Primary decomposition of the Mayr-Meyer ideal
Unedited collection of facts

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In "Primary decomposition of the Mayr-Meyer ideal" [S], partial primary decompositions were determined for the Mayr-Meyer ideals $J(n,d)$ for all $n \geq 2$, $d \geq 1$. While working on the primary decompositions of $J(n,d)$ I tried various approaches, and many of them did not (and should not) make it into the final version of [S].

In this paper I put together all the different facts and approaches for anybody who is interested. I add the disclaimer that after these approaches were abandoned, I stopped editing them – there may be flaws here. Throughout I assume that the reader is familiar with [S].

I thank Craig Huneke for all the conversations, computations, and enthusiasm regarding this material.

Here is the index to this hodge-podge:

Lemma 0.1: *Let x, y be elements of a ring R and I an ideal. Suppose that there are integers k and l such that for all m and n, $I : x^m y^n \subseteq I : x^k y^l$. Then*

$$I = \left(I : x^k y^l\right) \cap \left(I + (x^k y^l)\right). \qquad \blacksquare$$

Trying to get primary decomposition via Lemma 0.1

1. Primary components of J not containing a power of s

We'll prove in this section that $\cap_{i=0}^{n} p_i = J : s^\infty = J : s^3$.

Lemma 1.1: *For all $r = 0, 1, \ldots, n-1$,*

$$s^2 C_0 C_1 \cdots C_{r-1} D_r \subseteq J.$$

Furthermore, for all $r = 0, 1, \ldots, n-1$, and all $i_1, \ldots, i_r \in \{1, 2, 3, 4\}$,

$$\begin{aligned} s^2 C_0 C_1 \cdots C_r + J &= (sfc_{01} c_{1i_1} \cdots c_{ri_r}) + J \\ &= (sfc_{04} c_{1i_1} \cdots c_{ri_r}) + J \\ &= (s^2 c_{02} c_{1i_1} \cdots c_{ri_r}) + J \\ &= (s^2 c_{03} c_{1i_1} \cdots c_{ri_r}) + J, \end{aligned}$$

and when $r = 0$,

$$sC_0 + J = (fc_{01}) + J = (fc_{04}) + J = (sc_{02}) + J = (sc_{03}) + J.$$

Proof: The case $r = 0$ is straightforward due to $h_{01}, h_{02}, h_{04}, h_{13}, h_{14}$ and h_{15} (for example, $s^2(c_{01} - c_{02} b_{01}^d) \in (h_{01}, s(fc_{01} - sc_{02}) b_{01}^d) \subseteq J$).

Now let $r > 0$. Then $D_r = (c_{r4} - c_{r1}, c_{r3} - c_{r2}, c_{r2} - c_{r1})$. By above it suffices to prove only that $s^2 C_0 C_1 \cdots C_r + J \subseteq (sfc_{01} c_{1i_1} \cdots c_{ri_r}) + J$. By induction, $s^2 C_0 C_1 \cdots C_{r-1} D_r$ is contained in

$$sfc_{01} c_{11} \cdots c_{r-1,1} (c_{r3} - c_{r2}, c_{r2} - c_{r1}) + sfc_{01} c_{11} \cdots c_{r-2,1} c_{r-1,4} (c_{r4} - c_{r1}) + J.$$

First note that

$$sfc_{01} c_{11} \cdots c_{r-2,1} c_{r-1,4} (c_{r4} - c_{r1}) \in sfc_{01} c_{11} \cdots c_{r-2,1} c_{r-1,2} (c_{r3} - c_{r2}) + J,$$

and that $sfc_{01} c_{11} \cdots c_{r-1,1} (c_{r3} - c_{r2})$ is contained in J due to $h_{r+1,5}$. Finally,

$$sfc_{01} c_{11} \cdots c_{r-1,1} (c_{r2} - c_{r1}) \in sfc_{01} c_{11} \cdots c_{r-2,1} (c_{r-1,4} - c_{r-1,1}) c_{r1} + J$$

modulo $h_{r+1,3}$, and that is contained, by induction on r, in J. ∎

Lemma 1.2: $s^2 C_0 E \subseteq J$ *and* $sfc_{02} \left(b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d\right) \subseteq J$.

Proof: By Lemma 1.1,

$$\begin{aligned} sC_0 E &\subseteq fc_{01} \left(s - fb_{01}^d, b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d\right) + J \\ &\subseteq fc_{01} \left(b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d\right) + J. \end{aligned}$$

Thus as $sc_{01} \in (fc_{01}, h_{01})$, $s^2 C_0 E \subseteq f^2 c_{01}\left(b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d\right) + J$, and as $fc_{01} \in (sc_{02}, h_{13})$, it remains to prove the second part of the lemma. But $sfc_{02}\,(b_{01} - b_{04})$ equals $f(sc_{02}b_{01} - sc_{03}b_{04})$ modulo h_{15}, and that is in J. Also, $sfc_{02}\left(b_{02}^d - b_{03}^d\right) \in \left(sfc_{02}b_{02}^d - sfc_{03}b_{03}^d\right) + J = \left(s^2c_{02} - s^2c_{03}\right) + J = J$, and finally $sfc_{02}\left(b_{01}^d - b_{02}^d\right) \in \left(f^2c_{01}b_{01}^d - sfc_{02}b_{02}^d\right) + J = \left(sfc_{01} - s^2c_{02}\right) + J = J$. ∎

Lemma 1.3: *For all* $r = 2, \ldots, n$,

$$s^2 C_0 C_1 \cdots C_{r-1} \subseteq sfc_{04} C_1 \cdots C_{r-1} + J \subseteq sfc_{04} C_1 \cdots C_{r-1} b_{r-1,2}^d + J.$$

Also, $s^2 C_0 \subseteq (sfc_{02}b_{02}^d) + J$.

Proof: When $r = 1$, $s^2 C_0$ is contained in $(s^2 c_{02}) + J$ by Lemma 1.1, which, modulo h_{02} is contained in $(sfc_{02}b_{02}^d) + J$, as desired.

Now let $r \geq 2$. By Lemma 1.1 it suffices to prove the second inclusion.

When $r = 2$, By Lemma 1.1, $sfc_{04}C_1$ is contained in $(s^2c_{02}c_{12}) + J$, whence modulo h_{02} in $(sfc_{02}c_{12}b_{02}^d) + J$, and then modulo h_{18} in $(sfc_{02}c_{12}b_{12}^d) + J$, and then by Lemma 1.1 in $(f^2c_{04}C_1b_{12}^d) + J$.

Now let $r > 2$. By induction on r, $sfc_{04}C_1 \cdots C_{r-1} \subseteq sfc_{04}C_1 \cdots C_{r-1}b_{r-2,2}^d + J$. By Lemma 1.1, the latter is contained in $(sfc_{01}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}b_{r-2,2}^d) + J$, and modulo $h_{r-1,8}$, this equals $(sfc_{01}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}b_{r-1,2}^d b_{r-2,3}^d) + J$, whence is in $sfc_{04}C_1 \cdots C_{r-1}b_{r-1,2}^d + J$, as desired. ∎

Lemma 1.4: *For all* $r = 2, \ldots, n-1$, *and all* $i, j = 1, \ldots, 4$,

$$s^2 C_0 C_1 \cdots C_r (b_{rj} - b_{ri}) \subseteq J,$$
$$sfc_{04} C_1 \cdots C_r b_{r-1,2} (b_{rj} - b_{ri}) \subseteq J.$$

Also,

$$s^3 C_0 C_1 (b_{1j} - b_{1i}) \subseteq J,$$
$$s^2 f c_{04} C_1 b_{02} (b_{1j} - b_{1i}) \subseteq J.$$

Proof: By Lemma 1.3, $s^2 C_0 C_1 \cdots C_r \subseteq sfc_{04} C_1 \cdots C_r b_{r-1,2}^d + J$, so it suffices to prove the second inclusion. But

$$b_{r-1,2}(b_{rj} - b_{ri}) \in (b_{rj}(b_{r-1,2} - b_{ri}b_{r-1,3}), b_{ri}(b_{r-1,2} - b_{rj}b_{r-1,3})) + J,$$

whence by Lemma 1.1 and by using $h_{r,6+i}$ and $h_{r,6+j}$ we are done. The last part follows similarly. ∎

Remark 1.5: *A calculation by Macaulay2 shows that* s^3 *is needed in the lemma above. For this reason, we need* s^3 *also below (yet* s^2 *works below if* $r > 2$*):*

Lemma 1.6: *For all $r = 2, \ldots, n-1$, and all $i = 1, \ldots, 4$,*
$$s^3 C_0 C_1 \cdots C_r (1 - b_{ri}) \subseteq J.$$

Proof: By Lemma 1.3, $s^2 C_0 C_1 \cdots C_r \subseteq sfc_{04} C_1 \cdots C_r b_{r-1,2}^d + J$, so it suffices to prove that $sfc_{04} C_1 \cdots C_r b_{r-1,2}^d (1 - b_{ri})$ is contained in J. But

$$\begin{aligned} & sfc_{04} C_1 \cdots C_r b_{r-1,2}^d (1 - b_{ri}) \\ & \quad \subseteq sfc_{04} C_1 \cdots C_r b_{r-1,2}^{d-1} \left(b_{ri}(b_{r-1,3} - b_{r-1,2}), b_{r-1,2} - b_{ri} b_{r-1,3} \right) + J, \end{aligned}$$

where $sfc_{04} C_1 \cdots C_r b_{r-1,2}^{d-1} (b_{r-1,2} - b_{ri} b_{r-1,3})$ is contained in J by Lemma 1.1 and modulo $h_{r,6+i}$. Also,

$$s^2 f c_{04} C_1 \cdots C_r (b_{r-1,3} - b_{r-1,2}) \subseteq s^2 f c_{04} C_1 \cdots C_r b_{r-2,2} (b_{r-1,3} - b_{r-1,2}) + J$$

by Lemma 1.3, and the latter ideal is contained in J by Lemma 1.4. ■

Lemma 1.7: $s^3 C_0 C_1 F \subseteq J$.

