 & & \\ & t^2 & t & 1+t & 1 & 1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & t^2 & t & 1+t & 1 \\ & & & & t^2 & t & 1+t \end{bmatrix}.$$

Expanding this along the first column gives

$$\det((M_n)_{21})_{12} = t \det M_{n-3} - t^2 \det(((M_n)_{21})_{12})_{21}.$$

But $(((M_n)_{21})_{12})_{21} = (M_{n-2})_{21}$. Thus

$$\det(M_n)_{21} = \det M_{n-2} - t \det M_{n-3} + t^2 \det(M_{n-2})_{21}.$$

Similarly we analyze $(M_n)_{22}$:

$$(M_n)_{22} = \begin{bmatrix} 1 & 1 & & & & & & \\ 1+t & 1 & 1 & & & & & \\ t^2 & t & 1+t & 1 & 1 & & & \\ 0 & t^2 & t & 1+t & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & t^2 & t & 1+t & 1 & 1 \\ & & & & t^2 & t & 1+t & 1 \\ & & & & & t^2 & t & 1+t \end{bmatrix}.$$

Expansion along the first column yields

$$\begin{aligned} \det(M_n)_{22} &= \det(M_{n-1})_{21} - (1+t)\det((M_n)_{22})_{21} + t^2 \det((M_n)_{22})_{22} \\ &= \det(M_{n-1})_{21} - (1+t)\det M_{n-3} + t^2 \det M_{n-4}. \end{aligned}$$

All this combines to

$$\begin{aligned}\det M_n &= (1+t)\det M_{n-1} - t\det M_{n-2} + t^2\det M_{n-3} - t^3\det(M_{n-2})_{21}\\ &\quad + t^2\det(M_{n-1})_{21} - (1+t)t^2\det M_{n-3} + t^4\det M_{n-4}\\ &= (1+t)\det M_{n-1} - t\det M_{n-2} - t^3\det M_{n-3} + t^4\det M_{n-4}\\ &\quad + t^2\det(M_{n-1})_{21} - t^3\det(M_{n-2})_{21}.\end{aligned}$$

The two recursive formulations of $\det M_n$ and $\det(M_n)_{21}$ show that for all $n \geq 4$, the t-degree of $\det M_n$ is at most $n-2$, and that the t-degree of $\det(M_n)_{21}$ is at most $n-4$.

Let c_n be the coefficient of t^{n-2} in $\det M_n$. The recursive formula for $\det M_n$ then shows that for all $n \geq 8$,

$$c_n = c_{n-1} - c_{n-3} + c_{n-4}.$$

From $c_4 = 1$, $c_5 = c_6 = c_7 = 0$ it then follows that

$$c_n = \begin{cases} 1 & \text{if } n \equiv 2, 3 \text{ or } \ 4 \text{ modulo } 6,\\ 0 & \text{if } n \equiv 0, 1 \text{ or } \ 5 \text{ modulo } 6.\end{cases}$$

If n is congruent to 2, 3, or 4, then $\det M_n$ is a polynomial in t of degree exactly $n-2$. Thus as $\det M_n$ is non-zero and of degree at most $n-2$ for all n, this means that $s_0 = \frac{\det M_{n-1}}{\det M_n}$ is not an element of $k[t]$. ■

The previous four propositions imply:

Theorem 5: *Let k be a field of characteristic 2. Let R be the two-dimensional ring $\frac{k[t,x,y]}{(x^4+x^3y+x^2y^2+tx^2y^2+txy^3+t^2y^4)}$. With $I = (x,y)R$, the set of associated primes of all the Frobenius powers of I is infinite, where Frobenius powers of I are all the ideals of the form $(x^{2^e}, y^{2^e})R$, as e varies over the non-negative integers.*

If in addition the polynomial U^2+U+1 is irreducible over k, R is an integral domain.

