DISCRETE VALUATIONS CENTERED ON LOCAL DOMAINS

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Abstract. We study applications of discrete valuations to ideals in analytically irreducible domains, in particular applications to zero divisors modulo powers of ideals. We prove a uniform version of Izumi's theorem and calculate several examples illustrating it, such as for rational singularities. The paper contains a new criterion of analytic irreducibility, a new criterion of one-fiberedness, and a valuative criterion for when the normal cone of an ideal in an integrally closed domain is reduced.

Valuations of fields and field extensions play an important role in the study of algebras and algebraic varieties. The valuations of a function field of transcendence degree one, for instance, completely determine a smooth projective curve, giving a model of the function field. In higher dimensions the picture is much more complicated. We present here some higher dimensional ideal- and ring-theoretic properties determined by discrete valuations centered on local domains. Most of the results in this paper are in the spirit of Rees valuations and are about the information that discrete valuations contain about powers of ideals.

Section 1 explores relations among valuations, bounding one with respect to finitely and even infinitely many others either locally or globally. An example is Izumi's theorem (cf. [Iz]), which was given in an algebraic setting first by Rees [R_2], and is given in a somewhat more general version in this paper in Theorem 1.3. Rees' proof was based on Lipman's theory of intersection multiplicities [Li], and our proof uses Cutkosky's [C2] results on condition (E) of Heinzer and Lantz.

Section 2 treats zero divisors modulo powers of an ideal via valuations: if a product of two elements lies in a high power of an ideal, in what power does at least one of the elements have to be? Good results of this form only hold in analytically unramified rings, which gives a new criterion of analytic irreducibility. Section 2 also contains a new criterion of one-fiberedness.

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In order to bring the difficult proofs of Izumi's theorem down to earth, and in order to understand the bounding relations among valuations and zero divisors modulo high powers of an ideal better, we explicitly calculate several examples. Section 3 is thus devoted to examples. We develop some ad hoc techniques for calculating the graded ring associated to a discrete valuation. Our examples of Section 3 show that in general the results of Section 2 are sharp, but that in some specific cases they can be improved significantly.

One way of looking at zero divisors modulo high powers of an ideal is via examining the zero divisors in the normal cone of the ideal more carefully. We study the reducedness property of normal cones in Section 4, and give a valuative criterion for it. This criterion enables a short and very canonical proof of the fact that reduced normal cones of prime ideals in regular rings containing fields are domains (cf. Huneke, Simis and Vasconcelos [HSV]).

With the exception of Proposition 1.8, throughout this paper we use the term "valuation" to stand for discrete valuations of rank one or the discrete valuation ring associated to it. All rings in this paper are noetherian.

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§1. RELATIONS AMONG VALUATIONS

An interesting question in the study of valuations are the relations between two of them. Nagata's theorem on the analytic independence of valuations ([Na], (11.11)) implies that for any two valuations v, w on a field K and any integer n there exists an $x \in K \setminus \{0\}$ with $v(x) = n$, $w(x) = 0$. This changes dramatically in case there is some additional structure around. We prove in this section a strengthened form of Izumi's theorem, bounding linearly one valuation by valuations of a restricted kind. Izumi [Iz] originally proved this for integral analytic local algebras, and later Rees [R_2] proved it in an algebraic setting. We present here another proof, and in a more general context. At the end of the section we prove a result about pointwise relations among finitely many valuations.

Let $(R, \mathfrak{m})$ be an excellent local domain with field of fractions K. A valuation v of K is called *an* $\mathfrak{m}$*-valuation* if $v(x) \geq 0$ for all $x \in R \setminus \{0\}$, $v(x) > 0$ for $x \in \mathfrak{m} \setminus \{0\}$, and the transcendence degree of the residue field $k(v)$ of v over the residue field k of R is exactly the dimension of R minus 1. In this case the valuation ring V of v is essentially of finite type over R.

A valuation v is called *a Rees valuation* of an ideal I if its valuation ring is the localization of the normalization S of $R[It]$ (resp. $R[It, t^{-1}]$) at a minimal prime overideal over IS (resp. $t^{-1}S$). Every Rees valuation which is positive on $\mathfrak{m}$ is an $\mathfrak{m}$-valuation, and, moreover, every $\mathfrak{m}$-valuation is a Rees valuation of an $\mathfrak{m}$-primary ideal (by [R_2, Appendix]).

Throughout we let $\overline{\mathfrak{a}}$, as usual, denote the integral closure of an ideal $\mathfrak{a}$.

Lemma 1.1. *Let $(R,\mathfrak{m})$ be an analytically irreducible domain. Then every $\mathfrak{m}$-valuation on R extends naturally to an $\mathfrak{m}\hat{R}$-valuation, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R.*

Proof: By assumption $\hat{R}$ is an integral domain. Let w be the Rees–valuation of some $\mathfrak{m}$–primary ideal I. As $\widehat{R}/R$ is faithfully flat and I is $\mathfrak{m}$-primary, we have $\overline{I^n\widehat{R}} = \overline{I^n}\widehat{R}$. Hence

$$\overline{I^n\widehat{R}}/I\cdot\overline{I^n\widehat{R}} \cong \overline{I^n}\widehat{R}/I\cdot\overline{I^n}\widehat{R} \cong \overline{I^n}/I\cdot\overline{I^n} \quad \text{for all } n\in\mathbb{N}\,.$$

This implies that

$$\overline{R[It]}/I\cdot\overline{R[It]} \cong \overline{\widehat{R}[I\widehat{R}t]}/I\cdot\overline{\widehat{R}[I\widehat{R}t]},$$

therefore there exists a unique minimal prime ideal $\widehat{\mathfrak{P}}$ over $I\cdot\overline{\widehat{R}[I\widehat{R}t]}$ restricting to the minimal prime ideal $\mathfrak{P}$ over $I\cdot\overline{R[It]}$, where $\mathfrak{P}$ corresponds to the valuation w. The prime $\widehat{\mathfrak{P}}$ corresponds to a valuation $\widehat{w}$, which is then the natural extension of w. This proves the lemma.

For a valuation w on R set

$$I_n(w) := \{x\in R : x=0 \text{ or } w(x)\geq n\}.$$

Note that $I_n(w)$ is an integrally closed ideal of R, and in case w is positive on $\mathfrak{m}$,

$$\overline{\mathfrak{m}^n}\subseteq I_n(w). \tag{1.2}$$

In fact, $w(\mathfrak{m})>0$, hence $\mathfrak{m}\cdot W\subseteq\mathfrak{m}_W$, where W denotes the valuation ring of w, and therefore also $\mathfrak{m}^n\cdot W\subseteq\mathfrak{m}_W^n$, implying

$$\overline{\mathfrak{m}^n}\subseteq\mathfrak{m}^n\cdot W\cap R\subseteq\mathfrak{m}_W^n\cap R = I_n(w).$$

In the following we will prove that for each $\mathfrak{m}$-valuation v there exists an integer l such that for all integers n,

$$I_{ln}(v)\subseteq\overline{\mathfrak{m}^n}.$$

Theorem 1.3. *(Izumi's theorem, cf. [R_2]) Let $(R,\mathfrak{m})$ be an analytically irreducible excellent domain and let v be an $\mathfrak{m}$-valuation. Then there exists a constant $c=c(v)>0$ such that for any $\mathfrak{m}$-valuation w on R, and for any $x\in R\setminus\{0\}$,*

$$v(x)\leq c(v)\cdot w(x).$$

This result is a strengthening of Rees' version of Izumi's theorem (cf. [R_2]) which, however, can be deduced directly from it. We give here a new proof, using induction on $d=\dim(R)$. We start with a reduction:

Lemma 1.4. *Let $(R, \mathfrak{m})$ be an analytically irreducible domain, and v a valuation on the fraction field of R. Suppose that one of the conditions below is satisfied:*

1. *For every $\mathfrak{m}$-valuation w there exists a constant $c(v,w)$ such that for any $x \in R \setminus \{0\}$,*
$$v(x) \leq c(v,w) \cdot w(x).$$
2. *For every Rees valuation w of $\mathfrak{m}$, there exists a constant $c(v,w)$ such that for any $x \in R \setminus \{0\}$,*
$$v(x) \leq c(v,w) \cdot w(x).$$
3. *There exists a constant l such that for all $n \in \mathbb{N}$, $I_{ln}(v) \subseteq \overline{\mathfrak{m}^n}$.*

Then there exists a constant $c = c(v) > 0$ such that for any other $\mathfrak{m}$-valuation w on R, and for any $x \in R \setminus \{0\}$ we have

$$v(x) \leq c(v) \cdot w(x).$$

Proof: Suppose that the third condition is satisfied. Then by (1.2), for every valuation w positive on $\mathfrak{m}$, $I_{ln}(v) \subseteq I_n(w)$, whence for all $x \in R \setminus \{0\}$, $v(x) \leq (2l-1) \cdot w(x)$. Thus we may take $c(v) = 2l - 1$.

