IDEALS CONTRACTED FROM 1-DIMENSIONAL OVERRINGS WITH AN APPLICATION TO THE PRIMARY DECOMPOSITION OF IDEALS

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Abstract. We prove that each ideal of a locally formally equidimensional analytically unramified Noetherian integral domain is the contraction of an ideal of a one-dimensional semilocal birational extension domain. We give an application to a problem concerning the primary decomposition of powers of ideals in Noetherian rings. It is shown in [S2] that for each ideal I in a Noetherian commutative ring R there exists a positive integer k such that, for all $n \geq 1$, there exists a primary decomposition $I^n = Q_1 \cap \cdots \cap Q_s$ where each Q_i contains the nk-th power of its radical. We give an alternate proof of this result in the special case where R is locally at each prime ideal formally equidimensional and analytically unramified.

In this paper we prove that every ideal in a locally formally equidimensional analytically unramified Noetherian ring R is the contraction of an ideal of a one-dimensional semilocal extension which is essentially of finite type over R. If R is a domain, the extension may be taken to be birational, i.e., with the same field of fractions as R.

By passing to the extended Rees ring $R[It, t^{-1}]$ of an ideal I of R, these contraction properties give a type of uniform primary decomposition for the powers of I. This is based on the fact that the primary decomposition of a height-one ideal in a one-dimensional semilocal ring is unique, and the primary decomposition for powers of a fixed ideal in such a ring is obtained from just taking the powers of the primary components of the fixed ideal. Furthermore, contracting primary decompositions from an overring gives a primary decomposition for the contracted ideal. Our interest in establishing this result was motivated by a question, recently answered in [S2], concerning the primary decompositions of powers of an ideal.

All rings we consider are commutative and our notation is as in [AM] and [M].

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1. Powers of ideals and primary decompositions.

Let I be a proper ideal of a commutative Noetherian ring R. It is known that only finitely many prime ideals of R are associated primes of a power of I [Rat], and that all suitably large powers of I have the same associated primes [B]. In considering primary decompositions of the powers of I, it is natural to ask about the growth of the exponents of primary components of I^n, where the *exponent* of a primary ideal Q with $\text{rad}(Q) = P$ is the smallest positive integer e such that $P^e \subseteq Q$ [ZS, page 153]. If Q is a primary component associated to a minimal prime P of I, then $Q^{(n)}$, the inverse image in R of $Q^n R_P$, is the unique P-primary component of I^n and the exponent growth of the P-primary component of I^n is linearly bounded as a function of n in the sense that if Q has exponent e, then $P^{en} \subseteq Q^{(n)}$. The situation, however, for embedded associated primes of I is not as obvious [He], [S1]. By proving a version of the linear uniform Artin-Rees lemma in the spirit of Huneke's paper [Hu], it is shown in [S2] that there exist primary decompositions of the powers I^n of I for which the exponent growth of the primary components is linearly bounded. We present here an alternative approach to obtain a special case of this result.

If $I = Q_1 \cap \cdots \cap Q_s$ is a primary decomposition, then we clearly have

$$I^n \subseteq Q_1^n \cap \cdots \cap Q_s^n, \tag{1.1}$$

but in general the inclusion in (1.1) may be proper, and powers of a primary ideal need not be primary.

A case where equality holds in (1.1) is if the intersection of the Q_i is also their product. And a case where the intersection of ideals is their product is that of pairwise comaximal ideals. Thus if $\dim(R/I) = 0$, then the primary components of I are pairwise comaximal and for each positive integer n, $I^n = Q_1^n \cap \ldots Q_s^n$ is the unique irredundant primary decomposition of I^n. Our proof of a special case of the linearly bounded exponent growth result of [S2] is based on obtaining primary decompositions for the powers of I via descent from a regular principal ideal of a one-dimensional semilocal extension ring.

We use the following elementary lemma.

Lemma 1.2. Suppose R is a subring of a ring S and $x \in R$ is a regular element of S. If $xR = xS \cap R$, then $x^n R = x^n S \cap R$ for each positive integer n.