Proof: Recall that $F = \left(b_{02} - b_{11} b_{03}, b_{14} - b_{11}, b_{13} - b_{11}, b_{12} - b_{11}, 1 - b_{12}^d \right)$. By applying Lemma 1.4,
$$s^3 C_0 C_1 \left(b_{14} - b_{11}, b_{13} - b_{11}, b_{12} - b_{11} \right) \subseteq J,$$

and by Lemma 1.1,

$$s^2 C_0 C_1 \left(b_{02} - b_{11} b_{03} \right) \subseteq s^2 c_{02} c_{11} \left(b_{02} - b_{11} b_{03} \right) + J \subseteq sfc_{02} b_{02}^d c_{11} \left(b_{02} - b_{11} b_{03} \right) + J = J.$$

Finally,

$$\begin{aligned} s^2 C_0 C_1 \left(1 - b_{12}^d \right) & \subseteq s^2 c_{02} c_{12} \left(1 - b_{12}^d \right) + J \quad \text{(by Lemma 1.1)} \\ & \subseteq sfc_{02} c_{12} b_{02}^d \left(1 - b_{12}^d \right) + J \quad \text{(modulo } h_{02}) \\ & \subseteq sfc_{02} c_{12} \left(b_{02}^d - b_{12}^d b_{03}^d, b_{12}^d (b_{03}^d - b_{02}^d) \right) + J \\ & \subseteq sfc_{02} c_{12} b_{12}^d (b_{03}^d - b_{02}^d) + J \quad \text{(modulo } h_{18}) \\ & \subseteq J \quad \text{(by Lemma 1.2)}. \quad ■ \end{aligned}$$

Thus it follows from all these lemmas, that

Proposition 1.8: *Assume that $n \geq 2$. Then $s^3 \left(\bigcap_{i=0}^n p_i \right) \subseteq J$. Thus*
$$J = \left(\bigcap_{i=0}^n p_i \right) \bigcap \left(J + (s^3) \right),$$
so that in order to find a (possibly redundant) primary decomposition of J, it suffices to find a primary decomposition of $J + (s^3)$.

Proof: Lemmas , 1.1, 1.2, 1.6, and 1.7 show that $\bigcap_{i=0}^n p_i$ is contained in $J : s^\infty$. Thus as $J \subseteq \bigcap_{i=0}^n p_i$, $J : s^\infty = (\bigcap_{i=0}^n p_i) : s^\infty$. But s is a non-zerodivisor modulo $\bigcap_{i=0}^n p_i$, so that $J : s^\infty = \bigcap_{i=0}^n p_i$, and that equals $J : s^3$ by lemmas , 1.1, 1.2, 1.6, and 1.7. The rest follows from Lemma 0.1. ■

Now let L_1 be an ideal in $J : C_0$ such that no c_{0i} appears in any minimal generator of L_1. For example,

$$\begin{aligned}L_1 = \big(&s(s - fb_{01}^d), f(s - fb_{02}^d)b_{03}^d, f(s - fb_{03}^d)b_{02}^d, s(s - fb_{02}^d), s(s - fb_{03}^d), s(s - fb_{04}^d),\\ &fc_{1i}(b_{02} - b_{1i}b_{03})b_{03}^d, sc_{1i}(b_{02} - b_{1i}b_{03}) | i = 1, \ldots, 4\big).\end{aligned}$$

Then set

$$J_1 = J + \left(s, fb_{02}^d b_{03}^d\right)^3 + L_1.$$

Proposition 1.9: *For $n \geq 2$,*

$$J = \left(\bigcap_{i=0}^n p_i\right) \cap \left(J + \left(s, fb_{02}^d b_{03}^d\right)^3 + L_1\right),$$

so that in order to find a (possibly redundant) primary decomposition of J, it suffices to find a primary decomposition of $J_1 = J + \left(s, fb_{02}^d b_{03}^d\right)^3 + L_1$.

Proof: First note that $(\bigcap_{i=0}^n p_i) \bigcap J_1$ is contained in

$$\begin{aligned}(c_{01}, c_{02}, c_{03}, c_{04}) \cap J_1 &= J + (c_{01}, c_{02}, c_{03}, c_{04}) \bigcap \left(\left(s, fb_{02}^d b_{03}^d\right)^3 + L_1\right)\\ &= J + (c_{01}, c_{02}, c_{03}, c_{04}) \cdot \left(\left(s, fb_{02}^d b_{03}^d\right)^3 + L_1\right),\end{aligned}$$

and this is contained in $J + (c_{01}, c_{02}, c_{03}, c_{04})\,(s)^3$. Now the proposition follows from the previous one. ■

2. Primary components of $J_1 = J + \left(s, fb_{02}^d b_{03}^d\right)^3 + L_1$ not containing c_{01}

Lemma 2.1: *For all $r = 1, \ldots, n-1$, and all $i_1, \ldots, i_r \in \{1, 2, 3, 4\}$,*

$$fc_{01}^2 C_1 \cdots C_r + J = \left(fc_{01}^2 c_{1i_1} \cdots c_{ri_r}\right) + J.$$

Furthermore, for all $r = 1, \ldots, n-1$,

$$fc_{01}^2 C_1 \cdots C_{r-1} D_r \subseteq J.$$

Proof: When $r = 1$,

$(fc_{01}^2 c_{14}) + J$

$$
\begin{aligned}
&= (fc_{01}(c_{01} - c_{04})c_{14} + fc_{01}(c_{04}c_{14} - c_{01}c_{13}) + fc_{01}c_{01}c_{13}) + J \\
&= (sc_{02}(c_{04}c_{14} - c_{01}c_{13}), fc_{01}^2 c_{13}) + J \\
&= (fc_{01}^2 c_{13}) + J, \\
&= (c_{01}c_{13}(fc_{01} - sc_{02}) + sc_{01}c_{02}(c_{13} - c_{12}) - c_{01}c_{12}(fc_{01} - sc_{02}) + fc_{01}^2 c_{12}) + J \\
&= (fc_{01}^2 c_{12}) + J, \\
&= (fc_{01}(c_{01}c_{12} - c_{04}c_{11}) + fc_{01}(c_{04} - c_{01})c_{11} + fc_{01}^2 c_{11}) + J \\
&= (fc_{01}^2 c_{11}) + J,
\end{aligned}
$$

which proves the first part of the lemma for $r = 1$. Now let $r > 1$. Then

$$
\begin{aligned}
fc_{01}^2 C_1 \cdots C_{r-1} c_{r4} + J &= fc_{01}^2 c_{11} \cdots c_{r-2,1} c_{r-1,4} c_{r4} + J \quad \text{(by induction)} \\
&= sc_{01}c_{02}c_{11} \cdots c_{r-2,1} c_{r-1,4} c_{r4} + J \quad \text{(modulo } h_{13}\text{)} \\
&= sc_{01}c_{02}c_{11} \cdots c_{r-2,1} c_{r-1,1} c_{r3} + J \quad \text{(modulo } h_{r+1,4}\text{)} \\
&= fc_{01}^2 c_{11} \cdots c_{r-2,1} c_{r-1,1} c_{r3} + J \quad \text{(modulo } h_{13}\text{)} \\
&= fc_{01}^2 C_1 \cdots C_{r-1} c_{r3} + J, \\
fc_{01}^2 C_1 \cdots C_{r-1} c_{r3} + J &= fc_{01}^2 c_{11} \cdots c_{r-1,1} c_{r3} + J \quad \text{(by induction)} \\
&= sc_{01}c_{02}c_{11} \cdots c_{r-1,1} c_{r3} + J \quad \text{(modulo } h_{13}\text{)} \\
&= sc_{01}c_{02}c_{11} \cdots c_{r-1,1} c_{r2} + J \quad \text{(modulo } h_{r+1,5}\text{)} \\
&= fc_{01}^2 c_{11} \cdots c_{r-1,1} c_{r2} + J \quad \text{(modulo } h_{13}\text{)} \\
&= fc_{01}^2 C_1 \cdots C_{r-1} c_{r2} + J, \\
fc_{01}^2 C_1 \cdots C_{r-1} c_{r2} + J &= fc_{01}^2 c_{11} \cdots c_{r-1,1} c_{r2} + J \quad \text{(by induction)} \\
&= sc_{01}c_{02}c_{11} \cdots c_{r-1,1} c_{r2} + J \quad \text{(modulo } h_{13}\text{)} \\
&= sc_{01}c_{02}c_{11} \cdots c_{r-2,1} c_{r-1,4} c_{r1} + J \quad \text{(modulo } h_{r+1,3}\text{)} \\
&= fc_{01}^2 c_{11} \cdots c_{r-2,1} c_{r-1,4} c_{r1} + J \quad \text{(modulo } h_{13}\text{)} \\
&= fc_{01}^2 C_1 \cdots C_{r-1} c_{r1} + J.
\end{aligned}
$$

As each equality above of the form $(a) + J = (b) + J$, with a and b elements of the ring, actually means that $a - b \in J$, the last statement of the lemma follows by induction. ∎

Furthermore,

Lemma 2.2: *$fc_{01}D_0 \subseteq J$.*

Proof:

$$
\begin{aligned}
fc_{01}D_0 &= fc_{01}\left(c_{04} - c_{01}, c_{03} - c_{02}, c_{01} - c_{02}b_{01}^d\right) \\
&\subseteq fc_{01}\left(c_{03} - c_{02}, c_{01} - c_{02}b_{01}^d\right) + J \\
&= (sc_{02}(c_{03} - c_{02}), fc_{01}(c_{01} - c_{02}b_{01}^d)) + J \\
&= fc_{01}(c_{01} - c_{02}b_{01}^d) + J \\
&= (fc_{01}^2 - sc_{01}c_{02}) + J \\
&= J. \quad \blacksquare
\end{aligned}
$$