Now assume that the first or the second condition is satisfied. Let $v_1, \ldots, v_t$ be the Rees valuations of $\mathfrak{m}$. By assumption there exists a constant l such that

$$v(x) \leq \frac{l}{v_i(\mathfrak{m})} v_i(x) \quad \text{for all } x \in R \setminus \{0\} \text{ and all } i \in \{1, \ldots, t\}.$$

Thus for all $x \in R \setminus \{0\}$ with $v(x) \geq l \cdot n$ we have

$$v_i(x) \geq v_i(\mathfrak{m}) \cdot n \quad \text{for all } i \in \{1, \ldots, t\}$$

implying that $I_{ln}(v) \subseteq \overline{\mathfrak{m}^n}$. Thus we may apply the result for the third condition. This finishes the proof of the lemma.

So to prove Theorem 1.3, it suffices to prove that for any $\mathfrak{m}$-valuation w, there exists a constant $c(v,w)$ such that for all nonzero $x \in R$, $v(x) \leq c(v,w)w(x)$, or that there exists a constant l such that for every $n \in \mathbb{N}$, $I_{ln}(v) \subseteq \overline{\mathfrak{m}^n}$. This is what we do next:

Proof of Izumi's Theorem 1.3: As $(R, \mathfrak{m})$ is analytically irreducible, the integral closure $\overline{R}$ of R is local again with maximal ideal - say - $\overline{\mathfrak{m}}$, and any $\mathfrak{m}$-valuation is an $\overline{\mathfrak{m}}$-valuation as well. Thus we may assume that R is normal.

In case $d = 1$ there is only one $\mathfrak{m}$-valuation (as R is a discrete valuation ring in this case), and there is nothing to show.

Assume now that $d = 2$. By Lemma 1.1 we may pass to completion: if we can find $c(v)$ for $\hat{R}$, the same $c(v)$ will hold also in R. Thus we may assume that R is a two-dimensional complete local domain, and as in the first paragraph, that it is also integrally closed.

Set $Y := \operatorname{Spec}(R)$ and recall the following fact (cf. [Li], remark on p. 208): if $f : X \to Y$ is proper and birational, if $J \subseteq R$ is an ideal and if $\mathcal{J} \subseteq \mathcal{O}_X$ is a sheaf of ideals such that $\mathcal{J} \subseteq \overline{J \cdot \mathcal{O}_X}$ (the integral closure of $J \cdot \mathcal{O}_X$), then

$$\Gamma(X, \mathcal{J}) \subseteq \overline{J}. \tag{1.5}$$

As R is complete, we may apply Cutkosky's [C2, Theorem 2]: there exists an ideal $I \subseteq R$ such that, if $\pi : X_I \to Y$ is the normalized blow–up of I, then the center of v on X_I is an essential divisor and is equal to the reduced closed fiber $E = \pi^{-1}(\mathfrak{m})$ of π. (If I is $\mathfrak{m}$-primary, the last condition says that I is one-fibered. In general however I will not be $\mathfrak{m}$-primary.) Let $\mathcal{J} \subseteq \mathcal{O}_{X_I}$ be the sheaf of ideals of the reduced closed fiber E of π. As X_I is normal and E is the center of v on X_I, we have

$$I_n(w) = \overline{\Gamma(X, \mathcal{J}^n)}. \tag{1.6}$$

As $E = \pi^{-1}(\mathfrak{m})$ we have $\sqrt{\mathfrak{m} \cdot \mathcal{O}_{X_I}} = \mathcal{J}$, hence $\mathcal{J}^l \subseteq \mathfrak{m} \cdot \mathcal{O}_{X_I}$ for some $l \in \mathbb{N}$, so also $\mathcal{J}^{ln} \subseteq \mathfrak{m}^n \mathcal{O}_{X_I}$ for all $n \in \mathbb{N}$. Thus we get by (1.5):

$$\Gamma(X, \mathcal{J}^{ln}) \subseteq \overline{\mathfrak{m}^n} \quad \text{for all } n \in \mathbb{N},$$

and thus by (1.6)

$$I_{ln}(v) \subseteq \overline{\mathfrak{m}^n} \quad \text{for all } n \in \mathbb{N},$$

proving the case $d = 2$ with the help of Lemma 1.4.

Now let $d > 2$. Essentially this is Rees' version of Izumi's theorem, and the next step of our proof is inspired by the inductive step in Rees' proof [R_2]. For this we may assume that R has an infinite residue class field. Otherwise replace R by $R(X) := R[X]_{\mathfrak{m}R[X]}$ and note that every valuation of R extends trivially to a valuation of $R(X)$.

We first consider two $\mathfrak{m}$–valuations v and w. There exists an $\mathfrak{m}$-primary ideal $I = (x_1, \ldots, x_d)$ such that $v(I) = v(x_1) = \cdots = v(x_d)$ and $w(I) = w(x_1) = \cdots = w(x_d)$ and such that v and w are Rees valuations of I. For generic units $\lambda_1, \ldots, \lambda_d$ and $\mu_1, \ldots, \mu_d$ in R^* set $x = \sum \lambda_i x_i$ and $y = \sum \mu_i x_i$, and define

$$S = R(T)/(xT - y)$$

(i.e. S is the local ring of the generic point of the closed fiber of the blow–up of the ideal $I = (x, y) \subseteq R$, which we may assume to have height 2, generated by a regular sequence). Then $R \hookrightarrow S$ is a birational extension of excellent noetherian local domains with $\dim(S) = \dim(R) - 1$ and $\mathfrak{m}_S = \mathfrak{m} \cdot S$. Furthermore we have that the valuations v and w are $\mathfrak{m}_S$-valuations for x and y sufficiently generic (as $v(x) = v(y)$ and $w(x) = w(y)$. If S is analytically irreducible, then a constant $c_S(v, w)$ (working in S) as desired exists by the inductive assumptions and defines a constant working for R as well.

It remains to show that S is analytically irreducible. Clearly $R(T)$ is analytically irreducible, and clearly S is excellent, hence in particular analytically unramified. As

R (hence also $R(T)$) is normal, we conclude from the local version of Bertini's theorem, that $S_{\mathfrak{p}}$ is normal for $\mathfrak{p} \subsetneq \mathfrak{m}_S$ (as R has infinite residue field this follows from [Fl], (3.3) in combination with the techniques of [Fl], §4). Let $\overline{S}$ be the normalization of S. If $\overline{S}$ equals S, then S is analytically irreducible. So we assume that $\overline{S}$ and S are not equal. Hence the conductor $\mathfrak{C}_S$ of $\overline{S}/S$ is an $\mathfrak{m}_S$–primary ideal. Thus if $\widehat{S}$ (resp. $\widehat{\overline{S}}$) denotes the completions of S (resp. $\overline{S}$), then $\widehat{\overline{S}}$ is the normalization of $\widehat{S}$, and therefore the conductor $\mathfrak{C}_{\widehat{S}}$ of $\widehat{S}$ is an $\mathfrak{m}_{\widehat{S}}$–primary ideal. If we assume that $\widehat{S}$ is not a domain, then $\mathfrak{C}_{\widehat{S}} = I_1 + I_2$ with two ideals I_1, I_2 with $I_1 \cdot I_2 = (0)$. We already know that $\widehat{R(T)}$ is a domain. Taking preimages in $\widehat{R(T)}$, we find two ideals J_1, J_2 of $\widehat{R(T)}$ such that $J_1 + J_2$ is an $\mathfrak{m}_{\widehat{R(T)}}$–primary ideal with $J_1 \cdot J_2 \subseteq (xT - y)$. As $\dim(\widehat{R(T)}) = \dim(R) \geq 3$, this is impossible by the connectedness theorem of Faltings/Brodmann-Rung (cf. [Fa], [BR]).

This completes the proof of Izumi's theorem.

This theorem (or rather its proof for dimension 2) raises the following question (cf. "condition E" of Heinzer and Lantz [HL]):

Question 1.7. Let $(R, \mathfrak{m})$ be a complete normal local domain and let v be an $\mathfrak{m}$-valuation. Does there exist a normal scheme X together with a projective birational map $\pi : X \longrightarrow \operatorname{Spec}(R)$ such that the reduced closed fiber $\pi^{-1}(\{\mathfrak{m}\})$ is an essential prime divisor of X and the center of v on X?

Cutkosky [C2] proved this for dimension 2, thus answering the original question of Heinzer and Lantz, and we are asking if there exists a higher-dimensional analogue. Cutkosky's arguments do not generalize in an obvious way to the higher–dimensional situation.

Though our version of Izumi's theorem shows that there are close relations between any two $\mathfrak{m}$-valuations, there are still certain ways to distinguish valuations by the values they take on $\mathfrak{m}$, as the following proposition shows. However, in this proposition we use the more general definition of valuation, namely, we allow $\mathbb{Q}$-valued valuations, that is the valuations which take on rational values. A valuation v is said to dominate another valuation w if for all non-zero x, $v(x) \geq w(x)$.

