Associated primes of local cohomology modules and Frobenius powers

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Abstract

We construct normal hypersurfaces whose local cohomology modules have infinitely many associated primes. These include hypersurfaces of characteristic zero with rational singularities, as well as F-regular hypersurfaces of positive characteristic. As a consequence, we answer a question on the associated primes of certain families of ideals which arose from the localization problem in tight closure theory.

1. Introduction

Let R be a commutative Noetherian ring and $\mathfrak{a} \subset R$ an ideal. In [Hu1] Huneke asked whether the number of associated prime ideals of a local cohomology module $H^n_{\mathfrak{a}}(R)$ is always finite. In [Si] the first author constructed an example of a hypersurface

$$R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)$$

for which the local cohomology module $H^3_{(x,y,z)}(R)$ has a p-torsion element for every prime integer p, and consequently has infinitely many associated prime ideals. However this example does not address Huneke's question for rings containing a field, nor does it yield an example over a local ring. Recently Katzman constructed the following example in [Ka2]: Let K be an arbitrary field, and consider the hypersurface

$$S = K[s, t, u, v, x, y]/\big(su^2x^2 - (s + t)uxvy + tv^2y^2\big).$$

Katzman showed that the local cohomology module $H^2_{(x,y)}(S)$ has infinitely many associated prime ideals. Since the defining equation of this hypersurface factors, the ring in Katzman's example is not an integral domain. In this paper we generalize Katzman's construction and obtain families of examples which include examples over normal domains, and even over hypersurfaces with rational singularities:

Theorem 1.1. *Let K be an arbitrary field. Then there exists a standard graded normal hypersurface R with $[R]_0 = K$, and an ideal $\mathfrak{a} \subset R$, such that a local cohomology module $H^n_{\mathfrak{a}}(R)$ has infinitely many associated prime ideals.*

If K has characteristic zero, there exist such examples where, furthermore, R has rational singularities. If K has positive characteristic, we may choose R to be F-regular. In each case, if $\mathfrak{m}$ denotes the homogeneous maximal ideal of R, then $H^n_{\mathfrak{a}}\big(R_{\mathfrak{m}}\big)$ also has infinitely many associated prime ideals.

There are positive answers to Huneke's question if the ring R is regular but, as our theorem indicates, the hypothesis of regularity cannot be weakened substantially. The first results were obtained by Huneke and Sharp who proved that if R is a regular ring containing a field of prime

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characteristic, then the set of associated prime ideals of $H^n_{\mathfrak{a}}(R)$ is finite, [HS]. Lyubeznik established that $H^n_{\mathfrak{a}}(R)$ has finitely many associated prime ideals if R is a regular local ring containing a field of characteristic zero, or an unramified regular local rings of mixed characteristic, see [Ly1, Ly2] and [Ly3]. Marley showed that if R is a local ring with $\dim R \leqslant 3$, then for any finitely generated R-module M, a local cohomology module $H^n_{\mathfrak{a}}(M)$ has finitely many associated primes, [Ma]. For some of the other work on this question, we refer the reader to the papers [BF, BKS, BRS, He, KS, MV] and [TZ].

In §2 we establish a relationship between the associated primes of Frobenius powers of an ideal, and the associated primes of a local cohomology module over an auxiliary ring. Recall that for an ideal $\mathfrak{a}$ in a ring R of prime characteristic $p > 0$, the *Frobenius powers* of $\mathfrak{a}$ are the ideals $\mathfrak{a}^{[p^e]} = \big(x^{p^e} | x \in \mathfrak{a}\big)$ where $e \in \mathbb{N}$. The finiteness of the associated primes of the ideals $\mathfrak{a}^{[p^e]}$ is related to the localization problem in tight closure theory, discussed in §5 of this paper. In [Ka1] Katzman constructed the first example where the set $\bigcup_e \operatorname{Ass} R/\mathfrak{a}^{[p^e]}$ is infinite. The question however remained whether the set $\bigcup_e \operatorname{Ass} R/\big(\mathfrak{a}^{[p^e]}\big)^*$ is finite, or even if it has finitely many maximal elements—this has strong implications for the localization problem, see [AHH, Ho, Ka1, SN] or [Hu2, §12]. We settle this question in §5:

Theorem 1.2. *There exists an F-regular ring R of characteristic $p > 0$, with an ideal $\mathfrak{a}$, for which the set*

$$\bigcup_{e\in\mathbb{N}} \operatorname{Ass} \frac{R}{\mathfrak{a}^{[p^e]}} = \bigcup_{e\in\mathbb{N}} \operatorname{Ass} \frac{R}{\big(\mathfrak{a}^{[p^e]}\big)^*}$$

has infinitely many maximal elements.

2. General constructions

Let $\mathfrak{a} = (x_1, \dots, x_n)$ be an ideal of a ring R. For an integer $r \geqslant 0$, the local cohomology module $H^r_{\mathfrak{a}}(R)$ may be computed as the rth cohomology module of the extended Čech complex

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^{n} R_{x_i} \longrightarrow \bigoplus_{i<j} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1\cdots x_n} \longrightarrow 0.$$

For positive integers m_i and an element $f \in R$, we will use $[f + (x_1^{m_1}, \dots, x_n^{m_n})]$ to denote the cohomology class

$$\left[\frac{f}{x_1^{m_1}\cdots x_n^{m_n}}\right] \;\in\; H^n_{\mathfrak{a}}(R) \;=\; \frac{R_{x_1\cdots x_n}}{\sum R_{x_1\cdots\hat{x_i}\cdots x_n}}.$$

It is easily seen that $[f + (x_1^{m_1}, \dots, x_n^{m_n})] \in H^n_{\mathfrak{a}}(R)$ is zero if and only if there exist integers $k_i \geqslant 0$ such that

$$f x_1^{k_1}\cdots x_n^{k_n} \;\in\; \big(x_1^{m_1+k_1}, \dots, x_n^{m_n+k_n}\big)R.$$

Consequently $H^n_{\mathfrak{a}}(R)$ may also be computed as the direct limit

$$H^n_{\mathfrak{a}}(R) \cong \varinjlim_{m\in\mathbb{N}} R/(x_1^m, \dots, x_n^m)R,$$

where the maps in the direct limit system are induced by multiplication by the element $x_1\cdots x_n$. We may regard an element $[f + (x_1^m, \dots, x_n^m)] \in H^n_{\mathfrak{a}}(R)$ as the class of $f + (x_1^m, \dots, x_n^m)R$ in this direct limit.

We next record two results which illustrate the relationship between associated primes of local cohomology modules and associated primes of generalized Frobenius powers of ideals.

Proposition 2.1. *Let $\mathfrak{a} = (x_1, \dots, x_n)$ be an ideal of a ring R. Then, for any infinite set $\mathbb{S}$ of* *positive integers,*

$$\operatorname{Ass} H^n_{\mathfrak{a}}(R) \subseteq \bigcup_{m\in\mathbb{S}} \operatorname{Ass} \frac{R}{(x_1^m,\dots,x_n^m)}.$$

Proof. Let $\mathfrak{p} = \operatorname{ann}\eta$, where $\eta = [f + (x_1^m,\dots,x_n^m)]$ is an element of $H^n_{\mathfrak{a}}(R)$. Then for every element $z \in \mathfrak{p}$, there exists an integer $m + k \in \mathbb{S}$ such that

$$zf(x_1\cdots x_n)^k \in (x_1^{m+k},\dots,x_n^{m+k})R.$$

Since $\mathfrak{p}$ is finitely generated, we may choose $m + k \in \mathbb{S}$ such that

$$\mathfrak{p} \subseteq \big(x_1^{m+k},\dots,x_n^{m+k}\big)R :_R f(x_1\cdots x_n)^k. \qquad (*)$$

On the other hand, if $rf(x_1\cdots x_n)^k \in (x_1^{m+k},\dots,x_n^{m+k})R$, then $r\eta = 0$, and so actually have an equality in $(*)$. Consequently $\mathfrak{p}$ is an associated prime of $R/(x_1^{m+k},\dots,x_n^{m+k})$. □

It immediately follows that whenever $H^n_{\mathfrak{a}}(R)$ has infinitely many associated prime ideals, the set $\bigcup_m \operatorname{Ass} R/(x_1^m,\dots,x_n^m)$ is infinite as well. The converse is false, as we shall see in Remark 4.7.