Lemma 2.3: *For all $r = 1, \ldots, n-1$, and all $i_1, \ldots, i_r \in \{1, 2, 3, 4\}$,*

$$fc_{01}c_{02}b_{03}^d C_1 \cdots C_r \subseteq \left(fc_{01}c_{02}b_{03}^d c_{1i_1} \cdots c_{ri_r}\right) + J,$$
$$fc_{02}^2 b_{02}^{2d} C_1 \cdots C_r \subseteq \left(fc_{02}^2 b_{02}^{2d} c_{1i_1} \cdots c_{ri_r}\right) + J.$$

Also, modulo J,

$$fc_{01}c_{02}b_{03}^d \equiv sc_{02}^2 b_{03}^d \equiv sc_{02}c_{03}b_{03}^d \equiv fc_{01}c_{03}b_{03}^d \equiv sc_{01}c_{03} \equiv fc_{01}^2,$$
$$fc_{01}c_{02}b_{02}^d \equiv sc_{01}c_{02} \equiv fc_{01}^2.$$

Proof: The last congruences are clear. They imply that by Lemma 2.1, $fc_{01}c_{02}b_{03}^d C_1 \cdots C_r$ is contained in $\left(fc_{01}^2 c_{1i_1} \cdots c_{ri_r}\right) + J$, whence again by the congruences it is contained in $\left(fc_{01}c_{02}b_{03}^d c_{1i_1} \cdots c_{ri_r}\right) + J$. The second part follows similarly. ∎

Lemma 2.4: *For all $r = 1, \ldots, n$,*

$$fc_{01}^2 C_1 \cdots C_{r-1} \subseteq fc_{01}c_{02}C_1 \cdots C_{r-1} b_{03}^d b_{13}^d \cdots b_{r-2,3}^d b_{r-1,2}^d + J.$$

Proof: The case $r = 1$ holds as $fc_{01} \in (sc_{02}) + J \subseteq (fc_{02}b_{02}^d) + J$. When $r > 1$, by induction,

$$fc_{01}^2 C_1 \cdots C_{r-1} \subseteq fc_{01}c_{02}C_1 \cdots C_{r-1} b_{03}^d b_{13}^d \cdots b_{r-3,3}^d b_{r-2,2}^d + J.$$

Let $B = b_{13}^d \cdots b_{r-3,3}^d$. Then by Lemma 2.3, $fc_{01}^2 C_1 \cdots C_{r-1}$ is contained in

$$(fc_{01}c_{02}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}b_{03}^d B b_{r-2,2}^d) + J.$$

When $r = 2$, we are done by using h_{13}. Otherwise, for $r > 2$, it follows by Lemma 2.3 that $fc_{01}c_{02}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}b_{03}^d B b_{r-2,2}^d$ is contained in

$$\begin{aligned}
&\subseteq \left(sc_{01}c_{03}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}Bb_{r-2,2}^d\right) + J \\
&\subseteq \left(sc_{01}c_{03}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}Bb_{r-2,3}^d b_{r-1,2}^d\right) + J \quad (\text{modulo } h_{r-1,8}) \\
&\subseteq \left(fc_{01}c_{02}C_1 \cdots C_{r-1}b_{03}^d B b_{r-2,3}^d b_{r-1,2}^d\right) + J \quad (\text{by Lemma 2.3}).
\end{aligned}$$
∎

Lemma 2.5: *For all $r = 1, \ldots, n-1$, and all $i, j = 1, \ldots, 4$,*

$$fc_{01}^3 C_1 \cdots C_r(b_{rj} - b_{ri}) \subseteq J.$$

Proof: When $r = 1$, by Lemma 2.4, $fc_{01}^3 C_1(b_{1j} - b_{1i})$ is contained in $fc_{01}^2 c_{02} C_1 b_{02}^d (b_{1j} - b_{1i}) + J$. As

$$b_{02}(b_{1j} - b_{1i}) \in (b_{1j}(b_{02} - b_{1i}b_{03}), b_{1i}(b_{02} - b_{1j}b_{03})) + J,$$

it suffices to prove that for all $i = 1, \ldots, 4$, $fc_{01}^2 c_{02} C_1(b_{02} - b_{1i}b_{03})$ is contained in J. But by Lemma 2.1, $fc_{01}^2 c_{02} C_1(b_{02} - b_{1i}b_{03}) \in fc_{01}^2 c_{02} c_{1i}(b_{02} - b_{1i}b_{03}) + J$, which is in J modulo $h_{1,6+i}$.

Now let $r > 1$. By Lemma 2.4, $fc_{01}^2 C_1 \cdots C_r \subseteq fc_{01}c_{02}C_1 \cdots C_r b_{03}^d b_{r-1,2}^d + J$. As

$$b_{r-1,2}(b_{rj} - b_{ri}) \in (b_{rj}(b_{r-1,2} - b_{ri}b_{r-1,3}), b_{ri}(b_{r-1,2} - b_{rj}b_{r-1,3})) + J,$$

it follows that by Lemma 2.3, and by using $h_{r,6+i}, h_{r,6+j}$ and h_{13} we are done. ■

Lemma 2.6: *For all $r = 2, \ldots, n-1$, and all $i = 1, \ldots, 4$,*

$$fc_{01}^3 C_1 \cdots C_r (1 - b_{ri}) \subseteq J.$$

Proof: By Lemma 2.4, $fc_{01}^3 C_1 \cdots C_r \subseteq fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d b_{r-1,2} + J$, so it suffices to prove that $fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d b_{r-1,2}(1 - b_{ri})$ is contained in J. But

$$\begin{aligned} &fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d b_{r-1,2}(1 - b_{ri}) \\ &\quad \subseteq fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d \left(b_{ri}(b_{r-1,3} - b_{r-1,2}), b_{r-1,2} - b_{ri}b_{r-1,3}\right) + J. \end{aligned}$$

Note that by Lemma 2.3, by using $h_{r,6+i}$, $fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d \left(b_{r-1,2} - b_{ri}b_{r-1,3}\right)$ is in J. Also, $fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d b_{ri}(b_{r-1,3} - b_{r-1,2})$ is contained in J by Lemma 2.5. ■

Thus

Corollary 2.7: *For all $r = 2, \ldots, n-1$, $fc_{01}^3 C_1 \cdots C_r B_r \subseteq J$.* ■

Now define

$$\begin{aligned} H_1 = &\left(f^3, s - fb_{01}^d\right) + fE + fD_0 + fC_1 F \\ &+ \sum_{i=0}^{n-2} fC_1 \cdots C_i \left(D_{i+1} + B_i\right) + fC_1 C_2 \cdots C_{n-1} B_{n-1}. \end{aligned}$$

It is easy to see that H_1 contains J_1, that $H_1 = (s, f) \cap \bigcap_{i=1}^n \left(p_i + (f^3)\right)$, that each $p_i + (f^3)$ is the intersection of the ideals $P_{i\alpha} + (f^3)$, where the $P_{i\alpha}$ vary over the minimal primes over p_i, and that each $P_{i\alpha} + (f^3)$ is primary to $P_{i\alpha} + (f)$. In particular, c_{01} is a non-zerodivisor modulo H_1. Furthermore,

Proposition 2.8: *With the assumption that $n \geq 2$, $J_1 : c_{01}^\infty = H_1 = J_1 : c_{01}^3$. Thus $J_1 = J + \left(s, fb_{02}^d b_{03}^d\right)^3 + L_1 = H_1 \cap \left(J_1 + (c_{01}^3)\right)$ and*

$$J = \left(\bigcap_{i=-1}^n p_i\right) \cap \left(J_1 + (c_{01}^3)\right),$$

so that in order to find a (possibly redundant) primary decomposition of J_1 and of J, it suffices to find a primary decomposition of $J_1 + (c_{01}^3)$.

Proof: It suffices to prove that $c_{01}^3 H_1 \subseteq J + \left(s, fb_{02}^d b_{03}^d\right)^3 + L_1$. By Lemmas 2.1 and 2.2, and by Corollary 2.7, it only remains to prove that

$$c_{01}^3 \left(\left(f^3, s - fb_{01}^d\right) + fE + fC_1 F\right) \subseteq J + \left(s, fb_{02}^d b_{03}^d\right)^3 + L_1.$$

Clearly $c_{01}(s - fb_{01}^d)$ is in J, and $c_{01}^3 f^3 \in (s^3 c_{02}^3) + J$. Furthermore, by using Lemma 2.5 for $r = 1$, it now remains to prove that

$$fc_{01}^2\left((b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d) + C_1\left(b_{02} - b_{11}b_{03}, b_{12}^d - 1\right)\right) \subseteq J + \left(s, fb_{02}^d b_{03}^d\right)^3 + L_1.$$

This follows from:

$$\begin{aligned}
fc_{01}^2(b_{01} - b_{04}) &\in sc_{01}c_{02}(b_{01} - b_{04}) + J \\
&\subseteq sc_{01}(c_{02}b_{01} - c_{03}b_{04}) + J = J, \\
fc_{01}^2(b_{02}^d - b_{03}^d) &\in sc_{01}c_{02}(b_{02}^d - b_{03}^d) + J \\
&\subseteq sc_{01}(c_{02}b_{02}^d - c_{03}b_{03}^d) + J \\
&\subseteq fc_{01}b_{01}^d(c_{02}b_{02}^d - c_{03}b_{03}^d) + J \\
&\subseteq sc_{01}b_{01}^d(c_{02} - c_{03}) + J = J, \\
fc_{01}^2(b_{01}^d - b_{02}^d) &\in sc_{01}c_{02}(b_{01}^d - b_{02}^d) + J \\
&\subseteq fc_{01}c_{02}b_{02}^d(b_{01}^d - b_{02}^d) + J \\
&\subseteq sc_{01}c_{02}b_{02}^d(1 - 1) + J = J, \\
fc_{01}^2C_1\left(b_{02} - b_{11}b_{03}\right) &\in J \quad \text{(by Lemma 2.3)}, \\
fc_{01}^2C_1\left(b_{12}^d - 1\right) &\in fc_{01}c_{02}b_{02}^dC_1\left(b_{12}^d - 1\right) + J \quad \text{(by Lemma 2.4)} \\
&\subseteq fc_{01}c_{02}b_{02}^dc_{12}\left(b_{12}^d - 1\right) + J \quad \text{(by Lemma 2.1)} \\
&\subseteq fc_{01}c_{02}c_{12}\left(b_{02}^d - b_{12}^d b_{03}^d, b_{12}^d(b_{03}^d - b_{02}^d)\right) + J \\
&\subseteq fc_{01}c_{02}c_{12}b_{12}^d(b_{03}^d - b_{02}^d) + J \quad \text{(modulo } h_{18}\text{)} \\
&\subseteq sc_{02}^2(b_{03}^d - b_{02}^d) + J \\
&= sc_{02}(c_{03}b_{03}^d - c_{02}b_{02}^d) + J \\
&= fc_{01}(c_{03}b_{03}^d - c_{02}b_{02}^d) + J \\
&= sc_{01}(c_{03} - c_{02}) + J = J. \qquad \blacksquare
\end{aligned}$$