Proposition 1.8. *Let $v_1, \dots, v_r$ be distinct mutually non-dominating $\mathbb{Q}$-valued valuations on an integral domain R. Then there exist x and y in R and distinct integers i and j in $\{1, \dots, r\}$ such that*

$$v_i(x) < v_k(x) \quad \text{for all } k \neq i,$$
$$v_j(y) < v_k(y) \quad \text{for all } k \neq j.$$

Proof: The case $r = 2$ is true by the non-dominating assumption. Now let $r > 2$.

By induction there exists an element x such that (after reindexing)

$$v_1(x) < v_3(x), v_4(x), \ldots, v_r(x).$$

If also $v_1(x) < v_2(x)$, then $v_1(x) < v_k(x)$ for all $k \neq 1$, and x is one of the elements x that we searched for. Similarly, if $v_2(x) < v_1(x)$, then $v_2(x) < v_k(x)$ for all $k \neq 2$, and again x is one of the two elements needed. Thus it remains to consider the case $v_1(x) = v_2(x)$. By the non-dominating assumption there exists an element z in R such that $v_1(z) < v_2(z)$. Then for all sufficiently large integers n (say $n \geq v_1(z) - v_k(z)$), $v_1(x^n z) < v_k(x^n z)$ for all $k \neq 1$. Now after renaming $x^n z$ by x, we have $x \in R$ such that $v_1(x) < v_k(x)$ for all $k \neq 1$.

Thus we have found an element x such that, after reindexing,

$$v_1(x) < v_k(x) \quad \text{for all } k > 1.$$

By induction on r applied to the valuations $v_1, v_2, \ldots, v_{r-1}$ there exist elements $y, z \in R$ and distinct integers $i, j \in \{1, \ldots, r-1\}$ such that

$$v_i(y) < v_k(y) \quad \text{for all } k \in \{1, 2, \ldots, r-1\} \setminus \{i\},$$
$$v_j(z) < v_k(z) \quad \text{for all } k \in \{1, 2, \ldots, r-1\} \setminus \{j\}.$$

Either i or j is different from 1, so by possibly switching the names of i and j, y and z, we may assume that $i \neq 1$. Then after possibly reindexing the valuations $v_2, \ldots, v_{r-1}$ we can take that $i = 2$, so that

$$v_2(y) < v_1(y), v_3(y), \ldots, v_{r-1}(y).$$

If $v_2(y) < v_r(y)$ or $v_r(y) < v_2(y)$, we are done as in the first part. So we may assume that $v_2(y) = v_r(y)$. By non-domination there is an element z such that $v_2(z) < v_r(z)$. Again proceed as in the first part to finish the proof of the proposition.

In the rest of the paper we present several applications of these relations between valuations.

§2. ZERO DIVISORS MODULO POWERS OF AN IDEAL

In this section we give new criteria for analytic irreducibility (Proposition 2.2) and one-fiberedness (Proposition 2.8), and we use Rees valuations and their interactions to examine the following question, which came up in the second author's study of equivalence of adic and symbolic topologies:

Question 2.1. Let $(R, \mathfrak{m})$ be a noetherian local ring. Do there exist integers a and b such that for all integers n and all elements x and y in R such that $xy \in \mathfrak{m}^{an+b}$, necessarily either x or y lies in $\mathfrak{m}^n$?

If R is analytically irreducible, i.e., if the completion of R is an integral domain, the answer is yes [S, Theorem 3.4]. In fact, the analytically irreducible assumption is necessary as otherwise there is no such k:

Proposition 2.2. *(Criterion of analytic irreducibility) Let $(R, \mathfrak{m})$ be a local ring which is not analytically irreducible. Then for all positive integers a and b, there exist elements x and y in R and an integer $n \in \mathbb{N}$, such that $xy \in \mathfrak{m}^{an+b}$, but neither x nor y lies in $\mathfrak{m}^n$.*

Proof: Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. By assumption $\widehat{R}$ is not a domain, hence there exist nonzero elements x and y in $\widehat{R}$ such that $xy = 0$. By Krull's intersection theorem there is a large $n \in \mathbb{N}$ such that x and y are not in $\mathfrak{m}^n\widehat{R}$. There exists a Cauchy sequence $\{x_l\}$ in R which converges to x in $\widehat{R}$, and a Cauchy sequence $\{y_l\}$ in R which converges to y. Hence for all large enough l, x_l and y_l are not elements of $\mathfrak{m}^n$. But by continuity of multiplication $x_l y_l$ converges to $xy = 0$, so that for large l, $x_l y_l \in \mathfrak{m}^{an+b}$.

Thus a and b exist if and only if the ring is analytically irreducible. The question remains as to what are the lowest possible values of a and b. For simplicity we consider for that only the complete analytically irreducible rings, i.e., complete local domains:

Question 2.3. Let $(R, \mathfrak{m})$ be a complete local domain. Find the smallest possible integers a and b such that for all integers n and for all elements x and y in R such that $xy \in \mathfrak{m}^{an+b}$, necessarily either x or y lies in $\mathfrak{m}^n$.

We look at this more generally: let I be an $\mathfrak{m}$-primary ideal in a complete local domain. We want to find the smallest possible integers a and b such that for all integers n and for all elements x and y in R such that $xy \in I^{an+b}$, necessarily either x or y lies in I^n.

First let $(R, \mathfrak{m})$ be a complete local domain of dimension 1 and let $\overline{R}$ be its normalization. Then $\overline{R}/R$ is a module–finite extension, and $\overline{R}$ is local again with maximal ideal $\mathfrak{n}$. Let $\mathfrak{C} = \mathfrak{C}_{\overline{R}/R}$ be the conductor of $\overline{R}/R$ and write $\mathfrak{C} = \mathfrak{n}^{f_R}$ (as an ideal of $\overline{R}$). Then f_R is called the conductor degree of R.

For a rational number q we let $\lfloor q \rfloor$ denote the largest integer n with $n \leq q$.

Proposition 2.4. *Let $I \subseteq R$ be an ideal, $I \neq 0$, write $I \cdot \overline{R} = \mathfrak{n}^l$ and set $c(I) := \lfloor \frac{2 \cdot f_R}{l} \rfloor$. Whenever $\alpha, \beta \in R$ with $\alpha \cdot \beta \in I^{2n+c(I)}$, then either $\alpha \in I^n$ or $\beta \in I^n$.*

In particular: if I is any $\mathfrak{m}$–primary ideal and if $\alpha, \beta \in R$ with $\alpha \cdot \beta \in I^{2n+2f_R}$, then either $\alpha \in I^n$ or $\beta \in I^n$.

Proof: First note that we have for any ideal $J \subseteq R$:

$$\mathfrak{C} \cdot J \cdot \overline{R} \subseteq J.$$

If $\alpha \cdot \beta \in I^{2n+c(I)}$ then

$$\alpha \cdot \beta \in I^{2n+c(I)} \cdot \overline{R} = \mathfrak{n}^{l(2n+c(I))}.$$

As $\overline{R}$ is a discrete valuation ring, this implies that (without loss of generality)

$$\alpha \in \mathfrak{n}^{nl+\lfloor \frac{lc(I)}{2} \rfloor} \subseteq \mathfrak{n}^{nl+f_R} = \mathfrak{C} \cdot I^n \overline{R} \subseteq I^n,$$

and the proposition follows.

Example 2.5. Let k be a field and let $R = k[[X^2, X^{2n+1}]]$. Then $\overline{R} = k[[X]]$ and $f_R = 2n$. For the ideal $I = (X^2)$ we have that $c(I) = 2n$, and this is optimal as one sees by looking at $\alpha = \beta = X^{2n+1}$.

Proposition 2.4 answers Question 2.3 in dimension 1 without recourse to Izumi's theorem. In the rest of the section we analyze the higher dimensional cases, and for that we need Izumi's theorem.

Associated to I we have the Rees valuations $v_1, \ldots, v_r$. Renormalize the valuations by setting $w_i(x) = \frac{v_i(x)}{v_i(I)}$. Then the integral closure $\overline{I^n}$ of I^n equals

$$\overline{I^n} = \{x \in R : w_i(x) \geq n, i = 1, \ldots, r\}.$$

By Izumi's theorem (1.3) there exist constants $C(i,j) \in \mathbb{Q}_+$ with

$$w_i(x) \leq C(i,j) w_j(x) \qquad \text{for all nonzero } x \in R \text{ and all } j = 1, \ldots, R,$$

(where, for each i, j, we may assume that $C(i,j)$ is chosen as small as possible). Let C be the maximum of all the $C(i,j)$. We set $a = C + 1$. To determine b, recall that by [R$_1$] there exists an integer l such that for all $n \in \mathbb{N}$, $\overline{I^{n+l}} \subseteq I^n$. Then set $b = al$.