Proof. We clearly have $x^nR \subseteq x^nS \cap R$. Assume by induction that $n \geq 2$ and $x^{n-1}R = x^{n-1}S \cap R$. Then

$$x^nS \cap R = x^nS \cap xR = x(x^{n-1}S \cap R) = xx^{n-1}R = x^nR,$$

where equality in the middle step uses that x is a regular element of S. $\square$

Remark 1.3. With R, S, x as in (1.2), if S is one-dimensional and Noetherian, then each associated prime of xS is a maximal ideal of S and a minimal prime of xS. Hence the ideal xS has a unique irredundant primary decomposition, say $xS = Q_1 \cap \cdots \cap Q_s$, and for each positive integer n, $x^nS = Q_1^n \cap \cdots \cap Q_s^n$ is the unique irredundant primary decomposition of x^nS. If $xR = xS \cap R$, then by (1.2), we have

$$x^nR = (Q_1^n \cap R) \cap \cdots \cap (Q_s^n \cap R) \tag{1.4}$$

for each positive integer n. Since Q_i^n is primary in S, the ideal $Q_i^n \cap R$ is primary in R. The decomposition given in (1.4) may fail to be irredundant, but it can be shortened to an irredundant primary decomposition. Moreover, if $\operatorname{rad}(Q_i) = M_i$ and e_i is the exponent of Q_i, then for $k = \max\{e_1, \ldots, e_s\}$ we have $M_i^{kn} \subseteq Q_i^n$ for each i, and therefore $(M_i \cap R)^{kn} \subseteq (Q_i^n \cap R)$ for each i. This shows that the exponent growth of the primary components of x^nR in a primary decomposition obtained from (1.4) is linearly bounded.

Remark 1.5. Let I be an ideal of a Noetherian ring R and let t be an indeterminate over R. With $S = R[It, t^{-1}]$, the extended Rees ring of I, we clearly have $t^{-n}S \cap R = I^n$ for each positive integer n. Therefore to show the existence of primary decompositions of the powers of I with linearly bounded exponent growth, by passing from I to the principal ideal $t^{-1}S$, it suffices to consider the case where I is a principal ideal generated by a regular element.

In view of (1.3) and (1.5), we are led to ask:

Question 1.6. Suppose R is a Noetherian ring and $x \in R$ is a regular element. Does there exist a one-dimensional Noetherian extension ring S of R such that x is a regular element of S and $xR = xS \cap R$?

In §2 we present an affirmative answer to (1.6) for a restricted class of Noetherian rings by proving that ideals in this restricted class of rings contract from one-dimensional Noetherian ring extensions. We are aware of no example where (1.6) has a negative answer.

2. One-dimensional semilocal extension rings.

Let R be a Noetherian ring and let I by an ideal in R. We prove in this section that under certain assumptions on R, I contracts from a one-dimensional semilocal Noetherian ring extension. Let $I = Q_1 \cap \cdots \cap Q_s$ be an irredundant primary decomposition and let $P_i = \mathrm{rad}(Q_i)$. Our first step is to prove that each $Q_i R_{P_i}$ contracts from a Noetherian ring extension of R_{P_i} which has smaller dimension than R_{P_i} (see Theorem 2.1 for the precise statement). This and induction on dimension then imply that each $Q_i R_{P_i}$, and hence also Q_i, is contracted from a one-dimensional Noetherian ring extension. Theorems 2.3 and 2.4 then prove the contraction property for all ideals I in locally formally equidimensional analytically unramified Noetherian rings. Corollary 2.6 then gives the linear growth of exponents of primary components of powers of an ideal.

Theorem 2.1. Let $(R, \mathbf{m})$ be a reduced local ring and let Q be an $\mathbf{m}$-primary ideal. Assume that the integral closure R' of R in its total quotient ring is a finitely generated R-module and that the height of each maximal ideal of R' is at least two. Then there exist regular elements $a, b \in \mathbf{m}$ such that $\mathbf{m}R[a/b]$ is a nonmaximal prime ideal of $R[a/b]$, and $S = R[a/b]_{\mathbf{m}R[a/b]}$ is a local ring with $\dim(S) < \dim(R)$ and $QS \cap R = Q$.