Let L_2 by any ideal contained in $J_1 : s$ such that s and f do not appear in any minimal generator of L_2. For example,

$$L_2 = \left(c_{02} - c_{03}, c_{01} - c_{02}b_{01}^d, c_{04} - c_{03}b_{04}^d, c_{1i}(b_{02} - b_{1i}b_{02}) | i = 1, \ldots, 4\right).$$

Also, let L_2' be any ideal in contained in $J_1 : (s, f)$ such that s and f do not appear in any minimal generator of L_2'. For example,

$$L_2' = \left(c_{1i}(b_{02} - b_{1i}b_{02})b_{03}^d | i = 1, \ldots, 4\right).$$

Then define

$$J_2 = J_1 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right) L_2 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 + L_2'.$$

Proposition 2.9: *Then with notation as in Proposition 2.8, $J_1 = H_1 \cap J_2$, so that in order to find a (possibly redundant) primary decomposition of J_1 and of J, it suffices to find a primary decomposition of J_2 as*

$$J = \left(\bigcap_{i=-1}^{n} p_i \right) \cap J_2.$$

Proof: As $J_1 \subseteq H_1$ and H_1 equals the intersection of (s, f) with an ideal properly containing $p_0 \cap \cdots \cap p_n$, it suffices to prove that

$$H_1 \cap \left(\left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right) L_2 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 + L_2' \right) \subseteq J_1.$$

But $H_1 \subseteq (s, f)$, so that the intersection is contained in

$$(s, f) \left(\left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right) L_2 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 + L_2' \right).$$

As $sL_2 \in J$, $(s, f)L_2' \subseteq J_1$ and $f(c_{01} - c_{04}), fc_{01} - sc_{02} \in J$, the ideal above is contained in $J_1 + \left(fc_{01}^3, (s - fb_{01}^d)c_{04}^3\right)$. But

$$(s - fb_{01}^d)c_{04}^3 = c_{04}^2(sc_{04} - fc_{04}b_{01}^d) \in J + fc_{04}^3(b_{04}^d - b_{01}^d) = J + fc_{01}^3(b_{04}^d - b_{01}^d),$$

so that $H_1 \cap J_2$ is contained in $J_1 + H_1 \cap \left(c_{01}^3\right)$, which equals J_1 by the previous proposition. ■

3. Sixteen embedded components of J and J_2

We'll show that for every subset $\Lambda \subseteq \{1, 2, 3, 4\}$,

$$P_{-5,\Lambda} = (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01} - c_{03}b_{04}) + (c_{1i} | i \notin \Lambda) + (b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j} | i, j \in \Lambda)$$

is an associated prime of J and J_2, with its embedded component being

$$p_{-5,\Lambda} = \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}\right) + (c_{1i} | i \notin \Lambda) + (b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j} | i, j \in \Lambda).$$

It is clear that these ideals are prime, respectively primary, and that there are sixteen of each kind. By a proof similar to the one getting the intersection q_4 (see [S]),

$$p_{-5} = \bigcap_{\Lambda} p_{-5,\Lambda} = \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}\right)$$
$$+ (c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}) | i, j = 1, \ldots, 4).$$

Now observe that

$$
\begin{aligned}
J_2 : (c_{02}c_{03}(c_{02}-c_{03}))^\infty &= J_2 : c_{02}c_{03}(c_{02}-c_{03}) \\
&= \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, f(c_{02}b_{01}-c_{03}b_{04}), fc_{1i}(b_{02}-b_{1i}b_{03})\right) \\
&= (s, f, c_{01}, c_{04}, b_{02}^d, b_{03}^d) \cap (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01}-c_{03}b_{04}) \\
&\qquad \cap \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j})\right) \\
&= (s, f, c_{01}, c_{04}, b_{02}^d, b_{03}^d) \cap p_{-3} \cap p_{-5}.
\end{aligned}
$$

None of the components is redundant, which proves that J_2 has the specified embedded components $p_{-5,\Lambda}$. Furthermore,

$$
\begin{aligned}
J : (c_{02}c_{03}(c_{02}-c_{03}))^\infty &= J : c_{02}c_{03}(c_{02}-c_{03}) \\
&= \left(s, fc_{01}, fc_{04}, fb_{02}^d, fb_{03}^d, f(c_{02}b_{01}-c_{03}b_{04}), fc_{1i}(b_{02}-b_{1i}b_{03})\right) \\
&= (s, f) \cap \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, f(c_{02}b_{01}-c_{03}b_{04}), fc_{1i}(b_{02}-b_{1i}b_{03})\right) \\
&= (s, f) \cap (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01}-c_{03}b_{04}) \\
&\qquad \cap \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j})\right) \\
&= p_{-1} \cap p_{-3} \cap p_{-5},
\end{aligned}
$$

and here also the embedded components $p_{-5,\Lambda}$ are not redundant.

Thus so far we have obtained that

$$
J = \left(\bigcap_{i=-1}^{n} p_i\right) \cap p_{-3} \cap p_{-5} \cap J_3,
$$

where p_{-5} is the intersection of 16 embedded components and

$$
\begin{aligned}
J_3 &= J_2 + (c_{02}c_{03}(c_{02}-c_{03}) \\
&= J_1 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right) L_2 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 + L_2' + (c_{02}c_{03}(c_{02}-c_{03})) \\
&= J + \left(s, fb_{02}^d b_{03}^d\right)^3 + L_1 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right) L_2 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 \\
&\qquad + L_2' + (c_{02}c_{03}(c_{02}-c_{03})) \\
&= J + \left(s, fb_{02}^d b_{03}^d\right)^3 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 + (c_{02}c_{03}(c_{02}-c_{03})) \\
&\qquad + \left(s(s-fb_{01}^d), f(s-fb_{02}^d)b_{03}^d, f(s-fb_{03}^d)b_{02}^d, s(s-fb_{02}^d), s(s-fb_{03}^d), s(s-fb_{04}^d)\right) \\
&\qquad + \left(s, c_{01}, c_{04}, c_{02}b_{02}^d, b_{03}^d\right) \left(c_{1i}(b_{02}-b_{1i}b_{03}) | i = 1, \ldots, 4\right) \\
&\qquad + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right) \left(c_{02}-c_{03}, c_{01}-c_{02}b_{01}^d, c_{04}-c_{03}b_{04}^d\right).
\end{aligned}
$$

It remains to find a primary decomposition of J_3.

4. Primary components of J_2 not containing $c_{02}(c_{02}-c_{03})$

Set $H_2 = f\left(b_{02}^d, c_{03}b_{03}^d, b_{01}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}\left(b_{02} - b_{1i}b_{03}\right)|i = 1, \ldots, 4\right)$. For each subset $\Lambda \subseteq \{1, 2, 3, 4\}$, define

$$q_\Lambda = (c_{1i}|i \notin \Lambda) + (b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j}|i, j \in \Lambda) + \left(b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}\right),$$

each of which is a primary ideal containing H_2. By a computation similar to the one for q_{-4} (see [S]),

$$\bigcap_\Lambda q_\Lambda = \left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}|i, j = 1, \ldots, 4\right).$$

Then the following intersection of primary ideals, namely $(\bigcap_\Lambda (q_\Lambda + (c_{01}, c_{04}, s))) \cap p_{-3} \cap p_{-4} \cap (c_{01}, c_{04}, s, f)$, equals

$$\begin{aligned}
&= \left(c_{01}, c_{04}, s, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}(b_{02} - b_{1i}b_{03})|i\right) \cap p_{-4} \cap (c_{01}, c_{04}, s, f) \\
&= \left(\left(c_{01}, c_{04}, s, b_{02}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}(b_{02} - b_{1i}b_{03})|i\right) + (b_{03}^d) \cap p_{-4}\right) \cap (c_{01}, c_{04}, s, f) \\
&= \left(\left(c_{01}, c_{04}, s, b_{02}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}(b_{02} - b_{1i}b_{03})|i\right) + (b_{03}^d)p_{-4}\right) \cap (c_{01}, c_{04}, s, f) \\
&= \left(c_{01}, c_{04}, s, b_{02}^d, c_{03}b_{03}^d, b_{01}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}(b_{02} - b_{1i}b_{03})|i\right) \cap (c_{01}, c_{04}, s, f) \\
&= \left(c_{01}, c_{04}, s, fb_{02}^d, fc_{03}b_{03}^d, fb_{01}b_{03}^d, f(c_{02}b_{01} - c_{03}b_{04}), fc_{1i}(b_{02} - b_{1i}b_{03})|i\right) \\
&= H_2 + (c_{01}, c_{04}, s)
\end{aligned}$$

In particular, this proves that $c_{02}(c_{02} - c_{03})$ is not a zero-divisor modulo $H_2 + (c_{01}, c_{04}, s)$.

