Cores of Ideals in Two Dimensional Regular Local Rings

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The main result of this paper is the explicit determination of the core of integrally closed ideals in two-dimensional regular local rings. The core of an ideal I in a ring R was introduced by Judith Sally in the late 1980's and alluded to in Rees and Sally's paper [RS]. Recall that a reduction of I is any ideal J for which there exists an integer n such that $JI^n = I^{n+1}$ [NR]. In other words, J is a reduction of I if and only if I is integrally dependent on J. An ideal is integrally closed if it is not a reduction of any ideal properly containing it. With this,

(1.1) Definition: The core of an ideal I, denoted $\operatorname{core}(I)$, is the intersection of all reductions of I.

In general, the core seems extremely difficult to determine and there are few computed examples. A priori is it not clear whether it is zero. However, one can show that, in general, the core always contains a power of I. A proof of this for Buchsbaum rings can be found in [T, Proposition 5.1].

It is quite natural to study the core, partly due to the theorem of Briançon and Skoda (see [BS], [LS], [LT], [L4], [HH], [RS], [S], [AH1-2], [AHT]). A simple version of this theorem states that if R is a d-dimensional regular ring and I is any ideal of R, then the integral closure of I^d is contained in I. In particular, the integral closure of I^d is contained in $\operatorname{core}(I)$. It is an important question to understand how the core of I relates to I. More generally, how one can approximate general m-primary ideals in local rings (R, m) by intersections of parameter ideals is an interesting question.

We hope our results in dimension two will provide insight into the nature of the core in higher dimensions.

Some of the open questions regarding the core are:

- If I is integrally closed, is $\operatorname{core}(I)$ also integrally closed?
- If the completion $\hat{R}$ of R is equidimensional, does $\operatorname{core}(I)\hat{R}$ equal $\operatorname{core}(\hat{I})$? More generally, how does the core behave under faithfully flat maps ?

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– How does core (I^n) compare to core (I)?

– If I and J are integrally closed ideals and $I \subseteq J$, is core$(I) \subseteq$ core (J)?

We will assume throughout that R is a two-dimensional regular local ring with maximal ideal m. An ideal I in such a ring equals xJ for some $x \in R$ and some ideal J which is either R or m-primary. Let S be the set of all reductions of J. As every reduction of I is of the form xK for some reduction K of J,

$$\text{core}\,(I) = \bigcap_{K \in S} xK = x \bigcap_{K \in S} K = x \cdot \text{core}\,(J).$$

Therefore to compute the core of ideals in two-dimensional regular local rings we only have to consider m-primary ideals.

We will also assume throughout that the residue field of R is infinite. This assumption serves two purposes: it allows us to choose good quadratic transformations (see Section 2), and it makes sure that we have an abundant set of reductions of every ideal (see [NR, Theorem 1, p. 153]).

We determine in Section 3 the core of an integrally closed ideal I in terms of ideals of minors of any presentation of I. We prove that core $(I) = I \cdot \text{adj}(I)$, where adj$(I)$ is the adjoint of I as defined by Lipman in [L3]. We prove the equivalent statement that core (I) is $\mathfrak{F}_1(I)\mathfrak{F}_2(I)$, where $\mathfrak{F}_i(I)$ is the ith Fitting ideal of I. We also answer several of the questions above. For example, in regular local rings of dimension two with an infinite residue field we prove that if $J \subseteq I$ are integrally closed ideals, then core $(J) \subseteq$ core (I). We also prove (again for two-dimensional regular local rings) that the core of an integrally closed ideal is integrally closed.

In Section 4 we discuss the various relationships between the core of an ideal and the adjoint of an ideal. More specifically, we prove that $\text{adj}(I^n) = I^{n-1}\text{adj}(I)$, $\text{core}\,(I^n) = I^{2n-2}\text{core}\,(I)$, $\text{adj}(\text{core}\,(I)) = (\text{adj}(I))^2$, and $\text{core}\,(\text{core}\,(I)) = I\ (\text{adj}(I))^3$. Lipman also proved in [L4] that $\text{adj}(I^n) = I^{n-1}\text{adj}(I)$. Lipman's proof uses duality and a vanishing theorem. We give an elementary proof. For higher dimensions Lipman conjectures that $\text{adj}(I^n) = I\text{adj}(I^{n-1})$ for all n bigger than the analytic spread of I.

Section 2 contains the needed background about two-dimensional regular local rings.

2. Two-dimensional regular local rings

Throughout (R, m, k) will be a two dimensional regular local ring with maximal ideal m and an infinite residue field k. Two-dimensional regular local rings have many special properties which we will use heavily. We list these properties and the needed definitions. For an introductory treatment see [H], and for comprehensive treatments and related results see [Z], [ZS, Volume 2, Appendix 5], [K], [L1], [L2], [L3], [C], [G], [HS], [N], [R], [V].

For any matrix A and any integer n, we will denote the ideal generated by the $n \times n$ minors of A by $I_n(A)$.

For any nonzero ideal I, *the order of* I, denoted $\mathrm{ord}(I)$, is the largest integer n such that $I \subseteq m^n$. Since the associated graded ring is a polynomial ring, ord is a valuation. The order of a reduction of I is the same as the order of I.

The smallest number of elements generating an ideal I will be denoted as $\mu(I)$.

The multiplicity of an m-primary ideal I in an arbitrary Noetherian local ring (R, m) of dimension d is defined as follows: there exists a polynomial $p(n)$ of degree d such that for large n, $p(n) = \text{length}\,(R/I^n)$; the multiplicity of I is then defined to be the coefficient of $n^d/d!$ in $p(n)$.

We now recall some of the facts about two-dimensional regular local rings:

(2.1) (Hilbert-Burch) For every m-primary ideal I there is a free resolution

$$0 \to R^{n-1} \xrightarrow{A} R^n \xrightarrow{(x_1,\ldots,x_n)} R \to R/I \to 0,$$

where A is a $n \times (n-1)$ matrix such that the maximal minors of A generate I. We will refer to A as a presenting matrix of I. In fact, if $I = (x_1, \ldots, x_n)$, we may assume that the $(n-1) \times (n-1)$ minor of A obtained after deleting the ith row equals x_i. (Here, n needn't be the minimal number of generators of I.)