Proposition 2.2. *Let A be an $\mathbb{N}$-graded ring which is generated, as an A_0-algebra, by elements $t_1,\dots,t_n$ of degree 1 which are nonzerodivisors in A. Let R be the extension ring*

$$R = A[u_1,\dots,u_n,x_1,\dots,x_n]/(u_1x_1 - t_1,\dots,u_nx_n - t_n).$$

Let $m_1,\dots,m_n$ be positive integers, and $f \in A$ a homogeneous element. Then, for arbitrary integers $k_i \geqslant 0$,

$$\big(t_1^{m_1},\dots,t_n^{m_n}\big)A :_{A_0} f \;=\; \big(x_1^{m_1+k_1},\dots,x_n^{m_n+k_n}\big)R :_{A_0} fx_1^{k_1}\cdots x_n^{k_n}.$$

Consequently if we consider the element $\eta = [f + (x_1^{m_1},\dots,x_n^{m_n})]$ of the local cohomology module $H^n_{(x_1,\dots,x_n)}(R)$, then

$$\operatorname{ann}_{A_0}\eta \;=\; \big(t_1^{m_1},\dots,t_n^{m_n}\big)A :_{A_0} f.$$

Proof. The inclusion $\subseteq$ is easily verified. For the other inclusion, let $e_i \in \mathbb{Z}^{n+1}$ be the unit vector with 1 as its ith entry, and consider the $\mathbb{Z}^{n+1}$-grading on R where $\deg x_i = e_i$ and $\deg u_i = e_{n+1} - e_i$ for all $1 \leqslant i \leqslant n$. If $f \in A_r$ then, as an element of R, the degree of f is re_{n+1}. The subring A is a direct summand of R since

$$A_j = R_{(0,\dots,0,j)} \quad \text{for} \quad j \geqslant 0, \quad \text{and} \quad A = \oplus_{j\geqslant 0} R_{(0,\dots,0,j)}.$$

Now if $h \in A_0$ is an element such that $hfx_1^{k_1}\cdots x_n^{k_n} \in (x_1^{m_1+k_1},\dots,x_n^{m_n+k_n})R$, then there exist homogeneous elements $c_1,\dots,c_n \in R$ such that

$$hfx_1^{k_1}\cdots x_n^{k_n} \;=\; c_1x_1^{m_1+k_1} + \cdots + c_nx_n^{m_n+k_n}.$$

We must have $\deg c_1 = (-m_1,k_2,\dots,k_n,r)$, and so c_1 is an A_0-linear combination of monomials μ of the form

$$\mu = u_1^{l_1+m_1}\; u_2^{l_2}\;\cdots\; u_n^{l_n}\; x_1^{l_1}\; x_2^{l_2+k_2}\;\cdots\; x_n^{l_n+k_n},$$

where $l_i \geqslant 0$, and $m_1 + l_1 + \cdots + l_n = r$. Consequently

$$\begin{aligned} \mu x_1^{m_1+k_1} &= (u_1x_1)^{l_1+m_1}\;(u_2x_2)^{l_2}\;\cdots\;(u_nx_n)^{l_n}\;x_1^{k_1}\;\cdots\;x_n^{k_n} \\ &= t_1^{l_1+m_1}\;t_2^{l_2}\;\cdots\;t_n^{l_n}\;x_1^{k_1}\;\cdots\;x_n^{k_n}, \end{aligned}$$

and so $c_1x_1^{m_1+k} \in \big(x_1^{k_1}\cdots x_n^{k_n}t_1^{m_1}\big)R$. Similar computations for $c_2,\dots,c_n$ show that

$$hfx_1^{k_1}\cdots x_n^{k_n} \;\in\; x_1^{k_1}\cdots x_n^{k_n}\big(t_1^{m_1},\dots,t_n^{m_n}\big)R.$$

Multiplying by $u_1^{k_1}\cdots u_n^{k_n}$ and using that A is a direct summand of R, we get

$$\begin{aligned} hft_1^{k_1}\cdots t_n^{k_n} &\in t_1^{k_1}\cdots t_n^{k_n}\big(t_1^{m_1},\dots,t_n^{m_n}\big)R\cap A \\ &= t_1^{k_1}\cdots t_n^{k_n}\big(t_1^{m_1},\dots,t_n^{m_n}\big)A. \end{aligned}$$

Since the elements $t_i \in A$ are nonzerodivisors, the required result follows. □

We next record two results which will be used in the proof of Theorem 2.6.

Lemma 2.3. *Let M be a square matrix with entries in a ring R, such that the determinant of M is a nonzerodivisor. Then the minimal primes of the ideal $(\det M)R$ are precisely the minimal primes of the cokernel of the matrix M.*

Proof. Since $\det M$ is a nonzerodivisor, we have an exact sequence

$$0 \longrightarrow R^n \xrightarrow{\ M\ } R^n \longrightarrow C \longrightarrow 0,$$

where C is the cokernel of M. For a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, note that $C_{\mathfrak{p}} = 0$ if and only if $\det M$ is a unit in $R_{\mathfrak{p}}$. Consequently

$$V(\det M) = \operatorname{Supp} C.$$

□

Lemma 2.4. *Let R be an $\mathbb{N}$-graded ring, and M be a $\mathbb{Z}$-graded R-module. For every integer r and prime ideal $\mathfrak{p} \in \operatorname{Ass}_{R_0} M_r$, there exists a homogeneous prime ideal $\mathfrak{P} \in \operatorname{Ass}_R M$ such that $\mathfrak{P} \cap R_0 = \mathfrak{p}$. Consequently if the set $\operatorname{Ass}_{R_0} M$ is infinite, then so is the set $\operatorname{Ass}_R M$.*

Proof. Let $\mathfrak{p} = \operatorname{ann}_{R_0} m$ for some element $m \in M_r$. There is no loss of generality in replacing M by the cyclic module $R/\mathfrak{a} \cong mR$, in which case $\mathfrak{p} = \mathfrak{a} \cap R_0$. The isomorphism

$$R/(\mathfrak{a} + R_+) \cong R_0/\mathfrak{p}$$

shows that $\mathfrak{a} + R_+$ is a prime ideal of R. Let $\mathfrak{P}$ be a minimal prime of $\mathfrak{a}$ which is contained in $\mathfrak{a} + R_+$. Then $\mathfrak{P} \in \operatorname{Min}_R R/\mathfrak{a} \subseteq \operatorname{Ass}_R R/\mathfrak{a}$, and $\mathfrak{P} \cap R_0 = \mathfrak{p}$ since $(\mathfrak{a} + R_+) \cap R_0 = \mathfrak{p}$. □

Definition 2.5. Let d be a positive even integer, and $r_0, \dots, r_d$ be elements of a ring A_0. The nth *multidiagonal matrix with respect to* $r_0, \dots, r_d$ will refer to the $n \times n$ matrix

$$M_n = \begin{bmatrix} r_{\frac{d}{2}} & \cdots & r_0 & & & \\ \vdots & \ddots & & \ddots & & \\ & & \ddots & & \ddots & \\ r_d & & & & & \\ & \ddots & & \ddots & & r_0 \\ & & \ddots & & \ddots & \vdots \\ & & & r_d & \cdots & r_{\frac{d}{2}} \end{bmatrix},$$

where the elements $r_0, \dots, r_d$ occur along the $d+1$ central diagonals, and all the other entries are zero. (These multi-diagonal matrices are special cases of Toeplitz matrices.)

Theorem 2.6. *Let d be an even positive integer, $r_0, \dots, r_d$ elements of a domain A_0, $a \geqslant 0$ an integer, and M_n the nth multidiagonal matrix with respect to $r_0, \dots, r_d$. Let u, v, x, y be variables over A_0, and $\mathbb{S} \subseteq \mathbb{N}$ a subset such that*

$$\bigcup_{n \in \mathbb{S}} \operatorname{Min}\big(\det M_{n-a-d/2}\big)$$

is an infinite set. If

$$A = A_0[x, y]/(xy)^a \left(r_0 x^d + r_1 x^{d-1} y + \cdots + r_d y^d\right),$$

then $\bigcup_{n \in \mathbb{S}} \operatorname{Ass} A/(x^n, y^n)$*, is an infinite set.*

Furthermore, if r_0 *and* r_d *are nonzero elements of* A_0*, then for*

$$R = A_0[u, v, x, y]/\left(r_0(ux)^d + r_1(ux)^{d-1}(vy) + \cdots + r_d(vy)^d\right),$$

the local cohomology module $H^2_{(x,y)}(R)$ *has infinitely many associated primes.*