Theorem 2.6. *With notation as in the preceeding, let x and y be in R with $xy \in I^{an+b}$. Then either $x \in I^n$ or $y \in I^n$.*

Proof: Assume that $x \notin I^n$. Then $x \notin \overline{I^{n+l}}$, so that for some $i = 1, \ldots, r$, $w_i(x) < n+l$. As $xy \in I^{an+b}$, $w_i(y) > an + b - n - l = C \cdot (n+l)$. Hence by the choice of C, for all $j = 1, \ldots, r$, $w_j(y) \geq n + l$, so that y is in the integral closure of I^{n+l} and hence in I^n.

Some simplifications in the Theorem are immediate:

Corollary 2.7. *Suppose that I is a one-fibered ideal (i.e., I has only one Rees valuation).*

(1) *If $x, y \in R$ with $xy \in I^{2n+2l}$, then either $x \in I^n$ or $y \in I^n$.*
(2) *If in addition I and all of its powers are integrally closed, then $xy \in I^{2n}$ implies $x \in I^n$ or $y \in I^n$.*

A partial converse also holds:

Proposition 2.8. *(One-fiberedness criterion) Let I be an $\mathfrak{m}$-primary ideal in an analytically irreducible noetherian local domain $(R, \mathfrak{m})$. Then I is one-fibered if and only if* *there exists an integer b such that for all $n \in \mathbb{N}$ and all x, y such that $xy \in I^{2n+b}$, either x or y lies in I^n.*

Proof: By the Corollary it suffices to prove the necessity, i.e., that I is one-fibered with the given assumptions. Assume that $v_1, \ldots, v_r$ are the Rees valuations of I, with $r \geq 2$. Let $w_i(x) = v_i(x)/v_i(I)$ be the normalization of v_i. Then w_i is a $\mathbb{Q}$-valued function. By Lemma 1.8 there exist elements x and y in $\mathfrak{m}$ such that (after reindexing):

$$\begin{aligned} 0 < w_1(x) < w_k(x) \quad &\text{for all } k \neq 1, \\ 0 < w_2(y) < w_k(y) \quad &\text{for all } k \neq 2. \end{aligned}$$

As the w_i take on positive rational values on $\mathfrak{m}$, by raising x and y to powers we may assume that $w_1(x) = w_2(y) = C$. Thus there exists a positive rational number q such that

$$\begin{aligned} w_1(x) + q \leq w_k(x) \quad &\text{for all } k \neq 1, \\ w_2(y) + q \leq w_k(y) \quad &\text{for all } k \neq 2. \end{aligned}$$

If for some $n \in \mathbb{N}$, $x^n \in I^{nC+1}$, then $nC = nw_1(x) = w_1(x^n) \geq nC + 1$, which is a contradiction. Thus for all $n \in \mathbb{N}$, x^n is not in I^{nC+1}, and similarly y^n is not in I^{nC+1}. Now let n be such that $\lfloor nq \rfloor \geq 2 + b + l$, where l is such that for all integers m, the integral closure of I^{m+l} lies in I^m. Then $w_1(x^n y^n) = n(w_1(x) + w_1(y)) \geq 2nC + nq$, similarly $w_2(x^n y^n) \geq 2nC + nq$, and for all $i \geq 3$, $w_i(x^n y^n) \geq 2nC + nq$. Hence $x^n y^n$ lies in the integral closure of $I^{\lfloor 2nC+nq \rfloor}$, hence in $I^{\lfloor 2nC+nq-l \rfloor}$. But by the choice of n then $x^n y^n$ lies in $I^{2(nC+1)+b}$, but neither x^n nor y^n lies in I^{nC+1}.

This gives a converse of the first part of Theorem 2.6. We do not know whether the second part also has a converse:

Question 2.9. Let I be an $\mathfrak{m}$-primary ideal in an analytically irreducible noetherian local domain $(R, \mathfrak{m})$. Suppose that for all $n \in \mathbb{N}$ and all x, y such that $xy \in I^{2n}$, either x or y lies in I^n. Or even suppose that for all $x \in R$ such that $x^2 \in I^{2n}$, necessarily x lies in I^n. Is I then normal?

Recall that an ideal is called normal if all the powers of I are integrally closed ideals.

Under the conditions in the Question, if all large powers of I are integrally closed, then all the powers of I are integrally closed. For let k be a positive integer and let x be in the integral closure of I^k. Then for all large enough n, I^{k2^n} is integrally closed, so that $x^{2^n} \in I^{k2^n}$. But under the assumption in the Question, then $x^{2^{n-1}} \in I^{k2^{n-1}}$, $x^{2^{n-2}} \in I^{k2^{n-2}}, \ldots, x \in I^k$.

In any case, we have not yet answered whether the a and b from Theorem 2.6 are the smallest possible integers to answer Question 2.3. In particular, perhaps setting $a = \max\{C(i,j) : i,j\} + 1$ might be too generous in general. In fact, it is too generous in general, but sharp in some cases. We produce examples for this, and also examples that show that for any constant C_0, setting $a = \min\{C(i,j) : i,j\} + C_0$ is too small. As calculating the examples is quite involved, we dedicate a separate section to them, which comes next.

§3. EXAMPLES

From a computational point of view, the constants in Izumi's theorem and in the results of Section 2 in general cannot be computed effectively as there is as yet no effective algorithm for calculating integral closures, and, moreover, there is no effective way to calculate the various $C(i,j)$. However, there are many cases where $C(i,j)$ and a, b can be computed, and this is what we do in this section. Most importantly, the first example below shows that setting $a = \max\{C(i,j) : i,j\} + 1$ is too generous, and the second class of examples shows that for any constant C_0, setting $a = \min\{C(i,j) : i,j\} + C_0$ is too small. The last two examples in this section are of rational singularities, suggested by Mike O'Sullivan, and have sharp values for a and b as given in Theorem 2.6.

The calculations involve the Jacobian criterion, construction of Rees valuations, and many ad hoc procedures. We include most of the details.

Example 3.1. We show that $a = \max\{C(i,j) : i,j\} + 1$ is too generous, but that $a = \min\{C(i,j) : i,j\}$ is not large enough.

Let F be a field of characteristic different from 2 and 3, let x, y, z be variables over F, and $R = \frac{F[[x,y,z]]}{(xy^2 - z^9)}$. Let $\mathfrak{m}$ be the maximal ideal, namely $\mathfrak{m} = (x,y,z)R$.

First we prove that for all $n \in \mathbb{N}$, if the product of two elements is in $\mathfrak{m}^{6n}$, then one or the other element is in $\mathfrak{m}^n$. For this, let $C = \frac{F[[u,v,t]]}{(uv^2 - t^3)}$. The associated graded ring of C with respect to its maximal ideal $\mathfrak{m}_C$ is an integral domain, which means that $\mathfrak{m}_C$ is one-fibered and normal. Thus by Corollary 2.7, whenever the product of two elements in C lies in $\mathfrak{m}_C^{2n}$ then one of the two elements lies in $\mathfrak{m}_C^n$. Now let $\varphi : A \to C$ be the algebra homomorphism determined by

$$\begin{aligned} \varphi : \quad & x \mapsto u^9 \\ & y \mapsto v^9 \\ & z \mapsto t^3. \end{aligned}$$

This is a well-defined injection. If α, β are elements in R such that $\alpha\beta \in \mathfrak{m}^{6n}$, then $\varphi(\alpha)\varphi(\beta) \in \mathfrak{m}_C^{18n}$. Thus by above say $\varphi(\alpha) \in \mathfrak{m}_C^{9n}$, and hence α lies in $\varphi^{-1}(\mathfrak{m}_C^{9n}) \subseteq \mathfrak{m}^n$, as was to be proved.

Next we calculate the $C(i,j)$, for which we have to determine all the Rees valuations of $\mathfrak{m}$. Let $S = R[\mathfrak{m}t, t^{-1}]$, where t is a new indeterminate. The associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is isomorphic to $S/t^{-1}S$, which is isomorphic to $\frac{F[xt,yt,zt]}{(xt)(yt)^2}$. From this we can read off the two minimal primes over $t^{-1}S$: $\mathfrak{P}_1 = (xt, t^{-1})$ and $\mathfrak{P}_2 = (yt, t^{-1})$. The Rees valuations of $\mathfrak{m}$ correspond to the height one prime ideals in the integral closure of S localized at the complement of $\mathfrak{P}_1 \cup \mathfrak{P}_2$. There is only one such prime ideal lying over $\mathfrak{P}_1$, and it is equal to $\mathfrak{P}_1$. After localization at $\mathfrak{P}_1$, the equation $(xt)(yt)^2 = (zt)^9 t^{-6}$

gives us the following values for the corresponding Rees valuation v_1:

$$\begin{aligned}
&v_1(t^{-1}) = 1,\\
&v_1(yt) = 0, v_1(zt) = 0,\\
&v_1(y) = v_1(yt) + v_1(t^{-1}) = 1, v_1(z) = v_1(zt) + v_1(t^{-1}) = 1,\\
&v_1(xt) = 9v_1(zt) + 6v_1(t^{-1}) - 2v_1(yt) = 6,\\
&v_1(x) = v_1(xt) + v_1(t^{-1}) = 7.
\end{aligned}$$

As $(yt)^2 = \delta t^{-2}$ for some $\delta \in S_{\mathfrak{P}_2}$, it follows that there is only one other Rees valuation v_2. By normalizing this valuation we may assume that $v_2(t^{-1}) = 1$, and then we calculate (calculations as above) that

$$v_2(x) = 1, v_2(y) = 4, v_2(z) = 1.$$

Note that $C(1,2) \geq \frac{v_1(x)}{v_2(x)} = 7$ and that $C(2,1) \geq \frac{v_2(y)}{v_1(y)} = 4$. Then C, the maximum of all the $C(i,j)$, is at least 7. By the first part, $a \leq 6$, so that $\max\{C(i,j) : i,j\} + 1 \geq 8$ is too generous.