(2.2) For any $x \in m \setminus m^2$ and any height two prime ideal P in $R[\frac{m}{x}]$, the localization $R[\frac{m}{x}]_P$ is a regular local ring. We call any ring of this form a *quadratic transformation* of R.

If $S = R[\frac{m}{x}]$, $x \notin m^2$, and I is an ideal of R, then $IS = x^{ord(I)} I'$ for some ideal I' in S of height two. The images of I' in the quadratic transformations of R are called the *quadratic transforms of* I.

If J is a reduction of I, that is if there exists an integer n such that $JI^n = I^{n+1}$, then the same equality still holds after extending ideals to $R[\frac{m}{x}]$. As $\mathrm{ord}(I) = \mathrm{ord}(J)$, the same equality still holds for the transforms of J and I. This implies that the transform of a reduction of an ideal is a reduction of the transform of the ideal.

For any matrix A with elements in R, $\frac{A}{x}$ denotes the matrix obtained by dividing every element of A by x. If I is an ideal of order $n-1$ and with n generators, and if A is an $n \times (n-1)$ presenting matrix of I, then $\frac{A}{x}$ is a not necessarily minimal presenting matrix of the transform I' of I, as we now explain. Let $I = (x_1, \ldots, x_n)$ and A as in (2.1). Then consider the complex

$$0 \to S^{n-1} \xrightarrow{\frac{A}{x}} S^n \xrightarrow{\left(\frac{x_1}{x^{n-1}}, \ldots, \frac{x_n}{x^{n-1}}\right)} S \to S/I' \to 0.$$

Notice that $I_{n-1}(\frac{A}{x}) = I'$ since the $(n-1)$-minor of $\frac{A}{x}$ deleting the ith row is exactly $\frac{x_i}{x^{n-1}}$. Then by the Buchsbaum-Eisenbud criterion for exactness ([BE]), $\frac{A}{x}$ is a (possibly non-minimal) presentation of I'.

(2.3) An ideal I is contracted from $R[\frac{m}{x}]$ for some $x \in m \setminus m^2$ if and only if $\text{ord}(I) = \mu(I) - 1$ (see [H, Proposition 2.3]). Any x chosen such that its leading form in the associated graded ring $gr_m(R)$ does not divide the leading term of the image of some element of I of degree $\text{ord}(I)$ will suffice (see [H, proof of Proposition 2.3]). As the residue field is infinite, we can always choose such an x for a finite set of ideals at a time. Thus the appropriate quadratic transformations always exist and in the rest of the paper we ignore the issue of existence.

For any m-primary ideal I of R the multiplicity of I is strictly larger than the multiplicity of the transform I' of I in any localization of $R[\frac{m}{x}]$, if x is chosen such that I is contracted from $R[\frac{m}{x}]$ (see [H, Proposition 3.6], [K, page 33]). It will be implicit throughout that our transformations are chosen carefully.

After a finite number of quadratic transformations, an integrally closed m-primary ideal transforms to a power of the maximal ideal.

(2.4) We will need to use several more properties of integrally closed ideals:

Let I be integrally closed and m-primary. Then $\text{ord}(I) = \mu(I) - 1$ (see [H, Propositions 2.3 and 3.1]). Thus every integrally closed m-primary ideal in R contracts from some $R[\frac{m}{x}]$. (This is false for an arbitrary m-primary ideal.)

The product of two integrally closed ideals in R is integrally closed (see [Z, Part II, 12]).

For any reduction (x_1, x_2) of an integrally closed m-primary ideal I, $(x_1, x_2)I = I^2$ (see [LT, Corollary 5.4], [H, Theorem 5.1], [HS, Theorem 3.1]).

The quadratic transform of an integrally closed m-primary ideal is integrally closed and is either the whole ring or it has height two. (See [H].)

(2.5) More generally, let R be a two dimensional regular local ring and let $R \subseteq S$, where S is a two dimensional regular local ring between R and its field of fractions. If I is an ideal of R, we define $I^S := (\gcd(IS))^{-1}IS$, the S-transform of I. This will be a height two ideal of S (or possibly S itself). The *point basis* of I is the family of integers $\{\mathrm{ord}_S(I^S)\}_{R\subseteq S}$, where S runs over all the two-dimensional regular local rings containing R and contained in its fraction field. There are only finitely many S for which $\mathrm{ord}_S(I^S)$ is nonzero; these S are called the base points of I (see [L3, p. 225]). Two m-primary ideals of R have the same point basis if and only if their integral closures are equal [L2, p. 209, (1.10)]. It will follow from our theorems that the core of an integrally closed ideal I is exactly the integrally closed ideal whose point basis is $\left\{\max\{0, 2\,\mathrm{ord}_S(I^S) - 1\}\right\}_{R\subseteq S}$.

3. The core

We analyze a free resolution of an m-primary ideal $I = (x_1, \ldots, x_n)$ more carefully:

$$\mathbb{F} : 0 \to R^{n-1} \xrightarrow{A} R^n \xrightarrow{(x_1,\ldots,x_n)} R \to R/I \to 0, \tag{1}$$

where A is an n by $n-1$ matrix. So the columns of A generate all the relations on $x_1, \ldots, x_n$. Assume that (x_1, x_2) is a reduction of I and let B be the submatrix of A consisting of the last $n-2$ rows.

Recall that for ideals I and J in a ring R, $I : J$ denotes the ideal $\{r \in R | rJ \subseteq I\}$. With this we state a result from linkage theory:

(3.1) Lemma: With notation as above, $(x_1, x_2) : I = I_{n-2}(B)$.