Proposition 4.1: *If $c_{03} - c_{02} \in L$ (recall the definition of L from Proposition 2.9), then*

$$\begin{aligned}
&\left(J + \left(s, fb_{02}^d b_{03}^d\right)^3 + (c_{01}, c_{04})L + (c_{01}, c_{04})^3\right) : (c_{02}(c_{02} - c_{03}))^\infty \\
&\qquad = H_2 + (c_{01}, c_{04}, s) \\
&\qquad = \left(J + \left(s, fb_{02}^d b_{03}^d\right)^3 + (c_{01}, c_{04})L + (c_{01}, c_{04})^3\right) : c_{02}(c_{02} - c_{03}).
\end{aligned}$$

Proof: Note that

$$c_{02}(c_{02} - c_{03})\,(c_{01}, c_{04}, s) \subseteq J + (c_{01}, c_{04})L,$$

so it suffices to prove that $c_{02}(c_{02} - c_{03})H_2 \subseteq J + \left(s, fb_{02}^d b_{03}^d\right)^3 + (c_{01}, c_{04})L + (c_{01}, c_{04})^3$. The only element that requires some work is $fb_{01}b_{03}^d$:

$$\begin{aligned}
fb_{01}b_{03}^d c_{02}(c_{02} - c_{03}) &= fb_{03}^d c_{02}(b_{01}c_{02} - b_{04}c_{03}) + fb_{03}^d c_{02}c_{03}(b_{04} - b_{01}) \\
&\in J + (fb_{03}^d c_{02}c_{03}(b_{04} - b_{01})) \\
&\in J + (sc_{02}c_{03}(b_{04} - b_{01})) \\
&\subseteq J + (sc_{03}(c_{03}b_{04} - c_{02}b_{01})) \\
&\subseteq J + (fc_{03}b_{03}^d(c_{03}b_{04} - c_{02}b_{01})) = J. \qquad \blacksquare
\end{aligned}$$

Thus by taking into account all the components of $H_2 + (c_{01}, c_{04}, s)$ we have reduced to the following:

$$J = \left(\bigcap_{i \neq -2} p_i\right) \cap \left(J + \left(s, fb_{02}^d b_{03}^d\right)^3 + (c_{01}, c_{04})L + (c_{01}, c_{04})^3 + (c_{02}(c_{02} - c_{03}))\right),$$

so that in order to find a possibly redundant primary decomposition of J it suffices to find a primary decomposition of

$$J_3 = J + \left(s, fb_{02}^d b_{03}^d\right)^3 + (c_{01}, c_{04})L + (c_{01}, c_{04})^3 + (c_{02}(c_{02} - c_{03})).$$

Now,

$$J_3 : (c_{02} - c_{03})^\infty = J_3 : (c_{02} - c_{03})^2 = (s, c_{01}, c_{02}, c_{04}, fb_{03}^d, fb_{04}).$$

For this it suffices to verify that

$$(c_{02} - c_{03})^2 (s, c_{01}, c_{02}, c_{04}, fb_{03}^d, fb_{04}) \subseteq J_3.$$

The only elements requiring a bit of work are:

$$(c_{02} - c_{03})^2 fb_{03}^d \in J + (c_{02} - c_{03})(fb_{03}^d c_{02} - sc_{03}) \subseteq J + (c_{02} - c_{03})(fb_{03}^d c_{02} - sc_{02}) \subseteq J_3,$$
$$(c_{02} - c_{03})^2 fb_{04} = fc_{02}(c_{02} - c_{03})(b_{04} - b_{01}) + f(c_{02} - c_{03})(c_{02}b_{01} - c_{03}b_{04}) \in J_3,$$

which verifies the claim. The primary decomposition of $(s, c_{01}, c_{02}, c_{04}, fb_{03}^d, fb_{04})$ is

$$(s, c_{01}, c_{02}, c_{04}, f) \cap (s, c_{01}, c_{02}, c_{04}, b_{03}^d, b_{04}),$$

where the former ideal properly contains p_{-1}, the P_{-1}-primary component of J, and the latter ideal is p_{-2}, primary to the prime ideal P_{-2}. Thus so far we have found all the minimal components of J and many redundant ones:

$$J = \left(\bigcap_{i=-4}^{n} p_i\right) \cap J_4,$$

where $J_4 = J + \left(s, fb_{02}^d b_{03}^d\right)^3 + (c_{01}, c_{04})L + (c_{01}, c_{04})^3 + (c_{02}, c_{03})(c_{02} - c_{03})$. So it suffices to find a primary decomposition of J_4.

Now observe that $J_4 : c_{02}^\infty b_{02}^\infty = (s, f, c_{02} - c_{03}, c_{04}^3, c_{04}^2 b_{01}^d) = J_4 : c_{02}^3 b_{02}^{3d}$, and for this it suffices to prove that $c_{02}^3 b_{02}^{3d}(s, f, c_{02} - c_{03}, c_{04}^3, c_{04}^2 b_{01}^d)$ is in J, but that is straightforward. As this colon ideal contains the intersection of the minimal components of J, in order to find a primary decomposition of J it now suffices to find a primary decomposition of

$$J + \left(s, fb_{02}^d b_{03}^d\right)^3 + (c_{01}, c_{04})L + (c_{01}, c_{04})^3 + (c_{02}, c_{03})(c_{02} - c_{03}) + \left(c_{02}^3 b_{02}^{3d}\right).$$

Sections 2, 3, 4 repeated more greadily

5. Primary components of J not containing a power of s

Lemma 5.1: *For* $2 \le r \le n$,

$$p_1 \cap p_2 \cap \cdots \cap p_r = E + D_0 + C_1 F + \sum_{i=0}^{r-1} C_1 C_2 \cdots C_i \left(D_{i+1} + B_i\right) + C_1 C_2 \cdots C_r.$$

Proof: When $r = 2$,

$$\begin{aligned} p_1 \cap p_2 &= (C_1 + E + D_0) \cap (C_2 + E + F + D_0 + D_1 + B_1) \\ &= E + D_0 + C_1 \cap (C_2 + E + F + D_0 + D_1 + B_1) \\ &= E + D_0 + D_1 + C_1 \cap (C_2 + E + F + D_0 + B_1) \\ &= E + D_0 + D_1 + C_1 \cdot (C_2 + E + F + D_0 + B_1) \\ &= E + D_0 + D_1 + C_1 F + C_1 \cdot (C_2 + B_1)\,, \end{aligned}$$

which starts the induction. Now by induction assumption for some $r \ge 2$ and $r \le n-1$,

$$\begin{aligned} p_1 \cap \cdots \cap p_{r+1} &= \left(E + D_0 + C_1 F + \sum_{i=0}^{r-1} C_1 C_2 \cdots C_i \left(D_{i+1} + B_i\right) + C_1 \cdots C_r\right) \cap p_{r+1} \\ &= E + D_0 + C_1 F + \sum_{i=0}^{r-1} C_1 C_2 \cdots C_i \left(D_{i+1} + B_i\right) + C_1 \cdots C_r \cap p_{r+1}. \end{aligned}$$

But by multihomogeneity, the last intersection equals

$$C_1 \cdots C_r \left(C_{r+1} + E + F + B_r\right) + \sum_{j=1}^{r} \left(D_j \prod_{k \ne j}^{r} C_k\right).$$

Combining the last two displays proves the lemma. ■

Now it follows easily with a similar proof that

Lemma 5.2: *For* $n \ge 2$,

$$p_1 \cap p_2 \cap \cdots \cap p_n = E + D_0 + C_1 F + \sum_{i=0}^{n-1} C_1 C_2 \cdots C_i \left(D_{i+1} + B_i\right),$$

and $p_0 \cap p_1 \cap \cdots \cap p_n = D_0 + C_0 \cdot (p_1 \cap \cdots \cap p_n)$. ■

We'll prove in the rest of this section that $\cap_{i=0}^{n} p_i = J : s^\infty = J : s^3$.

Lemma 5.3: *For all* $r = 0, 1, \ldots, n-1$,

$$s^2 C_0 C_1 \cdots C_{r-1} D_r \subseteq J.$$

Furthermore, for all $r = 0, 1, \ldots, n-1$, *and all* $i_1, \ldots, i_r \in \{1, 2, 3, 4\}$,

$$\begin{aligned} s^2 C_0 C_1 \cdots C_r + J &= (sfc_{01} c_{1i_1} \cdots c_{ri_r}) + J \\ &= (sfc_{04} c_{1i_1} \cdots c_{ri_r}) + J \\ &= (s^2 c_{02} c_{1i_1} \cdots c_{ri_r}) + J \\ &= (s^2 c_{03} c_{1i_1} \cdots c_{ri_r}) + J, \end{aligned}$$

and when $r = 0$,

$$sC_0 + J = (fc_{01}) + J = (fc_{04}) + J = (sc_{02}) + J = (sc_{03}) + J.$$

Proof: The case $r = 0$ is straightforward due to $h_{01}, h_{02}, h_{04}, h_{13}, h_{14}$ and h_{15} (for example, $s^2(c_{01} - c_{02} b_{01}^d) \in (h_{01}, s(fc_{01} - sc_{02}) b_{01}^d) \subseteq J$).

Now let $r > 0$. Then $D_r = (c_{r4} - c_{r1}, c_{r3} - c_{r2}, c_{r2} - c_{r1})$. By above it suffices to prove only that $s^2 C_0 C_1 \cdots C_r + J \subseteq (sfc_{01} c_{1i_1} \cdots c_{ri_r}) + J$. By induction, $s^2 C_0 C_1 \cdots C_{r-1} D_r$ is contained in

$$sfc_{01} c_{11} \cdots c_{r-1,1} (c_{r3} - c_{r2}, c_{r2} - c_{r1}) + sfc_{01} c_{11} \cdots c_{r-2,1} c_{r-1,4} (c_{r4} - c_{r1}) + J.$$

First note that

$$sfc_{01} c_{11} \cdots c_{r-2,1} c_{r-1,4} (c_{r4} - c_{r1}) \in sfc_{01} c_{11} \cdots c_{r-2,1} c_{r-1,2} (c_{r3} - c_{r2}) + J,$$

and that $sfc_{01} c_{11} \cdots c_{r-1,1} (c_{r3} - c_{r2})$ is contained in J due to $h_{r+1,5}$. Finally,

$$sfc_{01} c_{11} \cdots c_{r-1,1} (c_{r2} - c_{r1}) \in sfc_{01} c_{11} \cdots c_{r-2,1} (c_{r-1,4} - c_{r-1,1}) c_{r1} + J$$

modulo $h_{r+1,3}$, and that is contained, by induction on r, in J. ■

Lemma 5.4: $s^2 C_0 E \subseteq J$ *and* $sfc_{02} \left(b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d \right) \subseteq J$.