If $(A_0, \mathfrak{m})$ *is a local domain or if* $(A_0, \mathfrak{m})$ *is a graded domain and* $\det M_n$ *is a homogeneous element for all* $n \geqslant 0$*, then these issues are preserved under localizations at the maximal ideals* $(\mathfrak{m} + (x, y))A$ *and* $(\mathfrak{m} + (u, v, x, y))R$*, respectively.*

Proof. Consider the A_0-module $[A/(x^n, y^n)]_{n-1+a+d/2}$ for $n > a+d$. A generating set for this module is given by the $n - a - d/2$ monomials

$$x^{a+d/2} y^{n-1},\ x^{a+d/2+1} y^{n-2},\ \ldots,\ x^{n-1} y^{a+d/2}.$$

There are $n - a - d/2$ relations amongst these monomials, arising from the equations

$$(xy)^a \big(r_0 x^d + r_1 x^{d-1} y + \cdots + r_d y^d\big) x^i y^{n-1-a-d/2-i}$$

where $0 \leqslant i \leqslant n - 1 - a - d/2$. Using this, it is easily checked that the presentation matrix for $[A/(x^n, y^n)]_{n-1+a+d/2}$ is precisely the multidiagonal matrix $M_{n-a-d/2}$. By Lemma 2.3, whenever $\det M_{n-a-d/2}$ is nonzero, its minimal primes are the minimal primes of $[A/(x^n, y^n)]_{n-1+a+d/2}$, and so

$$\bigcup_{n \in \mathbb{S}} \operatorname{Ass}_{A_0} \left[\frac{A}{(x^n, y^n)}\right]_{n-1+a+d/2}$$

is an infinite set. Using Lemma 2.4, the set $\bigcup_{n \in \mathbb{S}} \operatorname{Ass} A/(x^n, y^n)$ is infinite as well.

Note that xy is a nonzerodivisor in $A_0[x, y]/(r_0 x^d + r_1 x^{d-1} y + \cdots + r_d y^d)$ whenever r_0 and r_d are nonzero elements of A_0. The assertion regarding local cohomology now follows using Proposition 2.2. □

Remark 2.7. We demonstrate how Katzman's examples from [Ka1] and [Ka2] follow from Theorem 2.6. Let K be an arbitrary field, and consider the polynomial ring $A_0 = K[t]$. Let M_n be the nth multidiagonal matrix with respect to the elements $r_0 = 1$, $r_1 = -(1 + t)$, and $r_2 = t$. An inductive argument shows that

$$\det M_n = (-1)^n (1 + t + t^2 + \cdots + t^n) = (-1)^n \frac{t^{n+1} - 1}{t - 1} \quad \text{for all} \quad n \geqslant 1.$$

It is easily verified that $\bigcup_{n \in \mathbb{N}} \operatorname{Min}(\det M_n)$ is an infinite set and, if K has characteristic $p > 0$, that the set $\bigcup_{e \in \mathbb{N}} \operatorname{Min}(\det M_{p^e-2})$ is also infinite. Theorem 2.6 now gives us the main results of [Ka2]: the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated primes where

$$R = K[t, u, v, x, y]/\big(u^2 x^2 - (1 + t)uxvy + tv^2 y^2\big).$$

Similarly, standard graded or homogeneous examples may be obtained using

$$S = K[s, t, u, v, x, y]/\big(su^2 x^2 - (s + t)uxvy + tv^2 y^2\big),$$

in which case $H^2_{(x,y)}(S)$ and $H^2_{(x,y)}(S_{\mathfrak{m}})$ have infinitely many associated primes.

If K has characteristic $p > 0$, consider the hypersurface

$$A = K[t, x, y]/\big(xy(x^2 - (1 + t)xy + ty^2)\big),$$

where $a = 1$ in the notation of Theorem 2.6. The theorem now implies that the Frobenius powers of the ideal $(x, y)A$ have infinitely many associated primes, as first proved by Katzman in [Ka1].

3. Tridiagonal matrices

The results of the previous section show how multidiagonal matrices give rise to associated primes of local cohomology modules and of Frobenius powers of ideals. From this point of view, the first multidiagonal matrices of interest are those which are tridiagonal, i.e., those for which $d = 2$ in the notation of Definition 2.5. Consider the $n \times n$ multidiagonal matrices

$$M_n = \begin{bmatrix} r_1 & r_0 & & & \\ r_2 & r_1 & r_0 & & \\ & \ddots & \ddots & \ddots & \\ & & r_2 & r_1 & r_0 \\ & & & r_2 & r_1 \end{bmatrix}.$$

It is convenient to define $\det M_0 = 1$, and it is easily seen that

$$\det M_{n+2} = r_1 \det M_{n+1} - r_0 r_2 \det M_n \quad \text{for all} \quad n \geqslant 0.$$

While we will not be using it here, we mention that

$$\det M_n = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} r_1^{n-2i} (r_0 r_2)^i.$$

Consider the generating function for $\det M_n$,

$$G(x) = \sum_{n \geqslant 0} (\det M_n)\, x^n.$$

By the recursion formula,

$$\sum_{n \geqslant 0} (\det M_{n+2})\, x^{n+2} = r_1 \sum_{n \geqslant 0} (\det M_{n+1})\, x^{n+2} - r_0 r_2 \sum_{n \geqslant 0} (\det M_n)\, x^{n+2}$$

and substituting $G(x)$ and solving, we get

$$G(x) = \sum_{n \geqslant 0} (\det M_n)\, x^n = \frac{1}{1 - r_1 x + r_0 r_2 x^2}.$$

One of the goals of this paper is to construct an integral domain A of characteristic $p > 0$, with an ideal $\mathfrak{a}$, such that $\bigcup_e \operatorname{Ass} A/\mathfrak{a}^{[p^e]}$ is infinite. To obtain such examples directly using Theorem 2.6 we need the set $\bigcup_e \operatorname{Min}\left(\det M_{p^e - d/2}\right)$ to be infinite, since the domain hypothesis forces us to use $a = 0$ in the notation of the theorem. The following result shows that this is not possible if $d = 2$, see also [Ka1, Lemma 10]. In §6 we show that $\bigcup_e \operatorname{Min}\left(\det M_{p^e - d/2}\right)$ may be infinite when $d = 4$.

Proposition 3.1. *Let r_0, r_1, r_2 be elements of a ring R of prime characteristic $p > 0$. For each $n \in \mathbb{N}$, let M_n be the nth multidiagonal matrix with respect to r_0, r_1, r_2. Then, for any integer $e \geqslant 1$,*

$$\det M_{p^e - 1} = (\det M_{p-1})^{1 + p + \cdots + p^{e-1}}.$$

Consequently the set $\bigcup_e \operatorname{Min}(\det M_{p^e - 1})$ is finite.

Proof. Let $1 - r_1 x + r_0 r_2 x^2 = (1 - \alpha x)(1 - \beta x)$ for some elements α and β in a suitable extension of R. The generating function $G(x)$ can be written as

$$G(x) = \sum_{n\geqslant 0} (\det M_n)\, x^n = \frac{1}{(1-\alpha x)(1-\beta x)} = \sum_{i,j\geqslant 0} \alpha^i \beta^j x^{i+j},$$

and consequently

$$\det M_{p-1} = \sum_{i=0}^{p-1} \alpha^i \beta^{p-1-i} \qquad \text{and} \qquad \det M_{p^e-1} = \sum_{i=0}^{p^e-1} \alpha^i \beta^{p^e-1-i}.$$

Using this,

$$\begin{aligned}(\det M_{p-1})^{1+p+\cdots+p^{e-1}} &= \prod_{j=0}^{e-1}\left(\sum_{i=0}^{p-1}\alpha^i\beta^{p-1-i}\right)^{p^j} = \prod_{j=0}^{e-1}\left(\sum_{i=0}^{p-1}\alpha^{ip^j}\beta^{(p-1-i)p^j}\right)\\ &= \sum_{k=0}^{p^e-1}\alpha^k\beta^{p^e-1-k} = \det M_{p^e-1}.\end{aligned}$$

□

We next consider a special family of tridiagonal matrices: Let $K[s,t]$ be a polynomial ring over a field K, and consider the $n \times n$ multidiagonal matrices

$$M_n = \begin{bmatrix} t & s & & & \\ s & t & s & & \\ & \ddots & \ddots & \ddots & \\ & & s & t & s \\ & & & s & t \end{bmatrix}.$$

In the notation of Definition 2.5, we have $d = 2$, $r_1 = t$, and $r_0 = r_2 = s$. Setting $Q_n(s,t) = \det M_n$, we have

$$Q_0 = 1, \quad Q_1 = t, \quad \text{and} \quad Q_{n+2} = tQ_{n+1} - s^2 Q_n \quad \text{for all} \quad n \geqslant 0.$$

Note that the polynomials $Q_n(s,t)$ are relatively prime to s. Using the specialization $P_n(t) = Q_n(1,t)$, we get polynomials $P_n(t) \in K[t]$ satisfying the recursion

$$P_0(t) = 1, \quad P_1(t) = t, \quad \text{and} \quad P_{n+2}(t) = tP_{n+1}(t) - P_n(t) \quad \text{for all} \quad n \geqslant 0.$$

Each $P_n(t)$ is a monic polynomial of degree n, and in Lemma 3.3 we establish that the number of distinct irreducible factors of the polynomials $\{P_n(t)\}_{n\in\mathbb{N}}$ is infinite. As $Q_n(s,t) = s^n P_n(t/s)$ for all $n \geqslant 0$, this also establishes that the number of distinct irreducible factors of the polynomials $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ is infinite.