Proof: Let $\mathbb{K}$ be the Koszul complex

$$\mathbb{K} : 0 \to R \xrightarrow{\binom{x_2}{-x_1}} R^2 \xrightarrow{(x_1,x_2)} R \to R/(x_1, x_2) \to 0.$$

There is an obvious complex homomorphism $\lambda : \mathbb{K} \to \mathbb{F}$ with λ_0 being the identity on R and λ_1 taking (r_1, r_2) to $(r_1, r_2, 0, \ldots, 0)$. Let $\mathbb{M}$ be the mapping cone of the dual of $\lambda : \mathbb{K} \to \mathbb{F}$. It is easy to see that $\mathbb{M}$ is acyclic (see [PS, Proposition 2.6]), that $H_0(\mathbb{M}) = R/((x_1, x_2) : I)$ and that the presenting matrix of $H_0(\mathbb{M})$ is B^{tr} (the transpose of B). In particular, B^{tr} is injective and by the Hilbert-Burch theorem, $(x_1, x_2) : I = I_{n-2}(B)$. ∎

(3.2) Lemma: Let I be a contracted ideal. Then $\mu(I_{n-2}(B)) - 1 = \mathrm{ord}(I_{n-2}(B)) = n-2$, i.e. $I_{n-2}(B)$ is contracted from a quadratic transformation.

Proof: The elementary column operations on A do not change $I_i(A)$ or $I_i(B)$ for any integer i. The same is true for the elementary row operations on rows of A which do not modify the rows of B by subtracting multiples of the first two rows of A.

Suppose that some entry a_{ij} in A is a unit. This means that x_i is contained in the ideal $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Thus as x_1, x_2 are part of a minimal generating set of

I, any a_{1j} or a_{2j} being a unit would imply that for some $i \geq 3$, a_{ij} is also a unit. Thus if some a_{ij} is a unit, we may assume that $i \geq 3$. Then by elementary row and column operations as prescribed above, and by possibly changing the element x_i, we may without loss of generality assume that all the other entries of A in the ith row or in the jth column are zero. Then the matrix A' obtained from A by deleting the ith row and the jth column is still a presenting matrix of I, but we are leaving out the redundant ith generator of I. Note that for all integers i, $I_i(A) = I_{i-1}(A')$ and $I_i(B) = I_{i-1}(B')$, where B' is the submatrix of A' consisting of all but the first two rows.

Thus we may assume that $n = \mu(I)$. As all the entries of A lie in m, $\operatorname{ord}(I_{n-2}(B)) \geq n-2$. If $\operatorname{ord}(I_{n-2}(B)) \geq n-1$, then $I_{n-2}(B) \subseteq m^{n-1}$. By expanding x_1 (resp. x_2) (as determinants) along the first (resp. second) row, we see that $(x_1, x_2) \subseteq mI_{n-2}(B) \subseteq m^n$. Thus (now we use that I is contracted)

$$n - 1 = \mu(I) - 1 = \operatorname{ord}(I) = \operatorname{ord}((x_1, x_2)) \geq n,$$

which is a contradiction. So $\operatorname{ord}(I_{n-2}(B)) = n-2$.

Furthermore, B^{tr} presents $I_{n-2}(B)$ and every entry of B is in the maximal ideal, so by the Hilbert-Burch theorem, $\mu(I_{n-2}(B)) = n-1$. ■

Thus if x is such that $I_{n-2}(B)$ is contracted from $R[\frac{m}{x}]$, then this lemma says that $I_{n-2}(B) = I_{n-2}(B)R[\frac{m}{x}] \cap R = x^{n-2} \cdot I_{n-2}\left(\frac{B}{x}\right) R[\frac{m}{x}] \cap R$.

Furthermore,

(3.3) Proposition: Let I be an integrally closed m-primary ideal and let A and B be as in (1). Then the m-primary ideal $I_{n-2}(B)$ equals $I_{n-2}(A)$ and is integrally closed.

Proof: By the elementary row and column operations we may assume without loss of generality that $n = \mu(I)$ (see the discussion in the proof of Lemma 3.2). We know that $I_{n-2}(B)$ is contracted from some $R[\frac{m}{x}]$. As $\frac{A}{x}$ presents the integrally closed ideal $\frac{I}{x^{ord(I)}}$ in $R[\frac{m}{x}]$ and $\left(\frac{x_1}{x^{ord(I)}}, \frac{x_2}{x^{ord(I)}}\right)$ is a reduction of I', by induction on the multiplicity of I (cf. (2.3)), $I_{n-2}\left(\frac{A}{x}\right) R[\frac{m}{x}]_P = I_{n-2}\left(\frac{B}{x}\right) R[\frac{m}{x}]_P$ is integrally closed for any height two prime ideal P in $R[\frac{m}{x}]$. Now note that the order of $I_{n-2}(A)$ equals $\operatorname{ord}(I_{n-2}(B)) = n-2$. Hence $I_{n-2}(\frac{A}{x})$ (resp. $I_{n-2}(\frac{B}{x})$) is the transform of the m-primary ideal $I_{n-2}(A)$ (resp. $I_{n-2}(A)$). It follows that $I_{n-2}(\frac{A}{x})$ and $I_{n-2}(\frac{B}{x})$ have height at least two and so the above means that $I_{n-2}\left(\frac{A}{x}\right)$ equals $I_{n-2}\left(\frac{B}{x}\right)$ and that this ideal is integrally closed in $R[\frac{m}{x}]$. Thus

$$\begin{aligned} I_{n-2}(A) &\subseteq x^{n-2} I_{n-2}\left(\frac{A}{x}\right) R[\tfrac{m}{x}] \cap R \\ &= x^{n-2} I_{n-2}\left(\frac{B}{x}\right) R[\tfrac{m}{x}] \cap R \end{aligned}$$

$$
\begin{aligned}
&= I_{n-2}(B) \\
&\subseteq I_{n-2}(A),
\end{aligned}
$$

and equality holds throughout. Moreover, these containments show that $I_{n-2}(A)$ is contracted from the integrally closed ideal $x^{n-2}I_{n-2}\left(\frac{B}{x}\right)R[\frac{m}{x}]$, hence $I_{n-2}(A)$ is integrally closed. ■

We record here the following corollary:

(3.4) Corollary: Let I be an integrally closed m-primary ideal and let A and B be as in (1). Then $I_j(A)$ equals $I_j(B)$ and is integrally closed for all $j = 0, \ldots, n-2$.