Proof: Let $\tau_e = 1 + \sum_{i=1}^{2e} t^{2^{i-1}}$. By Proposition 2, $\tau_e x^{2^{2e}-1}y^2 \in (x^{2^{2e}}, y^{2^{2e}})R$, and by Proposition 4, whenever e is congruent to 1 or to 2 mod 3, then $x^{2^{2e}-1}y^2 \notin (x^{2^{2e}}, y^{2^{2e}})R$. Thus for each e, there is an irreducible monic factor ρ_e of τ_e such that the prime ideal (x, y, ρ_e) is associated to $(x^{2^e}, y^{2^e})R$. But the set of all the ρ_e is infinite. ■

Modulo each minimal prime ideal Katzman's ideal from [K1] is principal. Thus the tight closure of every Frobenius power of Katzman's ideal is the same as its integral closure (see Hochster-Huneke [HH, Corollary 5.8]), so that the set of associated primes of the tight closures of all the Frobenius powers of Katzman's ideal is finite. For the example in this paper, the ideal I is not principal modulo the minimal primes (as R can be a domain). Nevertheless, the set of associated primes of the tight closures of all the Frobenius powers of I is still finite:

Theorem 6: *Let k be a field of characteristic 2 over which the polynomial U^2+U+1 is irreducible. Let $R = \frac{k[t,x,y]}{(x^4+x^3y+x^2y^2+tx^2y^2+txy^3+t^2y^4)}$. Then $\bigcup_e \mathrm{Ass}\left(\frac{R}{(x^{2^e},y^{2^e})^*}\right)$ is finite, where $(x^{2^e},y^{2^e})^*$ denotes the tight closure of $(x^{2^e},y^{2^e})R$. Also, tight closure of $(x^{2^e},y^{2^e})R$ commutes with localization for all e.*

Proof: Let $S = R[\frac{x}{y}]$. Note that S is a module-finite extension of R, so that for any ideal J of R, $J^* = (JS)^* \cap R$ (see Huneke [Hu, Theorem 1.7]). Thus as any primary decomposition of $(JS)^*$ contracts to a possibly redundant primary decomposition of $(JS)^* \cap R = J^*$, for the first part it suffices to prove that $\bigcup_e \mathrm{Ass}\left(\frac{S}{((x^{2^e},y^{2^e})S)^*}\right)$ is finite. But $(x^{2^e},y^{2^e})S = (y^{2^e})S$ is a principal ideal, so that its tight closure is simply its integral closure (Hochster-Huneke [HH, Corollary 5.8]). Also, the set of associated primes of all the integral closures of Frobenius powers of a principal ideal in S (such as of yS) is finite. This proves the first part.

For the second part, by Lemmas 1 and 2 in Smith [Sm] it suffices to prove that tight closure of $(x^{2^e},y^{2^e})S = (y^{2^e})S$ commutes with localization for all e. But tight closure of a principal ideal is simply its integral closure, and that commutes with localization. ■

Thus the question remains:

Question 7: *Does there exist a commutative Noetherian ring R of positive prime characteristic p which contains an ideal I for which the set of associated primes of the tight closures of all the Frobenius powers of I is infinite?*

One can easily modify the constructed ideal with infinitely many associated primes of the Frobenius powers into a homogeneous ideal with infinitely many associated primes of the Frobenius powers: let $R' = k[x,y,t,s]/(g')$, where $g' = s^2(x^4+x^3y+x^2y^2) + st(x^2y^2+xy^3) + t^2y^4$ is a (bi)homogeneous polynomial in the variables x, y and s, t. As $g = g'|_{s=1}$, then clearly g' is irreducible whenever g is. Also, g is the polynomial $\frac{1}{s^2}g'$ after rewriting $\frac{t}{s}$ as a new variable t. Thus also the ideal $(x,y)R'$ satisfies the property that the set of associated prime ideals of all its Frobenius powers is infinite. In fact, the embedded primes of the eth Frobenius power of $(x,y)R'$ are of the form $(x,y,\rho_e)R'$, where ρ_e is an irreducible factor of the s-homogenization $s^{2^{2e-1}} + \sum_{i=1}^{2e} t^{2^{i-1}} s^{2^{i-1}(2^{2e-1}-1)}$ of τ_e.

As all these prime ideals are contained in the maximal homogeneous ideal $(x, y, s, t)R'$, this constructs a 3-dimensional Noetherian local domain $R'_{(x,y,s,t)}$ in which the two-generated ideal (x, y) satisfies the property that the set of associated primes of all of its Frobenius powers is infinite.