Lemma 3.2. *Let K be an algebraically closed field and consider the polynomials $P_n(t) = \det M_n \in K[t]$ for $n \geqslant 1$ as above.*

i) *If ξ is a nonzero element of K with $\xi \neq \pm 1$, then $P_n(\xi + \xi^{-1}) = 0$ if and only if $\xi^{2n+2} = 1$.*
ii) *The number of distinct roots of P_n which are different from 0 and ± 1 is half of the number of distinct $(2n+2)$th roots of unity different from ± 1.*
iii) *If $2n+2$ is invertible in K, then $P_n(t)$ has n distinct roots of the form $\xi + \xi^{-1}$ where $\xi^{2n+2} = 1$ and $\xi \neq \pm 1$.*
iv) *The elements 2 or -2 are roots of $P_n(t)$ if and only if the characteristic of K is a positive prime p which divides $n+1$.*

v) *If the characteristic of K is an odd prime p, then $P_{q-2}(t)$ has $q-2$ distinct roots for all $q = p^e$. If $p = 2$, then $P_{q-2}(t)$ has $q/2-1$ distinct roots.*

Proof. (1) Consider the generating function of the polynomials $P_n(t)$,

$$G(t,x) = \sum_{n\geqslant 0} P_n(t)x^n = \frac{1}{1-xt+x^2} \in K[t][[x]].$$

If $\xi \neq 0$ and $\xi \neq \pm 1$, then

$$\begin{aligned}\sum_{n\geqslant 0} P_n(\xi+\xi^{-1})x^n &= \frac{1}{1-x(\xi+\xi^{-1})+x^2} = \frac{1}{(\xi^{-1}-x)(\xi-x)}\\ &= \frac{1}{(\xi-\xi^{-1})(\xi^{-1}-x)} - \frac{1}{(\xi-\xi^{-1})(\xi-x)}\\ &= \frac{\xi}{\xi-\xi^{-1}}\sum_{n\geqslant 0}(\xi x)^n - \frac{\xi^{-1}}{\xi-\xi^{-1}}\sum_{n\geqslant 0}(\xi^{-1}x)^n \in K[[x]].\end{aligned}$$

Equating the coefficients of x^n, we have

$$P_n(\xi+\xi^{-1}) = \frac{\xi^{n+1}-\xi^{-(n+1)}}{\xi-\xi^{-1}} = \frac{\xi^{2n+2}-1}{\xi^n(\xi^2-1)},$$

and the assertion follows.

(2) We observe that

$$\xi+\frac{1}{\xi}-\left(\eta+\frac{1}{\eta}\right) = \xi-\eta-\frac{\xi-\eta}{\xi\eta} = (\xi-\eta)\left(1-\frac{1}{\xi\eta}\right),$$

and so $\xi+\xi^{-1} = \eta+\eta^{-1}$ if and only if ξ equals η or η^{-1}.

(3) Since $2n+2$ is invertible in K, the polynomial $X^{2n+2}-1=0$ has $2n$ distinct roots ξ with $\xi \neq \pm 1$. These give the n distinct roots $\xi+\xi^{-1}$ of the degree n polynomial $P_n(t)$.

(4) Using the generating function above,

$$G(2,x) = \frac{1}{1-2x+x^2} = (1-x)^{-2} = 1+2x+3x^2+\cdots,$$

and so $P_n(\pm 2) = 0$ if and only if $n+1=0$ in K.

(5) The case when p is odd follows immediately from (2). If $p=2$, the equation $X^{2q-2}-1 = (X^{q-1}-1)^2 = 0$ has $q-2$ distinct roots ξ with $\xi \neq 1$, which ensures that $P_{q-2}(t)$ has at least $q/2-1$ distinct roots. It follows from (4) that 0 is not a root of $P_{q-2}(t)$, so these must be all the roots. □

Lemma 3.3. *Let K be an arbitrary field. Then the number of distinct irreducible factors of the polynomials $\{P_n(t)\}_{n\in\mathbb{N}}$ is infinite. If K has characteristic $p>0$ and $q=p^e$ varies over the powers of p, then the polynomials $\{P_{q-2}(t)\}_{q=p^e}$ have infinitely many distinct irreducible factors.*

Consequently the number of distinct irreducible factors of the polynomials $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ as well as $\{Q_{q-2}(s,t)\}_{q=p^e}$ is also infinite.

Proof. It follows from Lemma 3.2 that $\{P_n(t)\}_n$ as well as $\{P_{q-2}(t)\}_{q=p^e}$ have infinitely many distinct irreducible factors in $K[t]$. □

4. Examples over integral domains

We can now construct a domain which has a local cohomology module with infinitely many associated primes:

Theorem 4.1. *Let K be an arbitrary field, and consider the integral domain*

$$R = K[s,t,u,v,x,y]/\big(su^2x^2 + tuxvy + sv^2y^2\big).$$

Then the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated prime ideals. Also, if we consider the local domain $R_{\mathfrak{m}}$ where $\mathfrak{m} = (s,t,u,v,x,y)R$, then $H^2_{(x,y)}(R_{\mathfrak{m}})$ has infinitely many associated primes.

If $\mathbb{S}$ is any infinite set of positive integers, then the set $\bigcup_{m\in\mathbb{S}} \operatorname{Ass} R/(x^m, y^m)$ is infinite; in particular, if K has characteristic $p > 0$, then $\bigcup_{e\in\mathbb{N}} \operatorname{Ass} R/(x^{p^e}, y^{p^e})$ is infinite. The same conclusions hold if we replace the hypersurface R by its specialization $R/(s-1)$ or by the localization $R_{\mathfrak{m}}$.

Proof. This assertions regarding local cohomology follow from Theorem 2.6 and Lemma 3.3. These, along with Proposition 2.1, then imply the results for generalized Frobenius powers of ideals; see also the remark below. □

Remark 4.2. Specializing $s = 1$ and working with the hypersurface

$$S = R/(s-1) = K[t,u,v,x,y]/\big(u^2x^2 + tuxvy + v^2y^2\big),$$

similar arguments show that $H^2_{(x,y)}(S)$ has infinitely many associated primes. This gives an example of a four dimensional integral domain S for which $H^2_{(x,y)}(S)$ has infinitely many associated prime ideals. However it remains an open question whether a local cohomology module $H^i_{\mathfrak{a}}(T)$ has infinitely many associated primes where T is a *local* ring of dimension four. This is of interest in view of Marley's results that the local cohomology of a Noetherian local ring of dimension less than four has finitely many associated primes, [Ma].

For the assertion regarding the associated primes of generalized Frobenius powers of an ideal, the hypersurface R of Theorem 4.1 can be modified to obtain a three-dimensional local domain, or a two-dimensional non-local domain:

Theorem 4.3. *Let K be an arbitrary field, and consider the integral domain*

$$A = K[s,t,x,y]/\big(sx^2 + txy + sy^2\big).$$

Then the set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n, y^n)$ is infinite. The same conclusion holds if we replace A by the specialization $A/(s-1)$ or by the localization $A_{(s,t,x,y)}$.

The proof of the theorem is again an immediate consequence of Theorem 2.6 and Lemma 3.3, but we feel it is of interest to explicitly determine the infinite set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n, y^n)$ at least in this one example, and we record the result as Theorem 4.6. If K has characteristic $p > 0$, this theorem also shows that the set $\bigcup_{e\in\mathbb{N}} \operatorname{Ass} A/(x^{p^e}, y^{p^e})$ is finite. We next record some preliminary computations which will be needed in determining the associated primes of the ideals $(x^n, y^n)A$, and will also be used later in §5.