Proof: If $n = 2$, there is nothing to show. Now assume that $n > 2$. By Proposition 3.3, $I_{n-2}(B) = I_{n-2}(A)$ is integrally closed. As B^{tr} presents $I_{n-2}(B)$, induction on n shows that $I_j(B) = I_j(B^{tr})$ is integrally closed for all $j = 0, \ldots, n-3$. By induction on multiplicity of ideals (cf. (2.3)),

$$
\begin{aligned}
I_j(A) &\subseteq x^j \cdot I_j\left(\frac{A}{x}\right) R[\tfrac{m}{x}] \cap R \\
&= x^j \cdot I_j\left(\frac{B}{x}\right) R[\tfrac{m}{x}] \cap R \\
&= I_j(B) \\
&\subseteq I_j(A). \quad ■
\end{aligned}
$$

Proposition 3.3, together with Lemma 3.1, gives:

(3.5) Proposition: For an m-primary integrally closed ideal I and any reduction (x_1, x_2) of I, $(x_1, x_2) : I = I_{n-2}(A)$ for any $n \times (n-1)$ presenting matrix A of I. ■

There are other ideals for which the results above also hold. For example, let I be (y^2, x^4, yx^3) in the power series ring in two variables x and y over a field. I has order two and is generated by three elements. It is easy to see that (y^2, x^4) is a reduction of I. A presenting matrix of I is

$$
A = \begin{pmatrix} x^3 & 0 \\ 0 & -y \\ -y & x \end{pmatrix}.
$$

The matrix B is just the last row of A. Note that $I_1(A) = (x, y) = I_1(B)$ but that I is not integrally closed as xy^2 is in the integral closure but not in I.

We now prove several corollaries for integrally closed ideals. For some statements we only need the weaker hypothesis that for a finite number of reductions (x_1, x_2) of I, the corresponding matrices B satisfy $I_{n-2}(B) = I_{n-2}(A)$. However, in order to avoid messy hypotheses, we state the results only for integrally closed ideals.

(3.6) Corollary: For an integrally closed ideal I, $I \cdot I_{n-2}(B) = (x_1, x_2)I_{n-2}(B) = (x_1, x_2)I_{n-2}(A)$ for any reduction (x_1, x_2) of I, where A and B are defined as in (1). In particular, $I_{n-2}(A)I$ is contained in core (I).

Proof: We have to show the first equality and in particular that the left hand side is contained in the right hand side. Note that $I \cdot I_{n-2}(B) = I \cdot ((x_1, x_2) : I)$ is contained in (x_1, x_2). In fact, when the matrices A and B are replaced by generic matrices A^* and B^*, $I^* \cdot I_{n-2}(B^*) \subseteq (x_1^*, x_2^*)I_{n-2}(A^*)$, where I^* is the ideal of $n-1$-minors and x_1^*, x_2^* are $(n-1)$-size minors, as in (1). (This can be seen easily by using the fact that $(x_1^*, ..., x_n^*)A^* = 0$, and summing the rows of this equation times suitable $n-3$ size minors of B^*, or alternatively by using the straightening relations on the minors.) Thus $I \cdot I_{n-2}(B) \subseteq (x_1, x_2) \cdot I_{n-2}(A)$. But in our case this equals the right hand side. The last claim follows as any reduction contains a reduction generated by two elements (see [NR, Theorem 1, p. 153]). ∎

(3.7) Corollary: Let I be integrally closed and let (x_1, x_2) be a reduction of I. Then $(x_1^2, x_2^2) : I^2 = I_{n-2}(B)(x_1, x_2)$.

Proof: We use corollary 3.6: $I_{n-2}(B)(x_1, x_2)I^2 = I_{n-2}(B)(x_1, x_2)^3 \subseteq (x_1^2, x_2^2)$, so the right hand side is contained in the left hand side. Now let $a \in (x_1^2, x_2^2) : I^2$. In particular, $ax_1x_2 \in (x_1^2, x_2^2)$, so $a \in (x_1, x_2)$. Write $a = a_1x_1 + a_2x_2$. Then $ax_2I \subseteq (x_1^2, x_2^2)$ implies that $a_1x_1x_2I \subseteq (x_1^2, x_2^2)$, so $a_1 \in (x_1^2, x_2^2) : x_1x_2I \subseteq (x_1, x_2) : I = I_{n-2}(B)$. Similarly, $a_2 \in I_{n-2}(B)$. Hence a is contained in the ideal on the right. ∎

In what follows we prove that $I_{n-2}(A)I$ is the core of I when I is integrally closed.

(3.8) Lemma: Let I be an integrally closed m-primary ideal in a two-dimensional regular local ring (R, m) with infinite residue field. Let x_1 be part of a minimal reduction of I, i.e., there exists x_2 (actually infinitely many by [NR, Theorem 1, p. 153]) such that (x_1, x_2) is a reduction of I. Then

$$\bigcap_{x_2}(x_1, x_2) = (x_1) + I_{n-2}(A)I$$

for any $n \times (n-1)$ presenting matrix A of I, where x_2 varies over all elements such that (x_1, x_2) is a reduction of I.

Proof: The right-hand side is contained in the left-hand side by Corollary 3.6. It is easy to see that, in general, if $J_1 \subseteq J_2$ are m-primary ideals, to prove the equality $J_1 = J_2$ it is necessary and sufficient to prove the equality $J_1 = J_2 \cap (J_1 : m)$. Thus to prove the lemma let α be arbitrary in the intersection of $\cap_{x_2}(x_1,x_2)$ and $\big((x_1)+I_{n-2}(A)I\big):m$. We will show that α is contained in $(x_1)+I_{n-2}(A)I$.