We finally prove that $\min\{C(i,j) : i,j\}$ is not large enough to be a. First of all, one can show with some extra work, mostly depending on having a canonical monomial basis of R over F, that $C(2,1)$ is indeed 4. Thus we are claiming that $a = 4$ is not large enough! For suppose, by contradiction, that for some integer $b \in \mathbb{N}$, whenever $\alpha\beta \in \mathfrak{m}^{4n+b}$ then either α or β lies in m^n. Let c be an even integer larger than $4 + b$. Then

$$x^c y^c = x^{\frac{c}{2}}(xy^2)^{\frac{c}{2}} = x^{\frac{c}{2}} z^{\frac{9c}{2}} \in \mathfrak{m}^{5c} \subseteq \mathfrak{m}^{4(c+1)+b},$$

yet neither x^c nor y^c lies in $\mathfrak{m}^{c+1}$.

The next very involved example shows that for any constant C_0, setting a to be $\min\{C(i,j) : i,j\} + C_0$ is not large enough.

Example 3.2. Let F be a perfect field of characteristic different from 2, and let k be a positive integer larger than 4. If the characteristic of F is not zero, we require that k and the characteristic of F be relatively prime. Let x, y, z, w be variables over F, and $R = \frac{F[[x,y,z,w]]}{(xyz - w^k - x^4)}$. Let $\mathfrak{m}$ be the maximal ideal of R, namely $\mathfrak{m} = (x, y, z, w)R$.

The calculations of this example take the next four pages. They are very illustrative of the workings of Izumi's theorem, so we present all steps explicitly.

By the Jacobian criterion one can verify that R is a normal domain. (Strictly speaking, the Jacobian criterion proves that $\frac{F[x,y,z,w]}{(xyz - w^k - x^4)}$ is normal, but then as this ring is excellent and analytically irreducible, its completion R is also normal.) Let $S = R[\mathfrak{m}t, t^{-1}]$. Then the associated graded ring of R, $S/t^{-1}S \cong \frac{F[xt,yt,zt,wt]}{(xt)(yt)(zt)}$, is reduced, so that $\mathfrak{m}$ is a normal ideal. Also, from $S/t^{-1}S$ we can read off the three Rees valuations v_1, v_2 and v_3 corresponding to $\mathfrak{m}$. The first Rees valuation v_1 corresponds to the minimal prime $\mathfrak{P}_1 = (t^{-1}, xt)S$ over $t^{-1}S$, hence $v_1(yt) = v_1(zt) = 0$. The equality

$(xt)\big((yt)(zt)-(xt)^3t^{-1}\big) = (wt)^k t^{-(k-3)}$ then gives that $v_1(t^{-1}) = 1$, $v_1(xt) = k-3$. Hence

$$v_1(x) = k-2,\ v_1(y) = v_1(z) = v_1(w) = 1.$$

Similarly, corresponding to the minimal prime $\mathfrak{P}_2 = (t^{-1}, yt)S$ over $t^{-1}S$ we have

$$v_2(y) = 2,\ v_2(x) = v_2(z) = v_2(w) = 1,$$

and, corresponding to the minimal prime $\mathfrak{P}_3 = (t^{-1}, zt)S$ over $t^{-1}S$ we have

$$v_3(z) = 2,\ v_3(x) = v_3(y) = v_3(w) = 1.$$

Say that k is a large odd integer. With notation of section 2, we prove first that a has to be strictly larger than $\frac{k-1}{2}$. For suppose that $a = \frac{k-1}{2}$, and let b be an arbitrary non-negative integer. Then for any integer c larger than $\frac{k-1}{4} + \frac{b}{2}$,

$$x^{2c}(yz-x^3)^c = x^c(xyz-x^4)^c = x^c(w^k)^c \in \mathfrak{m}^{c(k+1)} \subseteq \mathfrak{m}^{(2c+1)(\frac{k-1}{2})+b},$$

yet neither x^{2c} nor $(yz-x^3)^c$ lie in $\mathfrak{m}^{2c+1}$. Thus necessarily $a > \frac{k-1}{2}$.

Next we prove that $C(3,2)$ is 2 for all k. Then it will follow that in general, $a = \min\{C(i,j) +$ a little$\}$ does not work – as for k sufficiently large, 2 is a lot smaller than $\frac{k-1}{2}$.

Certainly $C(3,2) \geq \frac{v_3(z)}{v_2(z)} = 2$ but the other inequality is harder. We need some notation.

Let $(R,\mathfrak{m})$ be a complete local integral domain, $\mathfrak{m} = (x_1,\ldots,x_r)$, and v a valuation on the quotient field of R centered on $\mathfrak{m}$. For each non-unit $f \in R$, write $f = \sum r_\nu x_1^{\nu_1}\cdots x_r^{\nu_r}$ with each r_ν either zero or a unit in R. For each non-negative integer n let $[f]_n = \sum r_\nu x_1^{\nu_1}\cdots x_r^{\nu_r}$, where the sum is over all those terms such that $v(x_1^{\nu_1}\cdots x_r^{\nu_r}) = n$. Then $f = [f]_0 + [f]_1 + [f]_2 + \cdots$. This expansion of f depends on the given system of generators of $\mathfrak{m}$, and even after having fixed $x_1,\ldots,x_r$ this presentation is not unique. We always assume that for the smallest n for which if $[f]_n$ is nonzero, $[f]_n$ cannot be written as $[g]_{n+1} + [g]_{n+2} + \cdots$ for any $g \in R$. Thus for each f we may define $L(f)$ to be the smallest integer n for which $[f]_n$ is nonzero. We define $L(0) = \infty$.

We emphasize that whereas $L(f)$ is well-defined, $[f]_{L(f)}$ is in general not uniquely determined!

Note that for every nonzero element f of R, $v(f) \geq L(f)$.

Lemma 3.3. *If $v(f) = L(f)$ and $v(g) = L(g)$ then $v(fg) = L(fg)$.*

Proof: We may assume that $fg \neq 0$. Then

$$\begin{aligned} v(fg) &\geq L(fg) \\ &\geq L(f) + L(g) \\ &= v(f) + v(g) \\ &= v(fg). \end{aligned}$$

In our example, we will use the generators x, y, z, w of the maximal ideal, r_ν will always lie in F, and v will be the valuation v_3. For example, if $f = xyz$, then $L(f) = 4$ but $[f]_4$ may be either xyz or x^4. Thus up to addition of an element $g(= w^k)$ such that $v(g) > L(f)$, xyz and x^4 are the same. We will exploit this relation between the various representations of f.

We will need the special element $\alpha = yz - x^3$. Note that $L(\alpha) = 3$, $[\alpha]_3 = \alpha$, and that

$$v(\alpha) = v(x\alpha) - v(x) = v(w^k) - v(x) = k - 1.$$

The crucial step in our calculations is rewriting elements of R as multiples of α whenever possible:

Lemma 3.4. *For any f in R, either $v(f) = L(f)$ or $f = \alpha g + h$ for some $g, h \in R$ with $L(g) \geq L(f) - 3$ and $L(h) > L(f)$.*

Proof: It suffices to prove that either $v([f]_{L(f)}) = L(f)$ or that we may take $[f]_{L(f)} = \alpha g$ for some $g \in R$ with $L(g) \geq L(f) - 3$. Thus by replacing f by $[f]_{L(f)}$ we assume that $f = [f]_{L(f)}$. Let L be $L(f)$. Suppose that $v(f)$ is not L. Then necessarily $v(f) > L$. This means that $f \in t^{-(L+1)} S_{\mathfrak{P}_3}$. Write $f = \sum r_{abcd} x^a y^b z^c w^d$ where a, b, c, d are non-negative, and $a + b + 2c + d = L$. By rewriting x^4 as $xyz - w^k$ our definition of $L(f)$ guarantees that $L(f)$ is unchanged, so without loss of generality all a are 3 or smaller. We next rewrite $f = \sum r_{abcd} x^a y^b z^c w^d \in t^{-(L+1)} S_{\mathfrak{P}_3}$ by using the equalities

$$x = (xt)t^{-1}, y = (yt)t^{-1}, w = (wt)t^{-1}, z = (zt)t^{-1} = \frac{(xt)^4 - (wt)^k t^{-(k-4)}}{(xt)(yt)} t^{-2},$$

to get

$$\sum r_{abcd} (xt)^a (yt)^b \left(\frac{(xt)^4 - (wt)^k t^{-(k-4)}}{(xt)(yt)} \right)^c (wt)^d \in t^{-1} S_{\mathfrak{P}_3}.$$