Furthermore, as in the recent work of Singh [Si] and Katzman [K2], this example can be further modified to produce another example of a second local cohomology module (over a local ring) with infinitely many associated prime ideals. In fact, here is a general method for translating some examples of ideals with infinitely many associated primes of Frobenius powers to local cohomology modules with infinitely many associated primes:

Theorem 8: *Let R_0 be a Noetherian ring, d an even positive integer and $r_0, \ldots, r_d \in R_0$. For each $n \geq 1$, set M_n be the $(n-2) \times (n-2)$ matrix consisting of $d+1$ diagonals of non-zero entries: these diagonals are adjacent, ranging from the leftmost diagonal of r_ds to the rightmost diagonal of r_0s, and the main diagonal consisting of the entries $r_{d/2}$. Suppose that for some infinite set of integers n, the set of associated prime ideals of the ideals $(\det M_n)R_0$ is infinite.*

Let x, y, u, v be variables over R_0. Set $G = \sum_{i=0}^{d} r_i x^i y^{d-i} v^i u^{d-i} \in R_0[x, y, u, v]$, and let R be the graded ring $R_0[x, y, u, v]/(G)$. Then $H^2_{(x,y)}(R)$ has infinitely many associated prime ideals.

Furthermore, if for some prime ideal m of R_0, the set of associated prime ideals of the $(\det M_n)R_0$ contained in m is infinite, then $H^2_{(x,y)}(R')$ has infinitely many associated prime ideals, where R' is the localization of R at the prime ideal $(m + (x, y, u, v))R$.

Proof: The proof closely follows that from [K2]. By the localization properties of local cohomology it suffices to prove the non-localized case.

Set $S = R_0[x, y, u, v]$. Note that $H^2_{(x,y)}(R)$ is the cokernel of $H^2_{(x,y)}(S) \xrightarrow{G} H^2_{(x,y)}(S)$. But $H^2_{(x,y)}(S)$ is a graded free $R_0[u, v]$-module with a homogeneous basis $x^{-a}y^{-b}$, $a, b \geq 1$, and G is homogeneous. We order the basis by $x^{-a}y^{-b} > x^{-a'}y^{-b'}$ if $a + b > a' + b'$ or if $a + b = a' + b'$ and $a < a'$. Thus with $n > d$, the $(-n+1)$-graded component of $H^2_{(x,y)}(R)$ is the cokernel of the $(n-2) \times (n+d-2)$ matrix N_n whose non-zero entries lie on $d+1$ adjacent diagonals: all the entries in the ith diagonal (from the left) are $r_{d-i}u^{d-i}v^i$, with the last diagonal (of entries $r_d u^d$) passing through the $(1,1)$ entry of N_n:

$$N_n = \begin{bmatrix} r_d u^d & r_{d-1}u^{d-1}v & \cdots & r_0 v^d & & & & \\ & r_d u^d & r_{d-1}u^{d-1}v & \cdots & r_0 v^d & & & \\ & & r_d u^d & r_{d-1}u^{d-1}v & \cdots & r_0 v^d & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & r_d u^d & r_{d-1}u^{d-1}v & \cdots & r_0 v^d \end{bmatrix}.$$

We further impose another grading on the variables and N_n: $\deg(u) = (1,0)$, $\deg(v) = (1,1)$; and impose a grading on the free $R_0[u,v]$-modules by setting $\deg(u^a v^b \vec{e}_i) = (a+b, b+i)$. Then the columns of N_n are homogeneous elements of $R_0[u,v]^{n-2}$ of degrees

$$(d,1), (d,2), (d,3), \ldots, (d, n+d-2).$$

The cokernel of N_n is the direct sum of its graded components, each graded component being a R_0-module. In particular, for $n > d$, the component of the cokernel of N_n of degree $(n+d-3, n-2+\frac{d}{2})$ is the quotient of the free $k[s,t]$-module with basis