Lemma 4.4. *Consider the polynomial ring $K[s,t,x,y]$ and $m, n \geqslant 1$. Then*

- i) $xy^{n-1}Q_{n-1} \in (x^n, y^n, sx^2 + txy + sy^2)$,
- ii) $(x^n, y^n, sx^2 + txy + sy^2) : (s^m xy^{n-1}) = (x, y, Q_{n-1})$, *and*
- iii) $(x^n, y^n, x^2 + txy + y^2) : (xy^{n-1}) = (x, y, P_{n-1})$.

Proof. (1) The case $n = 1$ holds trivially. Using the equation $tQ_i = Q_{i+1} + s^2Q_{i-1}$ for $1 \leqslant i \leqslant n-2$, we get

$$(sx^2 + txy + sy^2)(sx)^{n-2-i}y^iQ_i = s^{n-1-i}x^{n-i}y^iQ_i + s^{n-1-i}x^{n-2-i}y^{i+2}Q_i + s^{n-2-i}x^{n-1-i}y^{i+1}Q_{i+1} + s^{n-i}x^{n-1-i}y^{i+1}Q_{i-1},$$

and taking an alternating sum gives us

$$\sum_{i=0}^{n-2}(-1)^i(sx^2+txy+sy^2)(sx)^{n-2-i}y^iQ_i$$
$$= s^{n-1}x^nQ_0 \;+\; (-1)^{n-2}xy^{n-1}Q_{n-1} \;+\; (-1)^{n-2}sy^nQ_{n-2}.$$

This shows that $xy^{n-1}Q_{n-1} \in (x^n, y^n, sx^2+txy+sy^2)$.

(2) If $n = 1$ we have the unit ideal on each side of the asserted equality, so we may assume $n \geqslant 2$ for the rest of this proof. It is easy to verify that

$$sxy^{n-1}(x,y) \;\subseteq\; (x^n, y^n, sx^2+txy+sy^2).$$

Let $h \in K[s,t]$ be an element such that

$$hs^mxy^{n-1} \;\in\; (x^n, y^n, sx^2+txy+sy^2).$$

Using the grading where $\deg s = \deg t = 0$ and $\deg x = \deg y = 1$, there exist elements α, β, and $d_0, \dots, d_{n-2}$ in $K[s,t]$ with

$$hs^mxy^{n-1} = (d_0x^{n-2} - d_1x^{n-3}y + \cdots + (-1)^{n-2}d_{n-2}y^{n-2})(sx^2+txy+sy^2)$$
$$+ \alpha x^n + \beta y^n.$$

Comparing coefficients of $x^{n-1}y, x^{n-2}y^2, \dots, xy^{n-1}$, we get

$$sd_1 - td_0 = 0,$$
$$sd_{i+2} - td_{i+1} + sd_i = 0 \quad \text{for all} \quad 0 \leqslant i \leqslant n-4,$$
$$(-1)^{n-2}(td_{n-2} - sd_{n-3}) = hs^m.$$

In particular,

$$d_1 = (t/s)d_0 \quad \text{and} \quad d_{i+2} = (t/s)d_{i+1} - d_i \quad \text{for all} \quad 0 \leqslant i \leqslant n-4,$$

and consequently $d_i = d_0P_i(t/s)$ for $0 \leqslant i \leqslant n-2$, where the P_i are the polynomials defined recursively in §3. This gives us

$$hs^m = (-1)^{n-2}\big(td_0P_{n-2}(t/s) - sd_0P_{n-3}(t/s)\big) = (-1)^{n-2}sd_0P_{n-1}(t/s),$$

and so $hs^{m+n-2} = (-1)^{n-2}d_0Q_{n-1}$. Since s and Q_{n-1} are relatively prime in $K[s,t]$, we see that h is a multiple of Q_{n-1}.

(3) is the inhomogeneous case of (2), and is left to the reader. □

Lemma 4.5. *Let $A = K[s,t,x,y]/(sx^2+txy+sy^2)$, and $n \geqslant 1$ be an arbitrary integer.*

i) *For all $1 \leqslant i \leqslant n$, we have $s^{i-1}x^iy^{n-i} \in (x^n, y^n, xy^{n-1})$. In particular,*

$$s^{n-1}(x,y)^n \subseteq (x^n, y^n, xy^{n-1}) \quad \text{and} \quad s^n(x,y)^n \subseteq (x^n, y^n, sxy^{n-1}).$$

ii) *Also, $t^n(x,y)^n \subseteq (x^n, y^n, sxy^{n-1})$.*

iii) *If $n \geqslant 2$, the ideal (x^n, y^n, sxy^{n-1}) has a primary decomposition*

$$(x^n, y^n, sxy^{n-1}) = (x,y)^n \cap (x^n, y^n, sxy^{n-1}, s^n, t^n).$$

Proof. For (1) we use induction on i to show that $s^{i-1}x^iy^{n-i} \in (x^n, y^n, xy^{n-1})$. This is certainly true if $i = 1$ and, assuming the result for integers less than i, observe that

$$s^{i-1}x^iy^{n-i} = -s^{i-2}x^{i-2}y^{n-i}(txy+sy^2) = -s^{i-2}tx^{i-1}y^{n-i+1} - s^{i-1}x^{i-2}y^{n-i+2}$$
$$\in (x^n, y^n, xy^{n-1}).$$

Next, the equation $txy = -(sx^2 + sy^2)$ gives us

$$t(x,y)^n \subseteq (x^n, y^n) + s(x,y)^n,$$

and using this inductively, we get

$$t^n(x,y)^n \subseteq (x^n, y^n) + s^n(x,y)^n \subseteq (x^n, y^n, sxy^{n-1}),$$

which proves (2).

We next use the grading on the hypersurface A where $\deg s = \deg t = 0$ and $\deg x = \deg y = 1$. If α and β are nonzero homogeneous elements of A with $\alpha s^n + \beta t^n \in (x,y)^n$, then α and β must have degree at least n, and therefore belong to the ideal $(x,y)^n$. This shows that

$$(s^n, t^n) \cap (x,y)^n = (s^n, t^n)(x,y)^n,$$

and using (1) and (2) we get

$$(s^n, t^n) \cap (x,y)^n \subseteq (x^n, y^n, sxy^{n-1}).$$

The intersection asserted in (3) follows immediately from this, and it remains to verify that the ideals $\mathfrak{q}_1 = (x,y)^n$ and $\mathfrak{q}_2 = (x^n, y^n, sxy^{n-1}, s^n, t^n)$ are indeed primary ideals. The radical of $\mathfrak{q}_2$ is the maximal ideal (s,t,x,y), so $\mathfrak{q}_2$ is a primary ideal. Using the earlier grading, any homogeneous zerodivisor in the ring $A/\mathfrak{q}_1$ must have positive degree, and hence must be nilpotent. Consequently $\mathfrak{q}_1$ is a primary ideal as well. □

Theorem 4.6. *Let $A = K[s,t,x,y]/(sx^2 + txy + sy^2)$ where K is a field. Then $\operatorname{Ass} A/(x^2, y^2) = \{(x,y), (t,x,y)\}$ and*

$$\operatorname{Ass} \frac{A}{(x^n, y^n)} = \{(x,y), (s,t,x,y)\} \cup \operatorname{Ass} \frac{A}{(x,y,Q_{n-1})} \quad \text{for} \quad n \geqslant 3.$$

In particular, $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n, y^n)$ is an infinite set. If K is an algebraically closed field, let

$$\mathcal{S} = \left\{ \left(x, y, t - s\xi - s\xi^{-1}\right) A \,|\, \xi \in K,\ \xi^n = 1 \text{ for some } n \geqslant 1, \text{ and } \xi \neq \pm 1 \right\}.$$

In the case that K has characteristic zero,

$$\bigcup_{n\geqslant 1} \operatorname{Ass} \frac{A}{(x^n, y^n)} = \{(x,y), (t,x,y), (s,t,x,y)\} \cup \mathcal{S},$$

and if K has positive characteristic, then

$$\bigcup_{n\geqslant 1} \operatorname{Ass} \frac{A}{(x^n, y^n)} = \{(x,y), (t,x,y), (s,t,x,y), (t-2s,x,y), (t+2s,x,y)\} \cup \mathcal{S}.$$