Thus as $k > 4$,

$$\sum r_{abcd} (xt)^a (yt)^b \left(\frac{(xt)^3}{(yt)} \right)^c (wt)^d \in t^{-1} S_{\mathfrak{P}_3}.$$

By Lemma 3.3 and the fact that α, yz is a regular sequence, neither the assumption nor the conclusion is changed if we multiply f by a power of y or a power of z, so we may without loss of generality assume that $0 < c < b$ for every 4-tuple (a, b, c, d). The display above is then

$$\sum r_{abcd} (xt)^{a+3c} (yt)^{b-c} (wt)^d \in t^{-1} S_{\mathfrak{P}_3} \cap S.$$

As $a + 3c > 0$ and xt is a unit in $S_{\mathfrak{P}_1}$, the expression above also lies in $t^{-1} S_{\mathfrak{P}_1}$. Similarly, as $b - c > 0$, the expression also lies in

$$t^{-1} S_{\mathfrak{P}_1} \cap t^{-1} S_{\mathfrak{P}_2} \cap t^{-1} S_{\mathfrak{P}_3} \cap S = t^{-1} S.$$

Thus

$$\sum r_{abcd} x^{a+3c} y^{b-c} w^d \in t^{-(1+a+3c+b-c+d)} S \cap R = t^{-(1+L)} S \cap R = \mathfrak{m}^{1+L}.$$

Each of the summands lies in $\mathfrak{m}^L$, but their sum is in $\mathfrak{m}^{L+1}$. But there is no such non-trivial relation among x, y and w alone. Thus for each A, B and D,

$$\sum r_{abcd} = 0,$$

where the sum is over all those (a, b, c, d) such that $a + 3c = A$, $b - c = B$ and $d = D$. In other words,

$$\sum_c r_{A-3c,B+c,c,D} = 0.$$

Recall that all the allowed indices satisfy $A + B + D = L$, $0 \le A - 3c \le 3$, so that the only non-trivial parts of f are when A is a positive multiple of 3 and c is allowed to have two values, $c = A/3$ and $c = A/3 - 1$. Thus f is the sum of F-multiples of

$$x^0 y^{B+\frac{A}{3}} z^{\frac{A}{3}} w^D - x^3 y^{B+\frac{A}{3}-1} z^{\frac{A}{3}-1} w^D = (yz - x^3)(y^{B+\frac{A}{3}-1} z^{\frac{A}{3}-1} w^D),$$

so that f is a multiple of $yz - x^3 = \alpha$, with the multiplier having L-value at least $B + \frac{A}{3} - 1 + 2(\frac{A}{3} - 1) + D = A + B + D - 3 = L(f) - 3$.

This proved the lemma. The proof also shows that for all $f \in F[[y, z, w]]$, $v(f) = L(f)$. Furthermore it shows that for every positive integer N, every nonzero f in R can be written in the form

$$f = g_0 + \alpha g_1 + \alpha^2 g_2 + \alpha^3 g_3 + \cdots + \alpha^N g_N, \tag{$*$}$$

where for all $i < N$, either g_i is zero or else $v(g_i) = L(g_i)$. Moreover, as in the proof we may assume that for all nonzero g_i, $L(g_i) \ge L(f) - 3i$.

Lemma 3.5. *Write f as in ($*$), with $N > v(f)$. Then*

$$v(f) = \min\{v(\alpha^i g_i) | i < N\}.$$

Proof: If $v(f) < v(\alpha) = k - 1$ then necessarily $v(f) = v(g_0)$, and the lemma is proved. If instead $v(f) \ge v(\alpha)$, we proceed by induction on $v(f)$. If $v(f - g_0) \le v(f)$, then by induction on $v(f)$,

$$\begin{aligned} v(f - g_0) &= v(\alpha(g_1 + \alpha g_2 + \alpha^2 g_3 + \cdots + \alpha^{N-1} g_N)) \\ &= v(\alpha) + v(g_1 + \alpha g_2 + \alpha^2 g_i + \cdots + \alpha^{N-1} g_N) \\ &= v(\alpha) + \min\{v(\alpha^{i-1} g_i) | 0 < i < N\} \\ &= \min\{v(\alpha^i g_i) | 0 < i < N\}. \end{aligned}$$

Thus the lemma is proved if $v(f) = v(g_0)$ or if $v(f) = v(f - g_0)$. So it remains to deal with the case $v(f) > v(g_0), v(f - g_0)$. Here necessarily $v(g_0) = v(f - g_0)$. This case is harder because in $(*)$, the L- and v-values of individual terms may differ. So we next remove this obstacle, and we accomplish this by multiplying $(*)$ through by x^N:

$$x^N f = x^N g_0 + x^{N-1} w^k g_1 + x^{N-2} w^{2k} g_2 + x^{N-3} w^{3k} g_3 + \cdots + w^{kN} g_N.$$

By Lemma 3.3, for each $i < N$, $L(x^{N-i} w^{ik} g_i) = v(x^{N-i} w^{ik} g_i)$. Let l be the minimum of these values. (As $N > v(f) \geq L(f)$ and $k > 2$, l is actually equal to $\min\{L(x^{N-i} w^{ik} g_i) | i \leq N\}$.) Set

$$s = \sum_i x^{N-i} w^{ik} [g_i]_{l-(N-i+ik)}.$$

and note that we may take $s = \sum [x^{N-i} w^{ik} g_i]_l$. Let $l_i = l - (N - i + ik)$. There exist some i such that $[g_i]_{l_i}$ is nonzero and necessarily for all these i also $l_i = L(g_i)$.

If $s = 0$, then

$$0 = s = x^N \sum_i \alpha^i [g_i]_{l_i},$$

so that $0 = \sum_i \alpha^i [g_i]_{l_i}$, whence we can rewrite f as in $(*)$ with a strictly larger value of l. But l cannot increase indefinitely as $l \leq L(x^N g_0) = N + v(g_0) < N + v(f) < 2v(f)$.

Thus we may assume that s is nonzero. Hence $L(s) = l$. If $v(s) = L(s)$, then as $v(x^N f - s) \geq L(x^N f - s) > l$, we have

$$N + v(f) = v(x^N f) = \min\{v(x^N f - s), v(s)\} = v(s) = l < N + v(f),$$

contradiction. So necessarily $L(s) < v(s)$. Then by Lemma 3.4, $s = \alpha h$ for some h with $L(h) \geq l - 3$. We multiply $s = \alpha h$ through by x to get $xs = w^k h$. As $L(w^k h) \geq k + L(h) \geq k + l - 3 > l + 1$,

$$0 = [xs]_{l+1} = x^{N+1} \sum \alpha^i [g_i]_{l_i}.$$

Thus $\sum \alpha^i [g_i]_{l_i} = 0$, and again we rewrite f as in (*) to increase the value of l. This proves the lemma.

Finally:

Proposition 3.6. $C(3, 2) = 2$.

Proof: Note that for any non-negative a, b, c, d,

$$2v_2(x^a y^b z^c w^d) = 2(a + 2b + c + d) \geq a + b + 2c + d = v_3(x^a y^b z^c w^d).$$

Thus for all f, $2v_2(f) \geq L(f)$. Now write f in the form (*). Then

$$\begin{aligned} 2v_2(f) &\geq 2\min\{v_2(\alpha^i g_i) : i \geq 0\} \\ &= 2\min\{v_2(\alpha^i) + v_2(g_i) : i \geq 0\} \\ &= 2\min\{v_3(\alpha^i) + v_2(g_i) : i \geq 0\} \\ &\geq \min\{v_3(\alpha^i) + 2v_2(g_i) : i \geq 0\} \\ &\geq \min\{v_3(\alpha^i) + L(g_i) : i \geq 0\} \\ &= \min\{v_3(\alpha^i) + v(g_i) : i \geq 0\} \\ &= \min\{v_3(\alpha^i g_i) : i \geq 0\} \\ &= v_3(f). \end{aligned}$$

This finishes the calculations for the second example, namely Example 3.2. These calculations raise two questions:

Question 3.7. Suppose $I \subseteq R$ is a two–fibered ideal with Rees valuations v_1 and v_2. Is it then possible to take $a = \min\{C(1,2), C(2,1)\} + 1$? This is to some extent the intermediate case between the previous two examples.