$$u^{\frac{d}{2}} v^{n-3+\frac{d}{2}} \vec{e}_1, u^{\frac{d}{2}+1} v^{n-4+\frac{d}{2}} \vec{e}_2, \ldots, u^{n-3+\frac{d}{2}} v^{\frac{d}{2}} \vec{e}_{n-2},$$

modulo the $k[s,t]$-submodule generated by the $u^{i-1-\frac{d}{2}} v^{n+\frac{d}{2}-i-2} \vec{c}_i$, $i = \frac{d}{2}+1, \ldots, n+\frac{d}{2}-2$, where $\vec{c}_i$ denotes the ith column of N_n. Thus $(\text{coker } N_n)_{(n+d-3, n-2+\frac{d}{2})} = \text{coker } M_n$, where M_n is as in the statement of the theorem, and explicitly equals

$$M_n = \begin{bmatrix} r_{\frac{d}{2}} & \cdots & r_0 & & & & \\ \vdots & \ddots & & \ddots & & & \\ r_d & \cdots & r_{\frac{d}{2}} & \cdots & r_0 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & r_d & \cdots & r_{\frac{d}{2}} & \cdots & r_0 \\ & & & \ddots & & \ddots & \vdots \\ & & & & r_d & \cdots & r_{\frac{d}{2}} \end{bmatrix}.$$

Then $\det M_n$ is a zero-divisor on $H^2_{(x,y)}(R)$, so by assumption on the $\det M_n$, $H^2_{(x,y)}(R)$ has infinitely many associated primes. ∎

This immediately applies to the constructions with the ideal I:

Corollary 9: *Let k be a field of characteristic 2, and R either $k[s,t,x,y,u,v]/(G)$ or the localization of $k[s,t,x,y,u,v]/(G)$ at the image of (s,t,x,y,u,v), where $G = s^2(x^4v^4 + x^3yv^3u + x^2y^2v^2u^2) + st(x^2y^2v^2u^2 + xy^3vu^3) + t^2y^4u^4$. Then $H^2_{(x,y)}(R)$ has infinitely many associated prime ideals.*

The same holds if s above is replaced everywhere by 1.

Proof: With notation as in the previous proof, $R_0 = k[s,t]$, and N_n equals

$$\begin{bmatrix} t^2u^4 & stu^3v & s(s+t)u^2v^2 & s^2uv^3 & s^2v^4 & & & \\ & t^2u^4 & stu^3v & s(s+t)u^2v^2 & s^2uv^3 & s^2v^4 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & t^2u^4 & stu^3v & s(s+t)u^2v^2 & s^2uv^3 & s^2v^4 \end{bmatrix},$$

(or this matrix with s set to 1), and M_n from the previous proof is an s-homogenization of the matrix M_n as in the proof Proposition 4. By Proposition 3, for infinitely many even n, both $\det M_n$ and $\tau_{\frac{n}{2}}$ multiply $\vec{e}_1$ into the image of the matrix M_n (see Proposition 2 and the proof of Proposition 4). Thus $\det M_n$ and $\tau_{\frac{n}{2}}$ have a factor in common. But by Proposition 3, these τ_n are not contained in a finite set of primes. Thus the corollary follows by the previous theorem. ■

References

[HH] M. Hochster and C. Huneke, Tight closure, invariant theory, and the Briançon-Skoda Theorem, *J. Amer. Math. Soc.*, **3** (1990), 31-116.

[H] C. Huneke, Tight closure and its applications, C.B.M.S. Regional Conference Series in Math. No **88** (1996).

[K1] M. Katzman, Finiteness of $\cup_e \text{Ass}\, F^e(M)$ and its connections to tight closure, *Illinois J. Math.*, **40** (1996), no.2, 330-337.

[K2] M. Katzman, An example of an infinite set of associated primes of a local cohomology module, preprint, 2002.

[R] L. J. Ratliff, Jr., On prime divisors of I^n, n large, *Michigan Math. J.*, **23** (1976), 337-352.

[Si] A. K. Singh, p-torsion elements in local cohomology modules, *Math. Research Letters*, **7** (2000), 165-176.

[Sm] K. E. Smith, Tight closure commutes with localization in binomial rings, *Proc. Amer. Math. Soc.*, **129** (2001), 667-669.

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