Proof. Since R is reduced, the total quotient ring of R is a finite product of fields and R' is a finite product of normal Noetherian domains, say $R' = R'_1 \times \cdots \times R'_m$. Let r be a positive integer such that $\mathbf{m}^r \subseteq Q$. By the Artin-Rees lemma, there exists a positive integer n such that $\mathbf{m}^n R' \cap R \subseteq \mathbf{m}^r$. Let $a, b \in \mathbf{m}^n$ be such that the ideal $(a, b)R'$ has height two. It follows that a and b are regular elements of R and the images of a, b in R'_i form a regular sequence for $1 \le i \le m$. Let t be an indeterminate over R' and let $\phi' : R'[t] \to R'[a/b]$ be the R'-algebra homomorphism such that $\phi'(t) = a/b$. Then $R'[t] = R'_1[t] \times \cdots \times R'_m[t]$. Since the images of a, b in each R'_i form a regular sequence, $\ker(\phi') = (bt - a)R'[t]$. Let $\phi : R[t] \to R[a/b]$ be the restriction of ϕ'. Since $\ker(\phi') \subset \mathbf{m}^n R'[t]$ and $\ker(\phi) = \ker(\phi') \cap R[t]$, $\ker(\phi) \subset \mathbf{m}^r R[t]$. Since $\mathbf{m}R[t]$ is a nonmaximal prime ideal of $R[t]$ with $\mathrm{ht}(\mathbf{m}) = \mathrm{ht}(\mathbf{m}R[t])$, and since $QR[t]$ is $\mathbf{m}R[t]$-primary and $\ker(\phi) \subset QR[t]$, it follows that $\mathbf{m}R[a/b]$ is a nonmaximal prime ideal and $QR[a/b]$ is $\mathbf{m}R[a/b]$-primary. Therefore $S = R[a/b]_{\mathbf{m}R[a/b]}$ is a local ring with $\dim(S) < \dim(R)$ and $QS \cap R = Q$. $\square$

Corollary 2.2. Let $(R, \mathbf{m})$ be a formally equidimensional analytically unramified local ring with $\dim(R) = d \geq 1$, and let Q be an $\mathbf{m}$-primary ideal. There exists a one-dimensional local extension ring T of R such that T is a subring of the total quotient ring of R and is essentially of finite type over R, and is such that Q contracts from T.[1]

Proof. The fact that R is analytically unramified implies that the integral closure R' of R in its total quotient ring is a finitely generated R-module, and that finitely generated R-subalgebras of the the total quotient ring of R also have this property [R1, Theorem 1.5]. The assumption that R is formally equidimensional implies that: (i) R is universally catenary, (ii) equidimensional local rings essentially of finite type over R are formally equidimensional, and (iii) all the maximal ideal of R' have height equal to $\dim(R) = d$ [M, Theorem 31.6]. If $d > 1$, then (2.2) implies the existence of regular elements $a, b \in \mathbf{m}$ such that $\mathbf{m}R[a/b]$ is a nonmaximal prime ideal and $S = R[a/b]_{\mathbf{m}R[a/b]}$ is a local ring with $QS \cap R = Q$. Since R is equidimensional and universally catenary, $\dim(S) = d - 1$, and S is equidimensional, and therefore formally equidimensional. A simple induction argument implies the existence of a one-dimensional local extension T of R such that T is essentially of finite type over R, a subring of the total quotient ring of R, and $QT \cap R = Q$. □

Now let again $I = Q_1 \cap \cdots \cap Q_s$ be a primary decomposition of I and let $P_i = \operatorname{rad}(Q_i)$. By (2.2) we know that each Q_i contracts from a one-dimensional local extension ring as long as R_{P_i} is formally equidimensional and analytically unramified. The following lemma proves that in this case then I is also contracted from a one-dimensional extension ring.

Lemma 2.3. With notation as above, assume there exists, for each i, $1 \leq i \leq s$, a one-dimensional local extension ring T_i of R_{P_i} such that $Q_iR_{P_i} = Q_iT_i \cap R_{P_i}$. Let T be the direct product $T_1 \times \cdots \times T_s$. Then T is a one-dimensional semilocal extension ring of R and I contracts from T, i.e., $I = IT \cap R$.

Proof. Since the canonical map of R into the direct product $R_{P_1} \times \cdots \times R_{P_s}$ is an injection, and R_{P_i} is a subring of T_i for $1 \leq i \leq s$, the canonical map of R into T is an injection.