Proof. It is easily checked that $(x,y)^2 \cap (x^2, y^2, t)$ is a primary decomposition of (x^2, y^2), so we need to compute $\operatorname{Ass} A/(x^n, y^n)$ for $n \geqslant 3$. By Lemma 4.4(2) we have an exact sequence

$$0 \longrightarrow \frac{A}{(x,y,Q_{n-1})} \xrightarrow{\cdot sxy^{n-1}} \frac{A}{(x^n, y^n)} \longrightarrow \frac{A}{(x^n, y^n, sxy^{n-1})} \longrightarrow 0,$$

and consequently

$$\operatorname{Ass} \frac{A}{(x,y,Q_{n-1})} \subseteq \operatorname{Ass} \frac{A}{(x^n, y^n)} \subseteq \operatorname{Ass} \frac{A}{(x,y,Q_{n-1})} \cup \operatorname{Ass} \frac{A}{(x^n, y^n, sxy^{n-1})}.$$

By Lemma 4.5 $\operatorname{Ass} A/(x^n, y^n, sxy^{n-1}) = \{(x,y), (s,t,x,y)\}$, and so it suffices to verify that the prime ideals $\mathfrak{p}_1 = (x,y)$ and $\mathfrak{p}_2 = (s,t,x,y)$ are indeed associated primes of $A/(x^n, y^n)$. This follows since $\mathfrak{p}_1$ is a minimal prime of (x^n, y^n) and

$$\mathfrak{p}_2 = (x^n, y^n) : (xy)^{n-1}.$$

If K is an algebraically closed field, the polynomials $Q_i(s,t)$ split into linear factors determined by the roots of $P_i(t)$, which are computed in Lemma 3.2. $\square$

Remark 4.7. If K is a field of characteristic $p>0$ and A is the hypersurface

$$A = K[s,t,x,y]/(sx^2+txy+sy^2)$$

as above, we saw that the set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n,y^n)$ is infinite. However, the set $\bigcup_{e\in\mathbb{N}} \operatorname{Ass} A/(x^{p^e},y^{p^e})$ is finite since, by Thereom 4.6,

$$\operatorname{Ass}\frac{A}{(x^{p^e},y^{p^e})} = \{(x,y),(s,t,x,y)\} \cup \operatorname{Ass}\frac{A}{(x,y,Q_{p^e-1})} \quad \text{for} \quad p^e \geqslant 3,$$

and Q_{p^e-1} is a power of Q_{p-1} by Proposition 3.1. Using Proposition 2.1, the set $\operatorname{Ass} H^2_{(x,y)}(R)$ is finite as well. Consequently we have a strict inclusion

$$\operatorname{Ass} H^2_{(x,y)}(R) \subsetneq \bigcup_{n\in\mathbb{N}} \operatorname{Ass}\frac{R}{(x^n,y^n)R}.$$

The set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n,y^n)$ has been explicitly computed in Theorem 4.6, and we next observe that the only associated prime of $H^2_{(x,y)}(R)$ is the maximal ideal $\mathfrak{m}=(s,t,x,y)$. The module $H^2_{(x,y)}(R)$ is generated over R by the elements $\eta_q = [1+(x^q,y^q)]$ for $q=p^e$, and it suffices to show that η_q is killed by a power of $\mathfrak{m}$. It is immediately seen that x^q and y^q kill η_q, and for the remaining cases note that

$$s^q\eta_q = [s^q x^{2q} + (x^{3q},y^q)] = 0 \quad \text{and} \quad t^q\eta_q = [t^q x^q y^q + (x^{2q},y^{2q})] = 0.$$

5. F-regular examples and tight closure

In Theorem 4.1 we saw that for the hypersurface

$$R = K[s,t,u,v,x,y]/\big(su^2x^2+tuxvy+sv^2y^2\big),$$

the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated prime ideals. This ring R, while a domain, is not normal. In Theorem 5.1 we construct examples with this behavior for local cohomology modules over normal hypersurfaces, in fact over hypersurfaces of characteristic zero with rational singularities, as well as over F-regular hypersurfaces of positive characteristic. F-regularity is a notion arising from the theory of tight closure developed by Hochster and Huneke in [HH1]. For details of the theory, we refer the reader to [HH1, HH2, HH3] and [Hu2].

Theorem 5.1. *Let K be an arbitrary field, and consider the hypersurface*

$$S = \frac{K[s,t,u,v,w,x,y,z]}{\big(su^2x^2+sv^2y^2+tuxvy+tw^2z^2\big)}$$

Then S is a normal domain for which the local cohomology module $H^3_{(x,y,z)}(S)$ has infinitely many associated prime ideals. This is preserved if we replace S by $S/(s-1)$ or by the localization $S_{(s,t,u,v,w,x,y,z)}$. If K has characteristic zero, then S has rational singularities. If K has characteristic $p>0$, then S is F-regular, and the set

$$\bigcup_{e\in\mathbb{N}} \operatorname{Ass} S/\big(x^{p^e},y^{p^e},z^{p^e}\big)S$$

is also infinite.

Proof. We defer the proof that S has rational singularities or is F-regular, see Lemma 5.3 below. Normality follows from this, or may be proved directly using the Jacobian criterion. Let B be the subring of S generated, as a K-algebra, by the elements $s, t, a = ux, b = vy$ and $c = wz$, i.e.,

$$B = K[s,t,a,b,c]/\big(sa^2 + sb^2 + tab + tc^2\big).$$

For a fixed integer $n \geqslant 1$, let

$$\eta_n = \big[s(ux)(vy)^{n-1} + (x^n, y^n, z)\big] \in H^3_{(x,y,z)}(S).$$

Using $S_0 = K[s,t]$ as the subring of elements of degree zero, Proposition 2.2 and Lemma 4.4(2) give us

$$\operatorname{ann}_{S_0} \eta_n = (a^n, b^n, c)B :_{S_0} sab^{n-1} = (Q_{n-1})S_0,$$

where the Q_i are the polynomials defined recursively in §3. Using Lemma 2.4 and Lemma 3.3, it follows that $H^3_{(x,y,z)}(S)$ has infinitely many associated prime ideals. The assertion regarding the associated primes of Frobenius powers of the ideal $(x,y,z)S$ now follows from Proposition 2.1. □

It remains to prove that the hypersurface S in Theorem 5.1 has rational singularities or is F-regular, depending on the characteristic. The results of [SW] provide a direct proof that the hypersurface S has rational singularities in characteristic zero. However, instead of relying on this, we prove here that if K has positive characteristic, then S is F-regular. Using [Sm, Theorem 4.3], it then follows that S has rational singularities when K has characteristic zero.

We first record an elementary lemma:

Lemma 5.2. *Let $(S, \mathfrak{m})$ be an $\mathbb{N}$-graded Gorenstein domain of dimension d, finitely generated over a field $[S]_0 = K$ of characteristic $p > 0$, and let $\eta \in H^d_{\mathfrak{m}}(S)$ denote a socle generator. Let $c \in R$ be a nonzero element such that S_c is regular. Then S is F-regular if and only if there exists an integer $e \geqslant 1$ such that η belongs to the S-span of $cF^e(\eta)$.*

Proof. If S is F-regular then the zero submodule of $H^d_{\mathfrak{m}}(S)$ is tightly closed, i.e., $0^*_{H^d_{\mathfrak{m}}(S)} = 0$, and so there exists a positive integer e such that $cF^e(\eta) \neq 0$. Since η generates the socle of $H^d_{\mathfrak{m}}(S)$, which is one-dimensional, η must belong to the S-span of $cF^e(\eta)$.

Conversely, assume that η belongs to the S-span of $cF^e(\eta)$ for some $e \geqslant 1$. Then $cF^e(\eta) \neq 0$, and so the Frobenius morphism $F : H^d_{\mathfrak{m}}(S) \longrightarrow H^d_{\mathfrak{m}}(S)$ is injective. It follows from [HR, Proposition 6.11] that the ring S is F-pure. By [HH2, Theorem 6.2], the element c has a power which is a test element but then, since S is F-pure, c itself must be a test element. The condition $cF^e(\eta) \neq 0$ implies that $\eta \notin 0^*_{H^d_{\mathfrak{m}}(S)}$. Consequently $0^*_{H^d_{\mathfrak{m}}(S)} = 0$, and it follows that S is F-regular. □

Lemma 5.3. *Let K be a field and consider the hypersurface*

$$S = \frac{K[s,t,u,v,w,x,y,z]}{\big(su^2x^2 + sv^2y^2 + tuxvy + tw^2z^2\big)}.$$

If K has characteristic $p > 0$, then S is F-regular. If K has characteristic zero, then S has rational singularities.