Proof: By Lemma 5.3,

$$\begin{aligned} sC_0 E &\subseteq fc_{01} \left(s - fb_{01}^d, b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d \right) + J \\ &\subseteq fc_{01} \left(b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d \right) + J. \end{aligned}$$

Thus as $sc_{01} \in (fc_{01}, h_{01})$, $s^2 C_0 E \subseteq f^2 c_{01} \left(b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d \right) + J$, and as $fc_{01} \in (sc_{02}, h_{13})$, it remains to prove the second part of the lemma. But $sfc_{02} (b_{01} - b_{04})$ equals $f(sc_{02} b_{01} - sc_{03} b_{04})$ modulo h_{15}, and that is in J. Also, $sfc_{02} \left(b_{02}^d - b_{03}^d \right) \in \left(sfc_{02} b_{02}^d - sfc_{03} b_{03}^d \right) + J = \left(s^2 c_{02} - s^2 c_{03} \right) + J = J$, and finally $sfc_{02} \left(b_{01}^d - b_{02}^d \right) \in \left(f^2 c_{01} b_{01}^d - sfc_{02} b_{02}^d \right) + J = \left(sfc_{01} - s^2 c_{02} \right) + J = J$. ■

Lemma 5.5: *For all* $r = 2, \ldots, n$,

$$s^2 C_0 C_1 \cdots C_{r-1} \subseteq sfc_{04} C_1 \cdots C_{r-1} + J \subseteq sfc_{04} C_1 \cdots C_{r-1} b_{r-1,2}^d + J.$$

Also, $s^2C_0 \subseteq (sfc_{02}b_{02}^d) + J$.

Proof: When $r = 1$, s^2C_0 is contained in $(s^2c_{02}) + J$ by Lemma 5.3, which, modulo h_{02} is contained in $(sfc_{02}b_{02}^d) + J$, as desired.

Now let $r \geq 2$. By Lemma 5.3 it suffices to prove the second inclusion.

When $r = 2$, By Lemma 5.3, $sfc_{04}C_1$ is contained in $(s^2c_{02}c_{12}) + J$, whence modulo h_{02} in $(sfc_{02}c_{12}b_{02}^d)+J$, and then modulo h_{18} in $(sfc_{02}c_{12}b_{12}^d)+J$, and then by Lemma 5.3 in $(f^2c_{04}C_1b_{12}^d) + J$.

Now let $r > 2$. By induction on r, $sfc_{04}C_1 \cdots C_{r-1} \subseteq sfc_{04}C_1 \cdots C_{r-1}b_{r-2,2}^d + J$. By Lemma 5.3, the latter is contained in $(sfc_{01}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}b_{r-2,2}^d) + J$, and modulo $h_{r-1,8}$, this equals $(sfc_{01}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}b_{r-1,2}^d b_{r-2,3}^d) + J$, whence is in $sfc_{04}C_1 \cdots C_{r-1}b_{r-1,2}^d + J$, as desired. ■

Lemma 5.6: *For all* $r = 2, \ldots, n-1$, *and all* $i, j = 1, \ldots, 4$,

$$s^2C_0C_1 \cdots C_r(b_{rj} - b_{ri}) \subseteq J,$$
$$sfc_{04}C_1 \cdots C_rb_{r-1,2}(b_{rj} - b_{ri}) \subseteq J.$$

Also,

$$s^3C_0C_1(b_{1j} - b_{1i}) \subseteq J,$$
$$s^2fc_{04}C_1b_{02}(b_{1j} - b_{1i}) \subseteq J.$$

Proof: By Lemma 5.5, $s^2C_0C_1 \cdots C_r \subseteq sfc_{04}C_1 \cdots C_rb_{r-1,2}^d + J$, so it suffices to prove the second inclusion. But

$$b_{r-1,2}(b_{rj} - b_{ri}) \in (b_{rj}(b_{r-1,2} - b_{ri}b_{r-1,3}), b_{ri}(b_{r-1,2} - b_{rj}b_{r-1,3})) + J,$$

whence by Lemma 5.3 and by using $h_{r,6+i}$ and $h_{r,6+j}$ we are done. The last part follows similarly. ■

Remark 5.7: *A calculation by Macaulay2 shows that* s^3 *is needed in the lemma above. For this reason, we need* s^3 *also below:*

Lemma 5.8: *For all* $r = 2, \ldots, n-1$, *and all* $i = 1, \ldots, 4$,

$$s^3C_0C_1 \cdots C_r(1 - b_{ri}) \subseteq J.$$

Proof: By Lemma 5.5, $s^2C_0C_1 \cdots C_r \subseteq sfc_{04}C_1 \cdots C_rb_{r-1,2}^d + J$, so it suffices to prove that $s^2fc_{04}C_1 \cdots C_rb_{r-1,2}^d(1 - b_{ri})$ is contained in J. But

$$\begin{aligned} &sfc_{04}C_1 \cdots C_rb_{r-1,2}^d(1 - b_{ri}) \\ &\quad \subseteq sfc_{04}C_1 \cdots C_rb_{r-1,2}^{d-1}\left(b_{ri}(b_{r-1,3} - b_{r-1,2}), b_{r-1,2} - b_{ri}b_{r-1,3}\right) + J, \end{aligned}$$

where $sfc_{04}C_1\cdots C_r b_{r-1,2}^{d-1}(b_{r-1,2}-b_{ri}b_{r-1,3})$ is contained in J by Lemma 5.3 and modulo $h_{r,6+i}$. Also,

$$s^2fc_{04}C_1\cdots C_r(b_{r-1,3}-b_{r-1,2})\subseteq s^2fc_{04}C_1\cdots C_r b_{r-2,2}(b_{r-1,3}-b_{r-1,2})+J$$

by Lemma 5.5, and the latter ideal is contained in J by Lemma 5.6. ■

Remark 5.9: *Note that in the lemma above, s^2 works if $r>2$.*

Lemma 5.10: *For all $i=1,\ldots,4$, $s^2C_0C_1\left(b_{02}-b_{1i}b_{03}, b_{1i}^d-1, b_{14}-b_{11}\right)\subseteq J$. If $n=2$, $s^2C_0C_1\left(b_{13}-b_{12}\right)\subseteq J$. In general, $s^3C_0C_1F\subseteq J$.*

Proof: Recall that $F=\left(b_{02}-b_{11}b_{03}, b_{14}-b_{11}, b_{13}-b_{11}, b_{12}-b_{11}, 1-b_{12}^d\right)$. By applying Lemma 5.6,

$$s^3C_0C_1\left(b_{14}-b_{11}, b_{13}-b_{11}, b_{12}-b_{11}\right)\subseteq J.$$

When $n=2$, by Lemma 5.3, $s^2C_0C_1\left(b_{13}-b_{12}\right)\subseteq J$ by using h_{27}. For arbitrary n, $s^2C_0C_1\left(b_{14}-b_{11}\right)\subseteq J$ by Lemma 5.3 by using h_{26}. Also by Lemma 5.3,

$$s^2C_0C_1\left(b_{02}-b_{1i}b_{03}\right)\subseteq s^2c_{02}c_{1i}\left(b_{02}-b_{1i}b_{03}\right)+J\subseteq sfc_{02}b_{02}^dc_{1i}\left(b_{02}-b_{1i}b_{03}\right)+J=J,$$

and finally,

$$\begin{aligned}
s^2C_0C_1\left(1-b_{1i}^d\right) &\subseteq s^2c_{02}c_{1i}\left(1-b_{1i}^d\right)+J \quad \text{(by Lemma 5.3)}\\
&\subseteq sfc_{02}c_{1i}b_{02}^d\left(1-b_{1i}^d\right)+J \quad (\text{modulo } h_{02})\\
&\subseteq sfc_{02}c_{1i}\left(b_{02}^d-b_{1i}^db_{03}^d, b_{1i}^d(b_{03}^d-b_{02}^d)\right)+J\\
&\subseteq sfc_{02}c_{1i}b_{1i}^d(b_{03}^d-b_{02}^d)+J \quad (\text{modulo } h_{18})\\
&\subseteq J \quad \text{(by Lemma 5.4).}
\end{aligned}$$ ■

Thus it follows from all these lemmas and especially Lemma 5.2, that

Proposition 5.11: *Assume that $n\geq 2$. Then $s^3\left(\bigcap_{i=0}^n p_i\right)\subseteq J$. Thus*

$$J=\left(\bigcap_{i=0}^n p_i\right)\bigcap\left(J+(s^3)\right),$$

so that in order to find a (possibly redundant) primary decomposition of J, it suffices to find a primary decomposition of $J+(s^3)$.