Fix one x_2 and write $\alpha = rx_1 + sx_2$. Note that for any $x \in m$, $xsx_2 = x\alpha - xrx_1 \in (x_1) + I_{n-2}(A)I = (x_1) + I_{n-2}(A)x_2$. The last equality follows from Corollary 3.6. As x_1, x_2 form a regular sequence, $xs \in I_{n-2}(A) + (x_1) = I_{n-2}(A)$. As this holds for all $x \in m$, $s \in I_{n-2}(A) : m$. We will show that s is contained in $I_{n-2}(A)$, which will finish the proof of 3.8.

Suppose that s is not in $I_{n-2}(A) = (x_1,x_2) : I$. Then there exists an element x_3 of I such that (x_1,x_3) is a minimal reduction of I and such that $s \notin (x_1,x_2) : x_3$. This also follows by [NR, Theorem 1, p. 153]: the set of all elements x_3 in I such that (x_1,x_3) is a reduction of I generates all of I and is Zariski-open in I/mI; the condition $s \notin (x_1,x_2) : x_3$ means we must also avoid the proper subideal $\big((x_1,x_2):s\big)\cap I$ in I. It also follows by [NR] that there exist infinitely many units u such that (x_1, x_2+ux_3) is a reduction of I. For each such u write $\alpha = r_u x_1 + s_u(x_2+ux_3)$.

For any $x \in m$, $xs_u(x_2+ux_3) = x\alpha - xr_ux_1 \in I_{n-2}(A)I + (x_1) = I_{n-2}(A)(x_2+ux_3) + (x_1)$. As x_1, x_2+ux_3 is a regular sequence, xs_u is contained in $I_{n-2}(A)$ for all u. Since this holds for all $x \in m$ and all allowable u, the s_u generate a R/m vector space in $R/I_{n-2}(A)$. Let $u_1,\dots,u_l$ be such that the images of $s_{u_1},\dots,s_{u_l}$ form a basis of this vector space. Now choose a u different from all the u_i. We can do this since we have infinitely many units at our disposal. Then there exist $l_i \in R$ such that $us_u - \sum l_i u_i s_{u_i} \in I_{n-2}(A)$.

First suppose that $s_u - \sum l_i s_{u_i}$ is contained in $I_{n-2}(A)$. Then in the vector space $(I_{n-2}(A):m)/I_{n-2}(A)$ (the images there marked with an overbar $^-$) we get $\overline{us_u} = \sum \overline{l_iu_is_{u_i}}$ and $\overline{s_u} = \sum \overline{l_is_{u_i}}$ which is only possible if all the $\overline{l_i}$ are zero and hence only if $\overline{s_u}=0$. But then $\alpha = r_ux_1 + s_u(x_2+ux_3)$ is contained in $(x_1)+I_{n-2}(A)I$, which proves the lemma in the case $s_u - \sum l_i s_{u_i} \in I_{n-2}(A)$.

Next suppose that $s_u - \sum l_i s_{u_i} \notin I_{n-2}(A)$. Let $l = \sum l_i - 1$. Then

$$\begin{aligned} lrx_1 + lsx_2 &= l\ \alpha \\ &= \sum l_i\alpha - \alpha \\ &= \sum l_i\big(r_{u_i}x_1 + s_{u_i}(x_2+u_ix_3)\big) - \big(r_ux_1 + s_u(x_2+ux_3)\big) \\ &= \sum l_ir_{u_i}x_1 - r_ux_1 + \sum l_is_{u_i}x_2 - s_ux_2 + \big(\sum l_iu_is_{u_i} - s_uu\big)x_3. \end{aligned}$$

As $\sum l_iu_is_{u_i} - s_uu$ is in $I_{n-2}(A)$, we may write $(\sum l_iu_is_{u_i} - s_uu)x_3 = t_1x_1 + t_2x_2$ for some $t_i \in I_{n-2}(A)$. Thus $lrx_1 + lsx_2 = \sum l_ir_{u_i}x_1 - r_ux_1 + \sum l_is_{u_i}x_2 - s_ux_2 + t_1x_1 + t_2x_2$. It follows that $x_1(lr - \sum l_ir_{u_i} + r_u - t_1) = x_2(\sum l_is_{u_i} - s_u - ls + t_2)$ and hence that

$\sum l_i s_{u_i} - s_u - ls + t_2 \in (x_1)$. This forces l to be a unit for otherwise ls is contained in $I_{n-2}(A)$, hence $\sum l_i s_{u_i} - s_u \in I_{n-2}(A)$, contradicting the hypotheses.

Thus l is a unit and the above says that $s \in (x_1, t_2) + (\{s_u, s_{u_i}\}) \subseteq I_{n-2}(A) + (\{s_u, s_{u_i}\})$. As $s_u, s_{u_i} \in (x_1, x_2) : x_3$, this proves that $s \in (x_1, x_2) : x_3$, which contradicts the choice of x_3 and finishes the proof of the lemma. ∎

A corollary is the main result of this paper:

(3.9) Theorem: If I is integrally closed in a regular local ring of dimension two with infinite residue field, and A is an $n \times (n-1)$ presenting matrix of I, then $\operatorname{core}(I) = I_{n-2}(A)I$.

Proof: By Corollary 3.6 we only have to prove that the left hand side is contained in the right hand side. So let α be an element of $\operatorname{core}(I)$. Then by Lemma 3.8, for any reduction (x_1, x_2) of I, $\alpha \in ((x_1) + I_{n-2}(A)I) \cap ((x_2) + I_{n-2}(A)I)$. So $\alpha = rx_1 + \alpha_1 = sx_2 + \alpha_2$ for some α_i in $I_{n-2}(A)I$. Thus by Corollary 3.6, $sx_2 - rx_1 = \alpha_1 - \alpha_2 = t_1x_1 + t_2x_2$ for some $t_i \in I_{n-2}(A)$. Hence $(s - t_2)x_2 = -(r + t_1)x_1$. It follows that $s \in (t_2) + (x_1) \subseteq I_{n-2}(A)$. Similarly, $r \in I_{n-2}(A)$. ∎

Thus once we find a presenting matrix of I, it is easy to calculate the core of I.