Proof. We first consider the case where K has characteristic $p > 0$. It is easily checked that S_{twz} is a regular ring. We may compute $H^7_{\mathfrak{m}}(S)$ using the Čech complex with respect to the system of parameters $s, u, x, v, y, w-t, z-t$. The socle of $H^7_{\mathfrak{m}}(S)$ is spanned by the element

$$\eta = [t^4 + (s, u, x, v, y, w-t, z-t)] \in H^7_{\mathfrak{m}}(S).$$

Since S_{twz} is regular it suffices, by Lemma 5.2, to show that η belongs to the S-span of $twzF^e(\eta)$ for some $e \geqslant 1$, i.e., that

$$t^4\big(suxvy(w-t)(z-t)\big)^{q-1} \in \big(twzt^{4q}, s^q, u^q, x^q, v^q, y^q, (w-t)^q, (z-t)^q\big)S \qquad (*)$$

for some $q = p^e$. We shall consider here the case $p \geqslant 5$. The interested reader may verify that $(*)$ holds with $q = 2^3$ and $q = 3^2$ in the remaining cases $p = 2$ and $p = 3$ respectively. It suffices to show that

$$t^4(suxvy)^{p-1} \in \big(t^{4p+3}, s^p, u^p, x^p, v^p, y^p, w-t, z-t\big)S.$$

Working in the polynomial ring $A = K[s,t,u,v,x,y]$, it is enough to check that $t^4(suxvy)^{p-1} \in \mathfrak{a} + (t^{5p-1})A$, where

$$\mathfrak{a} = (x^p, y^p, su^2x^2 + sv^2y^2 + tuxvy + t^5)A.$$

We observe that

$$\begin{aligned} t^{5p-1} &\equiv t^4\big(su^2x^2 + sv^2y^2 + tuxvy\big)^{p-1} \mod \mathfrak{a} \\ &= t^4 \sum_{i,j} \binom{p-1}{i}\binom{p-1-i}{j} \big(su^2x^2\big)^i \big(sv^2y^2\big)^j \big(tuxvy\big)^{p-1-i-j} \mod \mathfrak{a} \\ &= t^4 \sum_{i,j} \binom{p-1}{i}\binom{p-1-i}{j} s^{i+j}t^{p-1-i-j}(ux)^{p-1+i-j}(vy)^{p-1-i+j} \mod \mathfrak{a}. \end{aligned}$$

The only terms which contribute $\mod (x^p, y^p)$ are those for which $i = j$, and so

$$t^{5p-1} \equiv t^4 \sum_{i=0}^{(p-1)/2} \binom{p-1}{i}\binom{p-1-i}{i} s^{2i}t^{p-1-2i}(uxvy)^{p-1} \mod \mathfrak{a}.$$

When $2i < p-1$, the corresponding summand in the above expression is a multiple of $t^5(uxvy)^{p-1}$, which is an element of $\mathfrak{a}$. Thus

$$t^{5p-1} \equiv t^4 \binom{p-1}{(p-1)/2} s^{p-1}(uxvy)^{p-1} \mod \mathfrak{a}.$$

Since the binomial coefficient occurring above is a unit, $t^4(suxvy)^{p-1} \in \mathfrak{a} + (t^{5p-1})A$, which completes the proof that S is F-regular.

It remains to show that S has rational singularities in the case K has characteristic zero. By [Sm, Theorem 4.3], it suffices to show that S has F-rational type, i.e., that for all but finitely many prime integers p, the fiber over the $p\mathbb{Z}$ of the map

$$\mathbb{Z} \longrightarrow \frac{\mathbb{Z}[s,t,u,v,w,x,y,z]}{\big(su^2x^2 + sv^2y^2 + tuxvy + tw^2z^2\big)}$$

is an F-rational ring. While this is indeed true for *all* prime integers p, our earlier computation for $p \geqslant 5$ certainly suffices. □

The question of whether tight closure commutes with localization remains open even for finitely generated algebras over fields of positive characteristic. We briefly describe the situation studied in [AHH, Ho, Ka1, SN] and [Hu2, §12], though we limit our discussion here to the tight closures of ideals. Let R be a ring of characteristic $p > 0$ and W a multiplicative system in R. For an ideal $\mathfrak{a} \subset R$, it is easily seen that $W^{-1}\big(\mathfrak{a}^*\big) \subseteq \big(W^{-1}\mathfrak{a}\big)^*$. When this inclusion is an equality, the tight closure of $\mathfrak{a}$ is said to *commute with localization at* W. The interest in the associated primes of $\mathfrak{a}^{[q]}$ and $\big(\mathfrak{a}^{[q]}\big)^*$ arises from the results of [AHH, §3], which state that the tight closure of $\mathfrak{a}$ commutes with localization at any multiplicative system which is disjoint from $\bigcup_q \operatorname{Ass} R/\mathfrak{a}^{[q]}$ or from $\bigcup_q \operatorname{Ass} R/\big(\mathfrak{a}^{[q]}\big)^*$. This raises the questions:

Question 5.4. [Ho, page 90] Let R be a Noetherian ring of characteristic $p > 0$, and $\mathfrak{a}$ an ideal of R.

i) Does the set $\bigcup_q \operatorname{Ass} R/\mathfrak{a}^{[q]}$ have finitely many maximal elements?

ii) Does $\bigcup_q \operatorname{Ass} R/\big(\mathfrak{a}^{[q]}\big)^*$ have finitely many maximal elements?

iii) For a complete local domain $(R, \mathfrak{m})$ and an ideal $\mathfrak{a} \subset R$, is there a positive integer B such that

$$\mathfrak{m}^{Bq} H^0_{\mathfrak{m}} \left(R/(\mathfrak{a}^{[q]})^* \right) = 0 \quad \text{for all} \quad q = p^e\,?$$

In [Ka1, Theorem 6] it is proved that positive solutions to (2) and (3) would imply that tight closure commutes with localization. Some positive answers to these questions appear in [AHH] where, for example, it is proved that (2) and (3) hold whenever $R/\mathfrak{a}$ has finite phantom projective dimension. It should also be mentioned that the situation for *ordinary* powers is known: $\bigcup_n \operatorname{Ass} R/\mathfrak{a}^n$ is finite for any Noetherian ring R, see [Br] or [Ra]. However, Katzman showed that the maximal elements of $\bigcup_q \operatorname{Ass} R/\mathfrak{a}^{[q]}$ need not form a finite set, thereby settling (1). We recall the example from [Ka1] discussed earlier in Remark 2.7: If

$$A = K[t, x, y]/\big(xy(x-y)(x-ty)\big),$$

then the set $\bigcup_q \operatorname{Ass} R/(x^q, y^q)$ is infinite. In this example $(x^q, y^q)^* = (x, y)^q$ for all $q = p^e$ and so, in contrast, $\bigcup_q \operatorname{Ass} A/(x^q, y^q)^*$ is finite. The question remained whether $\bigcup_q \operatorname{Ass} R/\big(\mathfrak{a}^{[q]}\big)^*$ has finitely many maximal elements for arbitrary rings R of characteristic $p > 0$, and we next show that this has a negative answer:

Theorem 5.5. *Let K be a field of characteristic $p > 0$, and consider*

$$R = \frac{K[t, u, v, w, x, y, z]}{\big(u^2x^2 + v^2y^2 + tuxvy + tw^2z^2\big)}.$$

Then R is an F-regular ring, and the set

$$\bigcup_{e \in \mathbb{N}} \operatorname{Ass} \frac{R}{\big(x^{p^e}, y^{p^e}, z^{p^e}\big)} = \bigcup_{e \in \mathbb{N}} \operatorname{Ass} \frac{R}{\big(x^{p^e}, y^{p^e}, z^{p^e}\big)^*}$$

has infinitely many maximal elements.

Proof. Since the hypersurface S of Lemma 5.3 is F-regular, so is its localization

$$S_s = \frac{K[t/s, u, v, w, x, y, z, s, 1/s]}{\big(u^2x^2 + v^2y^2 + (t/s)uxvy + (t/s)w^2z^2\big)}.$$

The ring S_s has a $\mathbb{Z}$-grading where $\deg s = 1$, $\deg 1/s = -1$, and the remaining generators, $t/s, u, v, w, x, y, z$, have degree 0. By [HH1, Proposition 4.12] a direct summand of an F-regular ring is F-regular, and so $R \cong [S_s]_0$ is F-regular.