Proof: Lemmas 5.2, 5.3, 5.4, 5.8, and 5.10 show that $\bigcap_{i=0}^n p_i$ is contained in $J:s^\infty$. Thus as $J\subseteq\bigcap_{i=0}^n p_i$, $J:s^\infty=\left(\bigcap_{i=0}^n p_i\right):s^\infty$. But s is a non-zerodivisor modulo $\bigcap_{i=0}^n p_i$, so that $J:s^\infty=\bigcap_{i=0}^n p_i$, and that equals $J:s^3$ by lemmas 5.2, 5.3, 5.4, 5.8, and 5.10. The rest follows from Lemma 0.1. ■

Now let L_1 be an ideal in $J : C_0$ such that no c_{0i} appears in any minimal generator of L_1. For example,

$$L_1 = \left(s, fb_{03}^d, fb_{04}\right)(c_{1i}(b_{02} - b_{1i}b_{03})|i = 1, \ldots, 4) + \left(s(s - fb_{01}^d), s(s - fb_{04}^d)\right) + \left(s - fb_{02}^d\right)\left(s, fb_{03}^d, fb_{04}\right) + \left(s - fb_{03}^d\right)\left(s, fb_{02}^d\right)$$

(Note that we could enlarge L_1 by adding also $fb_{01}\left(s - fb_{03}^d\right)$, but this addition seems to add a redundant primary component in the next step.) Then set

$$J_1 = J + \left(s, fb_{02}^d b_{03}^d, fb_{02}^d b_{04}\right)^3 + L_1.$$

Proposition 5.12: *For $n \geq 2$,*

$$J = \left(\bigcap_{i=0}^{n} p_i\right) \cap \left(J + \left(s, fb_{02}^d b_{03}^d, fb_{02}^d b_{04}\right)^3 + L_1\right),$$

so that in order to find a (possibly redundant) primary decomposition of J, it suffices to find a primary decomposition of $J_1 = J + \left(s, fb_{02}^d b_{03}^d, fb_{02}^d b_{04}\right)^3 + L_1$.

Proof: First note that $(\bigcap_{i=0}^{n} p_i) \bigcap J_1$ is contained in

$$\begin{aligned}(c_{01}, c_{02}, c_{03}, c_{04}) \cap J_1 &= J + (c_{01}, c_{02}, c_{03}, c_{04}) \bigcap \left(\left(s, fb_{02}^d b_{03}^d, fb_{02}^d b_{04}\right)^3 + L_1\right) \\ &= J + (c_{01}, c_{02}, c_{03}, c_{04}) \cdot \left(\left(s, fb_{02}^d b_{03}^d, fb_{02}^d b_{04}\right)^3 + L_1\right),\end{aligned}$$

and this is contained in $J + (c_{01}, c_{02}, c_{03}, c_{04})(s)^3$. Now the proposition follows from the previous one. ■

6. Primary components of $J_1 = J + \left(s, fb_{02}^d b_{03}^d, fb_{02}^d b_{04}\right)^3 + L_1$ not containing c_{01}

Lemma 6.1: *For all $r = 1, \ldots, n-1$, and all $i_1, \ldots, i_r \in \{1, 2, 3, 4\}$,*

$$fc_{01}^2 C_1 \cdots C_r + J = \left(fc_{01}^2 c_{1i_1} \cdots c_{ri_r}\right) + J.$$

Furthermore, for all $r = 1, \ldots, n-1$,

$$fc_{01}^2 C_1 \cdots C_{r-1} D_r \subseteq J.$$

Proof: When $r = 1$,

$(fc_{01}^2 c_{14}) + J$

$$
\begin{aligned}
&= (fc_{01}(c_{01}-c_{04})c_{14} + fc_{01}(c_{04}c_{14}-c_{01}c_{13}) + fc_{01}c_{01}c_{13}) + J \\
&= (sc_{02}(c_{04}c_{14}-c_{01}c_{13}), fc_{01}^2c_{13}) + J \\
&= (fc_{01}^2c_{13}) + J, \\
&= (c_{01}c_{13}(fc_{01}-sc_{02}) + sc_{01}c_{02}(c_{13}-c_{12}) - c_{01}c_{12}(fc_{01}-sc_{02}) + fc_{01}^2c_{12}) + J \\
&= (fc_{01}^2c_{12}) + J, \\
&= (fc_{01}(c_{01}c_{12}-c_{04}c_{11}) + fc_{01}(c_{04}-c_{01})c_{11} + fc_{01}^2c_{11}) + J \\
&= (fc_{01}^2c_{11}) + J,
\end{aligned}
$$

which proves the first part of the lemma for $r = 1$. Now let $r > 1$. Then

$$
\begin{aligned}
fc_{01}^2C_1\cdots C_{r-1}c_{r4} + J &= fc_{01}^2c_{11}\cdots c_{r-2,1}c_{r-1,4}c_{r4} + J \quad \text{(by induction)} \\
&= sc_{01}c_{02}c_{11}\cdots c_{r-2,1}c_{r-1,4}c_{r4} + J \quad (\text{modulo } h_{13}) \\
&= sc_{01}c_{02}c_{11}\cdots c_{r-2,1}c_{r-1,1}c_{r3} + J \quad (\text{modulo } h_{r+1,4}) \\
&= fc_{01}^2c_{11}\cdots c_{r-2,1}c_{r-1,1}c_{r3} + J \quad (\text{modulo } h_{13}) \\
&= fc_{01}^2C_1\cdots C_{r-1}c_{r3} + J, \\
fc_{01}^2C_1\cdots C_{r-1}c_{r3} + J &= fc_{01}^2c_{11}\cdots c_{r-1,1}c_{r3} + J \quad \text{(by induction)} \\
&= sc_{01}c_{02}c_{11}\cdots c_{r-1,1}c_{r3} + J \quad (\text{modulo } h_{13}) \\
&= sc_{01}c_{02}c_{11}\cdots c_{r-1,1}c_{r2} + J \quad (\text{modulo } h_{r+1,5}) \\
&= fc_{01}^2c_{11}\cdots c_{r-1,1}c_{r2} + J \quad (\text{modulo } h_{13}) \\
&= fc_{01}^2C_1\cdots C_{r-1}c_{r2} + J, \\
fc_{01}^2C_1\cdots C_{r-1}c_{r2} + J &= fc_{01}^2c_{11}\cdots c_{r-1,1}c_{r2} + J \quad \text{(by induction)} \\
&= sc_{01}c_{02}c_{11}\cdots c_{r-1,1}c_{r2} + J \quad (\text{modulo } h_{13}) \\
&= sc_{01}c_{02}c_{11}\cdots c_{r-2,1}c_{r-1,4}c_{r1} + J \quad (\text{modulo } h_{r+1,3}) \\
&= fc_{01}^2c_{11}\cdots c_{r-2,1}c_{r-1,4}c_{r1} + J \quad (\text{modulo } h_{13}) \\
&= fc_{01}^2C_1\cdots C_{r-1}c_{r1} + J.
\end{aligned}
$$

As each equality above of the form $(a) + J = (b) + J$, with a and b elements of the ring, actually means that $a - b \in J$, the last statement of the lemma follows by induction. ∎

Furthermore,

Lemma 6.2: $fc_{01}D_0 \subseteq J$.

Proof:

$$
\begin{aligned}
fc_{01}D_0 &= fc_{01}\left(c_{04}-c_{01}, c_{03}-c_{02}, c_{01}-c_{02}b_{01}^d\right) \\
&\subseteq fc_{01}\left(c_{03}-c_{02}, c_{01}-c_{02}b_{01}^d\right) + J \\
&= (sc_{02}(c_{03}-c_{02}), fc_{01}(c_{01}-c_{02}b_{01}^d)) + J \\
&= fc_{01}(c_{01}-c_{02}b_{01}^d) + J \\
&= (fc_{01}^2 - sc_{01}c_{02}) + J \\
&= J. \quad ∎
\end{aligned}
$$

Lemma 6.3: *For all* $r = 1, \ldots, n-1$, *and all* $i_1, \ldots, i_r \in \{1, 2, 3, 4\}$,

$$fc_{01}c_{02}b_{03}^dC_1 \cdots C_r \subseteq \left(fc_{01}c_{02}b_{03}^dc_{1i_1} \cdots c_{ri_r}\right) + J,$$
$$fc_{02}^2b_{02}^{2d}C_1 \cdots C_r \subseteq \left(fc_{02}^2b_{02}^{2d}c_{1i_1} \cdots c_{ri_r}\right) + J.$$

Also, modulo J,

$$fc_{01}c_{02}b_{03}^d \equiv sc_{02}^2b_{03}^d \equiv sc_{02}c_{03}b_{03}^d \equiv fc_{01}c_{03}b_{03}^d \equiv sc_{01}c_{03} \equiv fc_{01}^2,$$
$$fc_{01}c_{02}b_{02}^d \equiv sc_{01}c_{02} \equiv fc_{01}^2.$$

Proof: The last congruences are clear. They imply that by Lemma 6.1, $fc_{01}c_{02}b_{03}^dC_1 \cdots C_r$ is contained in $\left(fc_{01}^2c_{1i_1} \cdots c_{ri_r}\right) + J$, whence again by the congruences it is contained in $\left(fc_{01}c_{02}b_{03}^dc_{1i_1} \cdots c_{ri_r}\right) + J$. The second part follows similarly. ∎

Lemma 6.4: *For all* $r = 1, \ldots, n$,

$$fc_{01}^2C_1 \cdots C_{r-1} \subseteq fc_{01}c_{02}C_1 \cdots C_{r-1}b_{03}^db_{13}^d \cdots b_{r-2,3}^db_{r-1,2}^d + J.$$

Proof: The case $r = 1$ holds as $fc_{01} \in (sc_{02}) + J \subseteq (fc_{02}b_{02}^d) + J$. When $r > 1$, by induction,

$$fc_{01}^2C_1 \cdots C_{r-1} \subseteq fc_{01}c_{02}C_1 \cdots C_{r-1}b_{03}^db_{13}^d \cdots b_{r-3,3}^db_{r-2,2}^d + J.$$

Let $B = b_{13}^d \cdots b_{r-3,3}^d$. Then by Lemma 6.3, $fc_{01}^2C_1 \cdots C_{r-1}$ is contained in

$$(fc_{01}c_{02}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}b_{03}^dBb_{r-2,2}^d) + J.$$