[1] An alternative proof of this corollary can be given using work of Rees. For simplicity let $(R, \mathbf{m})$ be a reduced equidimensional complete local ring, and let Q be an $\mathbf{m}$-primary ideal. There exists an ideal I, generated by parameters, such that the integral closure of I is contained in Q [R1]. By [R2], it follows that the equations defining the Rees algebra $R[It]$ have coefficients contained in Q, and it then follows that a suitable affine piece of the blowup of I, localized at the extension of the maximal ideal $\mathbf{m}$ satisfies the conclusion of (2.2).

It is clear that T is one-dimensional and semilocal. Since Q_i is primary it is the inverse image in R of $Q_iR_{P_i}$. Therefore $Q_iT \cap R = Q_i$ for $1 \le i \le s$. Hence

$$IT \cap R \subseteq (Q_1T \cap R) \cap \cdots \cap (Q_sT \cap R) = Q_1 \cap \cdots \cap Q_s = I. \quad \square$$

Thus every ideal in a locally analytically unramified and formally equidimensional Noetherian ring is contracted from a one-dimensional Noetherian ring extension which is essentially of finite type. In case R is an integral domain one can take the extension to be a domain by replacing the finite direct product in the preceding proof with an intersection. Theorem 2.4 is related to [GH, (3.21)] which applies to a Cohen-Macaulay domain.

Theorem 2.4. Let I be an ideal of a Noetherian integral domain R. Assume that for each $P \in \mathrm{Ass}(R/I)$ the local ring R_P is analytically unramified and formally equidimensional. Then there exists a one-dimensional semilocal birational extension T of R such that T is essentially of finite type over R and $IT \cap R = I$.

Proof. Let $\mathrm{Ass}(R/I) = \{P_i\}_{i=1}^s$, and let Q_i be a P_i-primary component of I. By (2.2) there exists a one-dimensional local extension domain T_i of R_{P_i} such that T_i is a subring of the fraction field of R and $QT_i \cap R_{P_i} = QR_{P_i}, 1 \le i \le s$. Since T_i has center P_i on R, for $i \neq j$, the one-dimensional local domains T_i and T_j are not dominated by a common valuation domain. Hence by [HO, (2.9) and (2.10)], $T = \cap_{i=1}^s T_i$ is a one-dimensional semilocal domain and each localization of T at a prime ideal is essentially of finite type over R. It follows that T is essentially of finite type over R, and $Q_iT \cap R = Q_i$ for each i, $1 \le i \le s$, so $IT \cap R = I$. $\square$

As a consequence of these results on contractions of ideals we obtain our results on exponents of primary components of powers of ideals:

Theorem 2.5. Let R be a Noetherian ring and let $x \in R$ be a regular element. Assume that for each associated prime P of $I = xR$, the local ring R_P is analytically unramified and formally equidimensional. Then there exists a positive integer k such that, for all $n \ge 1$, there exists a primary decomposition $I^n = Q_1 \cap \cdots \cap Q_s$ where each Q_i contains the nk-th power of its radical.

Proof. By (1.3), it suffices to show the existence of a one-dimensional semilocal extension ring S of R such that x is a regular element of S and $xR = xS \cap R$. This follows by (2.2) and (2.3). $\square$

Since the passage from a Noetherian ring to an extended Rees ring preserves the property of being locally formally equidimensional and analytically unramified, Remark 1.5 and Theorems 2.4 and 2.5 imply:

Corollary 2.6. Let R be a Noetherian ring that is locally at each prime ideal analytically unramified and formally equidimensional, and let I be an ideal of R. There exists a one-dimensional semilocal extension ring S of R which is essentially of finite type over R and is such that every power of I is contracted from a principal ideal in S. If R is an integral domain one can take S to be a domain. Also, there exists a positive integer k such that, for all $n \geq 1$, there exists a primary decomposition $I^n = Q_1 \cap \cdots \cap Q_s$ where each Q_i contains the nk-th power of its radical. $\square$

Remark 2.7. In general, an ideal I of a Noetherian integral domain R need not be the contraction of a principal ideal of a birational extension of R. For example, if K is a field, t is an indeterminate over K, and R is the localization of $K[t^3, t^4, t^5]$ at the maximal ideal $(t^3, t^4, t^5)K[t^3, t^4, t^5]$, then the ideal $I = (t^3, t^4)R$ is not the contraction of a principal ideal of a birational extension of R.

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