Question 3.8. Given an $\mathfrak{m}$-valuation v, when do there exist finitely many elements $r_i \in R$ such that for any other $\mathfrak{m}$-valuation w, $C(v,w) = \max\{\frac{v(r_i)}{w(r_i)}\}$?

The answer to this question is yes whenever the graded valuation algebra $\mathrm{gr}_v(R) = \oplus_{i\geq 0} \frac{I_i(v)}{I_{i+1}(v)}$ is noetherian. For in that case $\mathrm{gr}_v(R)$ is finitely generated as an algebra over R, and we set r_i to be the preimages of these algebra generators. We do not know whether the converse holds, i.e., whether the r_i, if they exist, also generate $\mathrm{gr}_v(R)$.

The ring $\mathrm{gr}_v(R)$ is noetherian if and only if the ring $\oplus I_i(v)$ is noetherian (the proof is straightforward using that R is complete). Noetherianness of $\oplus I_i(v)$ was studied by Muhly and Sakuma [MS], Göhner [G], and by Cutkosky [C1, C2]. Cutkosky proved in [C1, page 427] that for a two-dimensional complete local normal domain with an algebraically closed residue field of characteristic 0, $\oplus I_i(v)$ is noetherian for all $\mathfrak{m}$-valuations v if and only if the ring has a rational singularity. Thus whenever a good two-dimensional ring has a rational singularity, each $\mathrm{gr}_v(R)$ is finitely generated, guaranteeing the existence of the r_i, but in general $\mathrm{gr}_v(R)$ is not noetherian.

Finally, on the positive, calculable note, we calculate some least possible a and b for rational singularities, suggested by Mike O'Sullivan. Izumi also calculated in the appendix of [Iz] some related constants for some of these rational singularity rings.

Example 3.9. Let F be a field, x, y, z variables over F, k an integer strictly bigger than 1, and $R = \frac{F[[x,y,z]]}{(xy - z^k)}$. Let $\mathfrak{m} = (x,y,z)R$, the maximal ideal in R. As R is a rational singularity ring, the ring R and the ideal $\mathfrak{m}$ and all of its powers are integrally closed. This means that the constant l from the set-up of Theorem 2.6 equals 0. Let S be the extended Rees ring $R[xt, yt, zt, t^{-1}]$, where t is a new indeterminate. Then

$S/t^{-1}S$, the associated graded ring of $\mathfrak{m}$, is isomorphic to $\frac{F[xt,yt,zt]}{(xt)(yt)}$. So we have two Rees valuations: $v_1(x) = k-1$, $v_1(y) = 1$, $v_1(z) = 1$, and $v_2(x) = 1$, $v_2(y) = k-1$, $v_2(z) = 1$. Note that if $k = 2$, the two Rees valuations are identical, so that in that case $\mathfrak{m}$ is one-fibered. For higher k the ideal $\mathfrak{m}$ is two-fibered. As the two valuations are compatible with the natural grading on R, we see that $C = k-1$. Thus by Theorem 2.6, whenever the product of two elements in R lies in $\mathfrak{m}^{kn}$, then one or the other element has to lie in $\mathfrak{m}^n$.

This $a = k, b = 0$ is the best possible as $xy = z^k \in \mathfrak{m}^k$, but neither x nor y lies in $\mathfrak{m}^2$.

Example 3.10. Let R be a rational singularity of type D_n, $n \geq 4$, or of type E_6, E_7, E_8. In any of these cases, R equals $F[[x,y,z]]/(z^2 - f(x,y))$, where the lowest term of $f(x,y)$ has degree 3. As R has a rational singularity, the maximal ideal $\mathfrak{m}$ and all of its powers are integrally closed so that again $l = 0$. Also, the associated graded ring of R is $F[x,y,z]/(z^2)$ so that $\mathfrak{m}$ is one-fibered. Thus if α, β are elements of R such that $\alpha\beta \in \mathfrak{m}^{2n}$, then either α or β lies in $\mathfrak{m}^n$.

In this example again the bound is the best possible: $x^{n-2}z, y^{n-2}z$ are in $\mathfrak{m}^{n-1}$ but not in $\mathfrak{m}^n$, yet their product lies in $\mathfrak{m}^{2n-1}$.

§4. ON THE NORMAL CONE OF A REDUCED IDEAL

Valuations associated to an ideal are also useful in the study of general (not necessarily $\mathfrak{m}$–primary) ideals. Here we give a valuative criterion for the normal cone of an ideal in an integrally closed domain to be reduced. This criterion then provides another proof that for a prime ideal in a regular ring containing a field, the normal cone is reduced if and only if it is a domain. See Huneke, Simis and Vasconcelos [HSV] for a more general version.

Theorem 4.1. *Let $(R,\mathfrak{m})$ be an integrally closed local domain, and let $I \subseteq R$ be an ideal. Then the following are equivalent:*

(1) *The graded ring $S = R/\overline{I} \oplus \overline{I}/\overline{I^2} \oplus \overline{I^2}/\overline{I^3} \oplus \overline{I^3}/\overline{I^4} \oplus \cdots$ is reduced, where the line over an ideal denotes its integral closure.*
(2) *For each Rees–valuation v of I, $v(I) = 1$.*

Similarly, the following two statements are equivalent:

(1) *The associated graded ring $gr_I(R)$ is reduced.*
(2) *$I \subseteq R$ is a normal ideal (i.e. $R[It]$ is a normal domain) and for each Rees–valuation v of I we have $v(I) = 1$.*

Proof: Each Rees valuation v of I corresponds to a prime ideal $\mathfrak{P}$ in the normal ring $T = R \oplus \overline{I} \oplus \overline{I^2} \oplus \cdots$, this prime ideal being minimal over IT or equivalently over $\overline{I} \oplus \overline{I^2} \oplus \overline{I^3} \oplus \cdots$, and thus necessarily having height one. If $S = \frac{T}{\overline{I} \oplus \overline{I^2} \oplus \overline{I^3} \oplus \cdots}$ is reduced, then

$$\mathfrak{P}T_{\mathfrak{P}} = \left(\overline{I} \oplus \overline{I^2} \oplus \overline{I^3} \oplus \cdots\right) T_{\mathfrak{P}} = \left(I \oplus I\overline{I} \oplus I\overline{I^2} \oplus \cdots\right) T_{\mathfrak{P}} = IT_{\mathfrak{P}},$$

as $T_{\mathfrak{P}}$ is a discrete valuation ring, so that $v(I) = 1$. Thus (1) implies (2) in the first set of statements.

If instead $\mathrm{gr}_I(R)$ is reduced, certainly all the powers of I are integrally closed so that $R[It]$ is a normal ring and equal to T. Thus also in the second set of statements, (1) implies (2).

Conversely, assuming (2) let x be nilpotent in S (resp. in $\mathrm{gr}_I(R)$). Thus there exist integers $k \geq 1$ and l such that $x \in \overline{I^l} \setminus \overline{I^{l+1}}$ (resp. $x \in I^l \setminus I^{l+1}$) and such that $x^k \in \overline{I^{kl+1}}$ (resp. $x^k \in I^{kl+1}$). Then for each Rees valuation v of I, $kv(x) = v(x^k) \geq (kl+1)v(I) = kl+1$, so that $v(x) \geq l + \frac{1}{k}$. As $v(x)$ is an integer, necessarily $v(x) \geq l+1$. But then x lies in the integral closure $\overline{I^{l+1}}$ of I^{l+1}, contradicting the assumption for the first set of statements. For the second set of statements, (2) contains the extra assumption that I is normal, hence x actually lies in I^{l+1}, again contradicting the hypothesis.

We need another result before we can apply this valuative criterion:

Proposition 4.2. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field, let $I \subseteq R$ be an ideal and let $x \in I$ with $x \notin \mathfrak{m}^2$. Denoting by $\overline{}$ residue classes mod x, we have a (non–canonical) isomorphism of graded rings*

$$gr_I(R) \cong gr_{\overline{I}}(\overline{R})[T]$$

identifying $x + I^2$ with the indeterminate T.

Proof: We may replace R by its xR–adic completion without changing $\mathrm{gr}_I(R)$. Let $k \subseteq R$ be a perfect subfield. As $\overline{R}$ is regular again, it is formally smooth over k by [Mat], (28.M). Thus by [Ha], chapt. I (1.2) the canonical surjection $R \to \overline{R}$ has a section, inducing an isomorphism of rings

$$R \cong \overline{R}[[x]]$$

(as R is xR–adically complete). Via this isomorphism we have

$$I = \overline{I} \cdot \overline{R}[[x]] + x \cdot \overline{R}[[x]]$$

as $x \in I$, and from this the proposition follows easily.