Now we mention a construction from [RS] that is related to the core. Let $I = (a_1, \ldots, a_n)$ be an m-primary ideal. If d is the dimension of R, let $Y_{ij}, i = 1, \ldots, d; j = 1, \ldots, n$ be indeterminates over R. Then define the ring S to be the localization at the extension of m of the polynomial ring in all the Y_{ij} over R. With this, Rees and Sally define $X(I) = (\sum_{j=1}^n a_j Y_{ij} | i = 1, \ldots, d)S \cap R$ and they show that $X(I)$ is contained in the core of I. By using the theorem above and some technical lemmas from [RS] one can obtain also the reverse inclusion. Thus another characterization of the core is:

$$\operatorname{core}(I) = \Big(\sum_{j=1}^{n} a_j Y_{ij} | i = 1, \ldots, d\Big) R[Y_{11}, Y_{12}, \ldots, Y_{dn}]_{mR[\underline{Y}]} \cap R.$$

However, note that Rees-Sally's formulation of the core does not make the actual computations of the core any easier as one has to introduce many new variables.

Another consequence of the lemma preceding the theorem is that when a is part of a minimal reduction of I, then the core of $I/(a)$ equals the image of $\operatorname{core}(I)$ in $R/(a)$. Furthermore:

(3.10) Proposition: If (a, b) is a reduction of I, then $b \cdot I_{n-2}(A)T = I \cdot I_{n-2}(A)T$, where $T = R/(a)$. If C is any other ideal in R which satisfies $bCT = ICT$ for all such b, then C is contained in the integral closure of $I_{n-2}(A)T$.

Proof: The first statement follows from (2.4).

For the rest, note that $IC \subseteq \cap_b(a,b) = (a) + I\ I_{n-2}(A)$. Hence modulo (a), $ICT \subseteq II_{n-2}(A)T$. As IT contains a non-zerodivisor, this inclusion implies (using the determinant trick) that CT is contained in the integral closure of $I_{n-2}(A)T$. ■

We combine this with Lemma 3.2 from [RS] in the case when a is square-free. Let C be the ideal of all elements of R which multiply the integral closure of $R/(a)$ into $R/(a)$ (i.e., the conductor). Lemma 3.2 in [RS] says that C has the property as in the proposition. Thus C is contained in the integral closure of the image in $R/(a)$ of $I_{n-2}(A)$ for any m-primary integrally closed ideal I in R for which a is part of a minimal reduction. In particular, C is contained in $m^{ord(a)-1}$.

With the main theorem we are also able to calculate the multiplicity of an integrally closed ideal I using elementary means. Recall the definition of multiplicity of an m-primary ideal in a Noetherian local ring of dimension d given on page 3. In a Cohen-Macaulay ring (which regular local rings certainly are) the multiplicity of I also equals the length of $R/(x_1, \ldots, x_d)$, where $(x_1, \ldots, x_d)$ is a reduction of I. However, the calculation of multiplicities following either of these two definitions can be difficult. There is a simple method for calculating the multiplicity of integrally closed ideals in a two-dimensional regular local ring:

$$\text{multiplicity of } (I) = \text{length}\,(R/I^2) - 2\ \text{length}\,(R/I).$$

We give another simple method:

(3.11) Proposition: If I is integrally closed and m-primary in a two-dimensional regular local ring (R, m) with infinite residue field, then

$$\text{multiplicity of } (I) = \text{length}\,(R/\text{core}\,(I)) - 2\ \text{length}\,(R/I_{n-2}(A)).$$

Proof: Let (x_1, x_2) be a reduction of I. Then the multiplicity of I equals the length of $R/(x_1, x_2)$. We use that

$$\text{length}\,(R/(x_1,x_2)) = \text{length}\,(R/(x_1,x_2)I_{n-2}(A)) - \text{length}\,((x_1,x_2)/(x_1,x_2)I_{n-2}(A)).$$

Now, $(x_1, x_2)/(x_1, x_2)I_{n-2}(A)$ is isomorphic to $R/I_{n-2}(A) \oplus R/I_{n-2}(A)$ and the product $(x_1, x_2)I_{n-2}(A)$ equals core (I). ■

The main theorem also allows us to answer some of our questions about the core:

(3.12) Corollary: The core of an integrally closed ideal is integrally closed.

Proof: When I is integrally closed, $I_{n-2}(A)$ is also integrally closed by Proposition 3.3. As the product of two integrally closed ideals is integrally closed (see (2.4), Zariski), the result follows. ■

(3.13) Corollary: If R and S are two-dimensional regular local rings with infinite residue fields and S is a faithfully flat R-algebra, then $\text{core}\,(I)S = \text{core}\,(IS)$ whenever both I and IS are integrally closed. In particular, this holds if S is the completion of R. ■

In the next section we shall answer more questions, but we first want to introduce a simpler notation. So far we wrote about $I_{n-2}(A)$ for any ideal I and any $n \times n-1$ presenting matrix A of I. In [L4, Proposition 3.3] it is proved that the adjoint of an integrally closed ideal I in R is equal to $(x_1, x_2) : I$ for any minimal reduction x_1, x_2 of I. (See [L4] for the definition of the adjoint.) By our results, it follows that the adjoint of I is exactly $I_{n-2}(A)$. Thus from now on we will denote $I_{n-2}(A)$ as $\text{adj}(I)$ and we call it the adjoint of I. We refer the reader to [L3], [L4] for the rich theory of adjoints which Lipman develops.

Thus we may rephrase Theorem 3.9 in terms of adjoint ideals:

(3.14) Theorem: Let (R, m) be a two-dimensional regular local ring with infinite residue field. Let I be an integrally closed m-primary ideal. Then $\text{core}\,(I) = I \cdot \text{adj}(I) = \text{adj}(I^2)$.