For $q = p^e$, consider the ideals of R,

$$\mathfrak{a}_q \;=\; (x^q, y^q, z^q)R :_R t^q uv^{q-2}x^2y^{q-1}z^{q-1}.$$

Let $R_0 = K[t]$. As in the proof of Theorem 5.1, we may use Proposition 2.2 and Lemma 4.4(3) to verify that

$$\begin{aligned} \mathfrak{a}_q \cap R_0 \;&=\; (x^q, y^q, z^q)R :_{R_0} t^q(ux)(vy)^{q-2}xyz^{q-1} \\ &=\; (x^{q-1}, y^{q-1}, z)R :_{R_0} t^q(ux)(vy)^{q-2} \;=\; P_{q-2} :_{R_0} t^q, \end{aligned}$$

where the P_i are the polynomials defined recursively in §3. In particular, this shows that $\mathfrak{a}_q \neq R$ for $q \gg 0$. It is immediately seen that $x, y, z \in \sqrt{\mathfrak{a}_q}$, and we claim that $u, v, w \in \sqrt{\mathfrak{a}_q}$. To see that $u \in \mathfrak{a}_q$, note that

$$u(t^q uv^{q-2}x^2y^{q-1}z^{q-1}) \;=\; t^q(u^2x^2)v^{q-2}y^{q-1}z^{q-1} \in (y^q, z^q).$$

Next, observe that

$$(vy)^2 \in ux(ux,vy)R + zR, \quad \text{and so} \quad (vy)^{q-1} \in (ux)^{q-2}(ux,vy)R + zR.$$

Using this,

$$v(t^q uv^{q-2}x^2y^{q-1}z^{q-1}) \;=\; t^q(vy)^{q-1}ux^2z^{q-1} \in (x^q, z^q),$$

and so $v \in \mathfrak{a}_q$. Finally, it is easily verified that $w^{q-1} \in \mathfrak{a}_q$, i.e., that

$$w^{q-1}(st^q uv^{q-2}x^2y^{q-1}z^{q-1}) \in (x^q, y^q, z^q),$$

since $t^q(wz)^{q-1} \;\in\; (x^{q-2}, y)$. We have now established

$$\operatorname{Min}(\mathfrak{a}_q) \;=\; \operatorname{Min}\big((u,v,w,x,y,z)R + (P_{q-2} :_{R_0} t^q)R\big),$$

and so the minimal primes of $\mathfrak{a}_q$ are maximal ideals of R. By Lemma 3.3 the union $\bigcup_q \operatorname{Min}(\mathfrak{a}_q)$ is an infinite set, and so we conclude that $\bigcup_q \operatorname{Ass} R/\big(x^q, y^q, z^q\big)$ has infinitely many maximal elements. $\square$

6. Examples of small dimension

We analyze multidiagonal matrices with $d = 4$ and use these computations to obtain low-dimensional examples of integral domains of characteristic $p > 0$ where the set of associated primes of Frobenius powers of an ideal is infinite. The example in Theorem 4.1, after specializing $s = 1$, is an integral domain of dimension four. We construct here a hypersurface A of dimension two, which is an integral domain, and has an ideal $(x, y)A$ for which $\bigcup_e \operatorname{Ass} A/\big(x^{p^e}, y^{p^e}\big)$ is infinite. In view of Proposition 3.1, to construct such an example using Theorem 2.6, we need to consider multidiagonal matrices with $d \geqslant 4$.

We start with the polynomial ring $A_0 = K[t]$ over a field K. Let $d = 4$, and consider the matrices M_n of multidiagonal form with respect to $r_0 = r_4 = 1, r_2 = t$, and $r_1 = r_3 = 0$, i.e.,

$$M_n = \begin{bmatrix} t & 0 & 1 & & & & \\ 0 & t & 0 & 1 & & & \\ 1 & 0 & t & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 0 & t & 0 & 1 \\ & & & 1 & 0 & t & 0 \\ & & & & 1 & 0 & t \end{bmatrix}.$$

We again use the convention $\det M_0 = 1$, and have $\det M_1 = t$, $\det M_2 = t^2$, $\det M_3 = t^3 - t$, and the recursion

$$\det M_{n+4} = t \det M_{n+3} - t \det M_{n+1} + \det M_n \quad \text{for all} \quad n \geqslant 0.$$

Using this, the generating function for $\det M_n$ is easily computed to be

$$G(x) = \sum_{n \geqslant 0} (\det M_n)\, x^n = \frac{1}{1 - tx + tx^3 - x^4} = \frac{1}{(1-x)(1+x)(1-tx+x^2)}.$$

Set $F_n(t) = \det M_n$, which is a monic polynomial of degree n. We need to analyze the distinct irreducible factors of the polynomials $\{F_n(t)\}$.

Lemma 6.1. *Let K be an algebraically closed field, and consider the polynomials $F_n(t) = \det M_n \in K[t]$ as above.*

i) *Let ξ be a nonzero element of K with $\xi \neq \pm 1$. If n is an odd integer, then*

$$F_n(\xi + \xi^{-1}) = \frac{(\xi^{n+3} - 1)(\xi^{n+1} - 1)}{\xi^n(\xi^2 - 1)^2},$$

and so $F_n(\xi+\xi^{-1})=0$ *if and only if* $\xi^{n+3}=1$ *or* $\xi^{n+1}=1$.

ii) *If* n *is an odd integer and* $(n+3)(n+1)$ *is invertible in* K, *then the polynomial* $F_n(t)$ *has* n *distinct roots of the form* $\xi+\xi^{-1}$, *where* $\xi\neq\pm1$, *and either* $\xi^{n+3}=1$ *or* $\xi^{n+1}=1$.

iii) *If the characteristic of* K *is an odd prime* p, *then* $F_{q-2}(t)$ *has* $q-2$ *distinct roots for all* $q=p^e$.

Proof. (1) Consider the generating function for the polynomials $F_n(t)$,

$$G(x)=\sum_{n\geqslant 0}F_n(t)x^n=\frac{1}{(1-x)(1+x)(1-tx+x^2)}\in K[t][[x]].$$

If $\xi\in K$ with $\xi\neq 0$ and $\xi\neq\pm1$, then

$$\sum_{n\geqslant 0}F_n(\xi+\xi^{-1})x^n=\frac{1}{(1-x)(1+x)(1-\xi x)(1-\xi^{-1}x)}$$

$$=\frac{\sum x^n}{2(2-\xi-\xi^{-1})}+\frac{\sum(-x)^n}{2(2+\xi+\xi^{-1})}+\frac{\xi^3\sum(\xi x)^n}{(\xi^2-1)(\xi-\xi^{-1})}+\frac{\xi^{-3}\sum(\xi^{-1}x)^n}{(\xi^{-2}-1)(\xi^{-1}-\xi)}.$$

Comparing the coefficients of x^n and simplifying, we obtain the asserted formula for $F_n(\xi+\xi^{-1})$.

(2) As we observed earlier in the proof of Lemma 3.2(2), $\xi+\xi^{-1}=\eta+\eta^{-1}$ if and only if ξ equals η or η^{-1}. The only common roots of the polynomials $X^{n+3}-1=0$ and $X^{n+1}-1=0$ are ±1. Since $n+3$ is invertible in the field K, the polynomial $X^{n+3}-1=0$ has $n+1$ distinct roots ξ with $\xi\neq\pm1$. These give the $(n+1)/2$ distinct roots $\xi+\xi^{-1}$ of $F_n(t)$. Similarly, the roots of $X^{n+1}-1=0$ contribute $(n-1)/2$ other distinct roots of $F_n(t)$. But then we have $(n+1)/2+(n-1)/2=n$ distinct roots of the degree n polynomial $F_n(t)$ which, then, must be all its roots.

(3) Since $n=q-2$ is odd and $(n+3)(n+1)=(q+1)(q-1)$ is invertible in K, it follows from (2) that $F_{q-2}(t)$ has $q-2$ distinct roots. □

As a consequence of Lemma 6.1, we immediately have:

Lemma 6.2. *Let* K *be an arbitrary field of characteristic* $p>2$. *Then the polynomials* $\{F_{q-2}(t)\}_{q=p^e}$ *have infinitely many distinct irreducible factors.*

Theorem 6.3. *Let* K *be an arbitrary field of characteristic* $p>2$, *and consider the integral domain*

$$A=K[t,x,y]/\big(x^4+tx^2y^2+y^4\big).$$

Then the set $\bigcup_{e\in\mathbb{N}}\operatorname{Ass}A/\big(x^{p^e},y^{p^e}\big)$ *is infinite.*

Proof. The hypersurface A arises from Theorem 2.6 using the matrices M_n of multidiagonal form with respect to $r_0=r_4=1, r_2=t$, and $r_1=r_3=0$. By Lemma 6.2, the set $\bigcup_e\operatorname{Min}(\det M_{p^e-2})$ is infinite, and so the result follows. □

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