This result enables the use of induction in proving that reduced normal cones of primes in regular rings containing fields are domains:

Corollary [HSV] 4.3. *Let R be a regular ring containing a field, and let $I \subseteq R$ be an ideal of R such that $gr_I(R)$ is reduced. Then there is a one–to–one correspondence between the minimal prime divisors of I and the minimal prime ideals of $gr_I(R)$. In particular, if $I \subseteq R$ is a prime ideal and $gr_I(R)$ is reduced, then $gr_I(R)$ is already a domain.*

Proof: As $R/I \subseteq \mathrm{gr}_I(R)$, I is a radical ideal. Note that for each minimal prime divisor $\mathfrak{p}$ of I, $\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$ is a domain as $R_\mathfrak{p}$ is a regular local ring and $IR_\mathfrak{p}$ is its maximal ideal, hence

$$\mathfrak{P} := \ker(\mathrm{gr}_I(R) \longrightarrow \mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}))$$

is the unique minimal prime of $\mathrm{gr}_I(R)$ whose intersection with $\mathrm{gr}_I(R)_0 = R/I$ corresponds to the minimal prime divisor $\mathfrak{p}$ of I. We have to show that these are all the minimal primes of $\mathrm{gr}_I(R)$.

Let $\mathfrak{Q} \in \mathrm{Min}(\mathrm{gr}_I(R))$, and let $\mathfrak{q} := \mathfrak{Q} \cap R/I$. We may replace R by its localization in $\mathfrak{q}$ (resp. its preimage in R) and we may assume that $\mathfrak{q} = \mathfrak{m}$ is the maximal ideal of R/I. We have to show that $I = \mathfrak{m}$.

First suppose that I is not contained in $\mathfrak{m}^2$. Then by the Proposition there exists an element $x \in I \setminus \mathfrak{m}^2$ which induces an isomorphism $\mathrm{gr}_I(R) \cong \mathrm{gr}_{\overline{I}}(\overline{R})[T]$ of rings. Necessarily the image of x in $\mathrm{gr}_I(R)$ is not a zero-divisor and $\mathrm{gr}_{\overline{I}}(\overline{R})$ is reduced. As $\overline{R} = R/xR$ is regular, we can use induction. Let $\mathfrak{P}$ be a minimal prime ideal in $\mathrm{gr}_{\overline{I}}(\overline{R})$ which is contained in $\mathfrak{Q}$. By the isomorphism from the Proposition, and since all minimal prime ideals in polynomial rings are extended (cf. [Mat], (9.B)), $\mathfrak{P} = \mathrm{gr}_{\overline{I}}(\overline{R}) \cap \mathfrak{Q}$. By induction there exists a minimal prime ideal $\mathfrak{p}$ over $\overline{I}$ such that $\mathfrak{p}$ is the contraction of $\mathfrak{P}$. Hence this minimal prime ideal $\mathfrak{p}$ over I is also the contraction of $\mathfrak{Q}$, thus proving that $\mathfrak{p} = \mathfrak{q} = \mathfrak{m}$ is a minimal prime over I.

Now assume that $I \subseteq \mathfrak{m}^2$. As $\mathfrak{Q}$ is a minimal prime of $\mathrm{gr}_I(R)$, it corresponds to a minimal prime divisor of $IR[It]$ in $R[It]$, hence by Theorem (4.1) to a Rees–valuation v of I with $v(I) = 1$. On the other hand, v is positive on $\mathfrak{m}$ (as $\mathfrak{Q} \cap R/I = \mathfrak{m}$), and therefore

$$v(I) \geq v(\mathfrak{m}^2) = 2v(\mathfrak{m}) \geq 2,$$

a contradiction.

Remark 4.4. We also have a short direct proof, without using valuations, of why this corollary holds for regular rings even when the ring does not contain a field. The proof in [HSV] of a more general result is quite a bit longer and more technical, that is why we present this shorter proof.

As in the proof of the corollary, it suffices to show that every minimal prime ideal $\mathfrak{Q}$ of $\mathrm{gr}_I(R)$ contracts to a minimal prime over I in R. As in the proof, after localizing R if necessary, without loss of generality $\mathfrak{Q}$ contracts to the maximal ideal $\mathfrak{m}$ of R. As $\mathrm{gr}_I(R)$ is reduced, there exists an element $s^* \in \mathrm{gr}_I(R) \setminus \mathfrak{Q}$ such that $s^*\mathfrak{Q} = 0$. In particular, $s^*\mathfrak{m} = 0$ in $\mathrm{gr}_I(R)$. Without loss of generality s^* is a homogeneous element, say of degree n. Let s be an element of $I^n \subseteq R$ whose image in $\mathrm{gr}_I(R)$ is s^*. Then $s^*\mathfrak{m} = 0$ implies that $s\mathfrak{m} \subseteq I^{n+1}$. Define

$$\widetilde{\mathfrak{Q}} = \frac{\mathfrak{m}}{I} \oplus \frac{I \cap \mathfrak{m}^2 + I^2}{I^2} \oplus \frac{I^2 \cap \mathfrak{m}^3 + I^3}{I^3} \oplus \frac{I^3 \cap \mathfrak{m}^4 + I^4}{I^4} \oplus \cdots,$$

an ideal in $gr_I(R)$. Then $s^*\widetilde{\mathfrak{Q}} = 0$ in $\mathrm{gr}_I(R)$, as for any nonzero element $z^* \in (I^j \cap \mathfrak{m}^{j+1} + I^{j+1})/I^{j+1} \subseteq \mathrm{gr}_I(R)$, its preimage z in $I^j \cap \mathfrak{m}^{j+1} \setminus I^{j+1}$ in R satisfies

$$\begin{aligned}(sz)^{n+j+2} &= sz \cdot s^{n+j+1} \cdot z^{n+j+1} \\ &\in \left(I^{n+j} \cap \mathfrak{m}^{n+j+1}\right) \cdot s^{n+j+1} \cdot (I^j)^{n+j+1} \\ &\subseteq \left(I^{n+1}\right)^{n+j+1} \cdot (I^j)^{n+j+1} \\ &= I^{(n+j+1)(n+j+1)}.\end{aligned}$$

But $(n+j+2)(\deg s^* + \deg z^*) = (n+j+2)(n+j) < (n+j+1)(n+j+1)$, which forces s^*z^* to be nilpotent in the reduced ring $\mathrm{gr}_I(R)$, hence zero. Thus $s^*\widetilde{\mathfrak{Q}} = 0$, and, as $s^* \notin \mathfrak{Q}$, it follows that $\widetilde{\mathfrak{Q}} \subseteq \mathfrak{Q}$.

Now write $I = (x_1, \dots, x_l) + I \cap \mathfrak{m}^2$, where $x_1, \dots, x_l$ is part of a regular system of parameters of R. Then

$$\frac{\mathrm{gr}_I(R)}{\widetilde{\mathfrak{Q}}} = \frac{R}{\mathfrak{m}} \oplus \frac{I}{I \cap \mathfrak{m}^2 + I^2} \oplus \frac{I^2}{I^2 \cap \mathfrak{m}^3 + I^3} \oplus \cdots,$$

where the ith part is an $R/\mathfrak{m}$-vector space, whose natural basis is any minimal set of generators of $(x_1, \dots, x_l)^i$. Thus $\frac{\mathrm{gr}_I(R)}{\widetilde{\mathfrak{Q}}}$ is a polynomial ring in l variables over $R/\mathfrak{m}$. In particular, $\widetilde{\mathfrak{Q}}$ is a prime ideal inside $\mathfrak{Q}$, so that by the minimality of $\mathfrak{Q}$, $\mathfrak{Q} = \widetilde{\mathfrak{Q}}$. But then

$$l = \dim\left(\frac{\mathrm{gr}_I(R)}{\widetilde{\mathfrak{Q}}}\right) = \dim\left(\frac{\mathrm{gr}_I(R)}{\mathfrak{Q}}\right) = \dim R,$$

the last equality as $\mathrm{gr}_I(R)$ is equidimensional. But $l = \dim R$ forces I to be equal to $\mathfrak{m}$, contradicting the choice of $\mathfrak{Q}$.

This proves the one-to-one correspondence in the Remark.

Proposition (4.2) raises the following two questions:

Question 4.5. Let $(R, \mathfrak{m})$ be a regular local ring, let $I \subseteq R$ be an ideal and assume that $I \nsubseteq \mathfrak{m}^2$. Does there exist an $x \in I \setminus \mathfrak{m}^2$ such that, if $\overline{}$ denotes residue classes mod x, there is an isomorphism of graded rings

$$\mathrm{gr}_I(R) \cong \mathrm{gr}_{\overline{I}}(\overline{R})[X],$$

under which $x + I^2$ maps to X?

Question 4.6. Let R be a local ring which is homomorphic image of a regular local ring P, say $P/I = R$. Is the conormal module I/I^2 of R a stable invariant of R?

Remark 4.7. A positive answer to question 4.5 gives a positive answer to question 4.6. For this one may assume that R is complete and that the regular ring mapping onto R is complete as well (using faithfully flat descent). Then any two complete regular local rings mapping onto R can be dominated by a third one, mapping onto both of them, and we are in the situation of question 4.5.

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