Proof: The first equality is simply a change of notation. The second equality follows from the fact that (x_1^2, x_2^2) is a reduction of I^2 and from Corollary 3.7. ■

The definition of the adjoint which Lipman gives in [L4] makes it clear that $\text{adj}(J) \subseteq \text{adj}(I)$ if J and I are integrally closed and J is contained in I. Thus we get:

(3.15) Proposition: Let R be a two-dimensional regular local ring with infinite residue field. If J and I are integrally closed ideals and J is contained in I, then $\text{core}\,(J) \subseteq \text{core}\,(I)$. ■

This result is interesting as the inclusions on cores do not hold in general. For example, let I be an integrally closed m-primary ideal of order bigger than one and let $J = (a, b)$ be a minimal reduction of I. Then $\text{core}\,(J) = J$ and $\text{core}\,(I) \subseteq m^{ord(I)-1}I$, which does not contain an ideal of order equal to $\text{ord}(I)$.

It is worthwhile to rephrase some of our results in terms of the Fitting ideals of I. Recall that if M is a finitely presented module with presenting matrix A which is n by m (so that under our conventions, M is the cokernel of the map from R^m to R^n determined by A), then the ith Fitting ideal of M, denoted $\mathfrak{F}_i(M)$, is the ideal generated by the $n-i$ size minors of A. By convention, the ideal generated by the k size minors of A is equal to R if $k \leq 0$, and is 0 for $k > \min\{m, n\}$. The Fitting ideals are independent of the presenting matrix. Our results are summarized in:

(3.16) Proposition: Let (R, m) be a two dimensional regular local ring with infinite residue field. Let I be an integrally closed m primary ideal. The following hold:

a) $\mathfrak{F}_1(I) = I$, $\mathfrak{F}_2(I) = \text{adj}(I)$, $\mathfrak{F}_1(I)\mathfrak{F}_2(I) = \text{core}\,(I)$.

b) $\mathfrak{F}_{j+1}(I) = \mathfrak{F}_j(\text{adj}(I))$ for $j \geq 1$.

c) $\mathfrak{F}_{j+1}(I) = \text{adj}(\mathfrak{F}_j(I))$ for $j \geq 1$.

Proof: Part a) is immediate from the definition of the Fitting ideals and the results above. Only b) and c) require comment. By Corollary 3.4, if A is a presenting matrix of I and B is defined as in (1), we obtain that $I_j(A) = I_j(B)$ for all $j \leq n-2$, where A is an n by $n-1$ matrix. Moreover, Lemma 3.1 proves that B is a presenting matrix for $\text{adj}(I)$. Hence $\mathfrak{F}_j(\text{adj}(I)) = I_{(n-1)-j}(B) = I_{n-(j+1)}(A) = \mathfrak{F}_{j+1}(I)$, proving b). Part c) follows by repeated use of b); it follows easily from b) that $\mathfrak{F}_j(I) = \text{adj}(\text{adj}(...(I))...)$, where we take the adjoint $j-1$ times. (By Corollary 3.4, all of these Fitting ideals are integrally closed.) Taking adjoints once more on each side yields that $\text{adj}(\mathfrak{F}_j(I)) = \text{adj}(\text{adj}(...(I))...)$, now adj taken j times, which again by b) is equal to $\mathfrak{F}_{j+1}(I)$. ∎

4. The Arithmetic of Cores and Adjoints

Throughout this section R is a two-dimensional regular local ring with maximal ideal m and with an infinite residue field.

(4.1) Lemma: Let (x_1, x_2) be an m-primary ideal which is a reduction of the integrally closed ideal I. Then $(x_1, x_2)^n : I = (x_1, x_2)^{n-1}\text{adj}(I)$.

Proof: Note that $(x_1, x_2)^{n-1}\text{adj}(I)I = (x_1, x_2)^n\text{adj}(I)$ by Corollary 3.6, so the right hand side is always contained in the left hand side.

Proposition 3.5 gives the lemma for $n = 1$. Now let $n > 1$ and assume that the result is true for $n-1$. Then any $\alpha \in (x_1, x_2)^n : I$ is contained in $(x_1, x_2)^{n-2}\text{adj}(I)$. Write $\alpha = \sum r_i x_1^i x_2^{n-2-i}$ for some $r_i \in \text{adj}(I)$. By assumption, $\alpha x_2 = \sum s_i x_1^i x_2^{n-i}$ for some $s_i \in R$. Hence $\alpha x_2 = \sum r_i x_1^i x_2^{n-1-i} = \sum s_i x_1^i x_2^{n-i}$ says that $r_i \in (x_1, x_2)$ for all i. Thus we may write $\alpha = \sum t_i x_1^i x_2^{n-1-i}$ for some $t_i \in R$. Now let c be any element of I. then $\alpha c = \sum s_i x_1^i x_2^{n-i}$ for some $s_i \in R$. As x_1, x_2 form a regular sequence, $\alpha c = \sum t_i c x_1^i x_2^{n-1-i} = \sum s_i x_1^i x_2^{n-i}$ says that $t_i c \in (x_1, x_2)$. Since this holds for all $c \in I$, all the t_i lie in $\text{adj}(I)$ and hence α lies in $(x_1, x_2)^{n-1}\text{adj}(I)$. ∎

The next results are leading towards the determination of the core of the core and the core of higher powers of the ideals.

(4.2) Proposition: If I is integrally closed, then $\operatorname{adj}(I^n) = I^{n-1}\operatorname{adj}(I)$.

Proof: Let (x_1, x_2) be a reduction of I. Then (x_1^n, x_2^n) is a reduction of the integrally closed ideal I^n, so $\operatorname{adj}(I^n) = (x_1^n, x_2^n) : I^n$ by Proposition 3.5. By (2.4), $I^n = (x_1, x_2)^{n-1}I$, thus $\operatorname{adj}(I^n) = (x_1^n, x_2^n) : (x_1, x_2)^{n-1}I$, which equals $((x_1^n, x_2^n) : (x_1, x_2)^{n-1}) : I = (x_1, x_2)^n : I$. Thus by the previous lemma, $\operatorname{adj}(I^n) = (x_1, x_2)^{n-1}\operatorname{adj}(I)$, and we are done by Corollary 3.6. ■

(4.3) Remark: The proposition above gives a proof in dimension two of a conjecture of Lipman ([L4, 1.6]). The conjecture states that if R is a regular local ring of any dimension and I is an ideal of R, then $\operatorname{adj}(I^n) = I \operatorname{adj}(I^{n-1})$ if $n \geq \ell$, where ℓ is the analytic spread of I. Lipman proves this in the two-dimensional case [L4, 2.3]; however, his proof uses some machinery including duality and a vanishing theorem. Proposition 4.2 furnishes an elementary proof.

(4.4) Proposition: If I is integrally closed, then $\operatorname{core}(I^n) = I^{2n-1}\operatorname{adj}(I) = I^{2n-2}\operatorname{core}(I)$.

Proof: We know that $\operatorname{core}(I^n) = I^n\operatorname{adj}(I^n)$, and we finish by using the previous proposition. ■

Now we want to understand $\operatorname{core}(\operatorname{core}(I))$ and higher iterates too. We first need a lemma:

(4.5) Lemma: For an integrally closed ideal I, $\operatorname{adj}(\operatorname{core}(I)) = (\operatorname{adj}(I))^2$.

Proof: Let (x_1, x_2) be a reduction of I. Then $\operatorname{core}(I) = (x_1, x_2)\operatorname{adj}(I)$. If $n = \mu(I)$, let A be a $n \times (n-1)$ presenting matrix of I whose first two rows correspond to x_1 and x_2 respectively. Let the ijth entry be a_{ij}, let B be the submatrix composed of the last $n-2$ rows, and let R_i be the ith row. Let t_i be the signed $n-2$ minor of B after deleting the ith column. Thus $\operatorname{adj}(I) = (t_1, \ldots, t_{n-1})$ and $x_1 = \sum_i a_{2i}t_i$, $x_2 = -\sum_i a_{1i}t_i$. As $\mu(\operatorname{core}(I)) = \operatorname{ord}(\operatorname{core}(I)) + 1 = \operatorname{ord}(I) + \operatorname{ord}(\operatorname{adj}(I)) + 1 = 2n-2$, it is not hard to show that the t_ix_1 and the t_ix_2 minimally generate $\operatorname{core}(I)$. We want to find the relations on $t_1x_1, \ldots, t_{n-1}x_1, t_1x_2, \ldots, t_{n-1}x_2$. As B^{tr} is a presenting matrix of $t_1, \ldots, t_{n-1}$ (see the proof of 3.1), the columns of

$$C = \left[\begin{array}{c|c|c} B^{tr} & 0 & R_2^{tr} \\ \hline 0 & B^{tr} & R_1^{tr} \end{array}\right]$$

are some of the relations on the given generators of $\operatorname{core}(I)$. We now prove that they generate all the relations.

Let $\sum r_i t_i x_1 + \sum s_i t_i x_2 = 0$. Then there exists $\beta \in R$ such that $\sum r_i t_i = \beta x_2$ and $\sum s_i t_i = -\beta x_1$. As $x_2 = -\sum a_{1i} t_i$, the $(n-1)$-tuple $(r_1 + \beta a_{11}, \dots, r_{n-1} + \beta a_{1,n-1})$ is a relation on the t_i. Similarly, $(s_1 + \beta a_{21}, \dots, s_{n-1} + \beta a_{2,n-1})$ is a relation on the t_i. Thus our relation $(r_1, \dots, r_{n-1}, s_1, \dots, s_{n-1})$ lies in the span of the column space of C. This proves that C is a presenting matrix of core (I).

By Proposition 3.5, $\mathrm{adj}(\mathrm{core}\,(I)) = I_{2n-4}(C)$. Thus $\mathrm{adj}(\mathrm{core}\,(I)) = (\mathrm{adj}(I))^2$. ∎

(4.6) Remark: The proposition above follows quite easily from the work of Lipman [L4, 3.1.2] where he proves that if I is integrally closed, then $\mathrm{adj}(I)$ is the unique integrally closed ideal with point basis $\{\max\{0, ord_S(I^S) - 1\}\}_{R \subseteq S}$. (See (2.5) for remarks concerning the point basis.) It is worth noting that this characterization for the adjoint of I also follows easily from our work. One uses that $\mathrm{adj}(I)$ is the second Fitting ideal of I, and the order of this Fitting ideal is one less than the order of I by Lemma 3.2. These properties are preserved under quadratic transformations. Similarly, it also follows that the adjoint commutes with transforms, which is Corollary 3.1.3 in [L4]. (Here we use the factorization theorem of Zariski which says that any two dimensional regular local ring birationally dominating R can be reached by a finite number of quadratic transformations. See [Sa] for a generalization.)

Now we introduce some notation: $\mathrm{core}^{1}(I) = \mathrm{core}\,(I)$, and for $n > 1$, $\mathrm{core}^{n}(I) = \mathrm{core}^{n-1}(\mathrm{core}\,(I))$. With this we get:

(4.7) Proposition: For an integrally closed ideal I, $\mathrm{core}^{n}(I) = I(\mathrm{adj}(I))^{2^n - 1}$. In particular, $\mathrm{core}\,(\mathrm{core}\,(I)) = I\ (\mathrm{adj}(I))^3$.

Proof: If $n = 1$, this is just the main theorem. Now let $n > 1$ and assume that the proposition holds for $n-1$. Then

$$\begin{aligned} \mathrm{core}^{n}(I) &= \mathrm{core}^{n-1}(\mathrm{core}\,(I)) \\ &= \mathrm{core}\,(I)(\mathrm{adj}(\mathrm{core}\,(I)))^{2^{n-1}-1} \quad \text{by induction} \\ &= I\ \mathrm{adj}(I)((\mathrm{adj}(I))^2)^{2^{n-1}-1} \quad \text{by Lemma 4.5} \\ &= I\ (\mathrm{adj}(I))^{2^n - 1}. \quad \blacksquare \end{aligned}$$

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