When $r = 2$, we are done by using h_{13}. Otherwise, for $r > 2$, it follows by Lemma 6.3 that $fc_{01}c_{02}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}b_{03}^dBb_{r-2,2}^d$ is contained in

$$\begin{aligned}
&\subseteq \left(sc_{01}c_{03}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}Bb_{r-2,2}^d\right) + J \\
&\subseteq \left(sc_{01}c_{03}c_{11} \cdots c_{r-4,1}c_{r-3,4}c_{r-2,2}c_{r-1,2}Bb_{r-2,3}^db_{r-1,2}^d\right) + J \quad (\text{modulo } h_{r-1,8}) \\
&\subseteq \left(fc_{01}c_{02}C_1 \cdots C_{r-1}b_{03}^dBb_{r-2,3}^db_{r-1,2}^d\right) + J \quad (\text{by Lemma 6.3}).
\end{aligned}$$
∎

Lemma 6.5: *For all* $r = 1, \ldots, n-1$, *and all* $i, j = 1, \ldots, 4$,

$$fc_{01}^3C_1 \cdots C_r(b_{rj} - b_{ri}) \subseteq J.$$

Proof: When $r = 1$, by Lemma 6.4, $fc_{01}^3C_1(b_{1j} - b_{1i})$ is contained in $fc_{01}^2c_{02}C_1b_{02}^d(b_{1j} - b_{1i}) + J$. As

$$b_{02}(b_{1j} - b_{1i}) \in (b_{1j}(b_{02} - b_{1i}b_{03}), b_{1i}(b_{02} - b_{1j}b_{03})) + J,$$

it suffices to prove that for all $i = 1, \ldots, 4$, $fc_{01}^2c_{02}C_1(b_{02} - b_{1i}b_{03})$ is contained in J. But by Lemma 6.1, $fc_{01}^2c_{02}C_1(b_{02} - b_{1i}b_{03}) \in fc_{01}^2c_{02}c_{1i}(b_{02} - b_{1i}b_{03}) + J$, which is in J modulo $h_{1,6+i}$.

Now let $r > 1$. By Lemma 6.4, $fc_{01}^2 C_1 \cdots C_r \subseteq fc_{01}c_{02}C_1 \cdots C_r b_{03}^d b_{r-1,2}^d + J$. As

$$b_{r-1,2}(b_{rj} - b_{ri}) \in (b_{rj}(b_{r-1,2} - b_{ri}b_{r-1,3}), b_{ri}(b_{r-1,2} - b_{rj}b_{r-1,3})) + J,$$

it follows that by Lemma 6.3, and by using $h_{r,6+i}, h_{r,6+j}$ and h_{13} we are done. ∎

Lemma 6.6: *For all $r = 2, \ldots, n-1$, and all $i = 1, \ldots, 4$,*

$$fc_{01}^3 C_1 \cdots C_r (1 - b_{ri}) \subseteq J.$$

Proof: By Lemma 6.4, $fc_{01}^3 C_1 \cdots C_r \subseteq fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d b_{r-1,2} + J$, so it suffices to prove that $fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d b_{r-1,2}(1 - b_{ri})$ is contained in J. But

$$\begin{aligned} &fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d b_{r-1,2}(1 - b_{ri}) \\ &\quad \subseteq fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d \left(b_{ri}(b_{r-1,3} - b_{r-1,2}), b_{r-1,2} - b_{ri}b_{r-1,3}\right) + J. \end{aligned}$$

Note that by Lemma 6.3, by using $h_{r,6+i}$, $fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d \left(b_{r-1,2} - b_{ri}b_{r-1,3}\right)$ is in J. Also, $fc_{01}^2 c_{02} C_1 \cdots C_r b_{03}^d b_{ri}(b_{r-1,3} - b_{r-1,2})$ is contained in J by Lemma 6.5. ∎

Thus

Corollary 6.7: *For all $r = 2, \ldots, n-1$, $fc_{01}^3 C_1 \cdots C_r B_r \subseteq J$.* ∎

Now define

$$\begin{aligned} H_1 = &\left(f^3, s - fb_{01}^d\right) + fE + fD_0 + fC_1 F \\ &+ \sum_{i=0}^{n-2} fC_1 \cdots C_i \left(D_{i+1} + B_i\right) + fC_1 C_2 \cdots C_{n-1} B_{n-1}. \end{aligned}$$

It is easy to see that H_1 contains J_1, that $H_1 = (s, f) \cap \bigcap_{i=1}^n \left(p_i + (f^3)\right)$, that each $p_i + (f^3)$ is the intersection of the ideals $P_{i\alpha} + (f^3)$, where the $P_{i\alpha}$ vary over the minimal primes over p_i, and that each $P_{i\alpha} + (f^3)$ is primary to $P_{i\alpha} + (f)$. In particular, c_{01} is a non-zerodivisor modulo H_1. Furthermore,

Proposition 6.8: *With the assumption that $n \geq 2$, $J_1 : c_{01}^\infty = H_1 = J_1 : c_{01}^3$. Thus $J_1 = J + \left(s, fb_{02}^d b_{03}^d, fb_{02}^d b_{04}\right)^3 + L_1 = H_1 \cap \left(J_1 + (c_{01}^3)\right)$ and*

$$J = \left(\bigcap_{i=-1}^{n} p_i\right) \cap \left(J_1 + (c_{01}^3)\right),$$

so that in order to find a (possibly redundant) primary decomposition of J_1 and of J, it suffices to find a primary decomposition of $J_1 + (c_{01}^3)$.

Proof: It suffices to prove that $c_{01}^3 H_1 \subseteq J_1$. By Lemmas 6.1 and 6.2, and by Corollary 6.7, it only remains to prove that

$$c_{01}^3 \left(\left(f^3, s - fb_{01}^d\right) + fE + fC_1 F\right) \subseteq J_1.$$

Clearly $c_{01}(s - fb_{01}^d)$ is in J, and $c_{01}^3 f^3 \in (s^3 c_{02}^3) + J$. Furthermore, by using Lemma 6.5 for $r = 1$, it now remains to prove that

$$fc_{01}^2 \left((b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d) + C_1 \left(b_{02} - b_{11} b_{03}, b_{12}^d - 1 \right) \right) \subseteq J_1.$$

This follows from:

$$\begin{aligned}
fc_{01}^2(b_{01} - b_{04}) &\in sc_{01}c_{02}(b_{01} - b_{04}) + J \\
&\subseteq sc_{01}(c_{02}b_{01} - c_{03}b_{04}) + J = J, \\
fc_{01}^2(b_{02}^d - b_{03}^d) &\in sc_{01}c_{02}(b_{02}^d - b_{03}^d) + J \\
&\subseteq sc_{01}(c_{02}b_{02}^d - c_{03}b_{03}^d) + J \\
&\subseteq fc_{01}b_{01}^d(c_{02}b_{02}^d - c_{03}b_{03}^d) + J \\
&\subseteq sc_{01}b_{01}^d(c_{02} - c_{03}) + J = J, \\
fc_{01}^2(b_{01}^d - b_{02}^d) &\in sc_{01}c_{02}(b_{01}^d - b_{02}^d) + J \\
&\subseteq fc_{01}c_{02}b_{02}^d(b_{01}^d - b_{02}^d) + J \\
&\subseteq sc_{01}c_{02}b_{02}^d(1 - 1) + J = J, \\
fc_{01}^2 C_1 \left(b_{02} - b_{11} b_{03} \right) &\in J \quad \text{(by Lemma 6.3)}, \\
fc_{01}^2 C_1 \left(b_{12}^d - 1 \right) &\in fc_{01}c_{02}b_{02}^d C_1 \left(b_{12}^d - 1 \right) + J \quad \text{(by Lemma 6.4)} \\
&\subseteq fc_{01}c_{02}b_{02}^d c_{12} \left(b_{12}^d - 1 \right) + J \quad \text{(by Lemma 6.1)} \\
&\subseteq fc_{01}c_{02}c_{12} \left(b_{02}^d - b_{12}^d b_{03}^d, b_{12}^d(b_{03}^d - b_{02}^d) \right) + J \\
&\subseteq fc_{01}c_{02}c_{12}b_{12}^d(b_{03}^d - b_{02}^d) + J \quad \text{(modulo } h_{18}) \\
&\subseteq sc_{02}^2(b_{03}^d - b_{02}^d) + J \\
&= sc_{02}(c_{03}b_{03}^d - c_{02}b_{02}^d) + J \\
&= fc_{01}(c_{03}b_{03}^d - c_{02}b_{02}^d) + J \\
&= sc_{01}(c_{03} - c_{02}) + J = J. \qquad \blacksquare
\end{aligned}$$

Let L_2 by any ideal contained in $J_1 : s$ such that s and f do not appear in any minimal generator of L_2. For example,

$$L_2 = \left(c_{02} - c_{03}, c_{01} - c_{02} b_{01}^d, c_{04} - c_{03} b_{04}^d, c_{1i}(b_{02} - b_{1i} b_{02}) | i = 1, \ldots, 4 \right).$$

Also, let L_2' be any ideal in contained in $J_1 : (s, f)$ such that s and f do not appear in any minimal generator of L_2'. For example,

$$L_2' = (c_{02}, b_{03}^d, b_{04}) \left(c_{1i}(b_{02} - b_{1i} b_{03}) | i = 1, \ldots, 4 \right) + C_0 \left(c_{02} b_{01} - c_{03} b_{04}, c_{01} - c_{04} \right).$$

Then define

$$J_2 = J_1 + \left(c_{01}, c_{04}, c_{02} b_{02}^d, c_{03} b_{03}^d \right) L_2 + \left(c_{01}, c_{04}, c_{02} b_{02}^d, c_{03} b_{03}^d \right)^3 + L_2'.$$

Proposition 6.9: *With notation as in Proposition 6.8, $J_1 = H_1 \cap J_2$, so that*

$$J = \left(\bigcap_{i=-1}^{n} p_i \right) \cap J_2.$$

Hence in order to find a (possibly redundant) primary decomposition of J_1 and of J, it suffices to find a primary decomposition of J_2.

Proof: By Lemma 0.1 and as H_1 equals the intersection of $p_{-1} = (s, f)$ with an ideal properly containing $p_0 \cap \cdots \cap p_n$, it suffices to prove that $J_1 = H_1 \cap J_2$. As $J_1 \subseteq H_1$, it suffices to prove that

$$H_1 \cap \left(\left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right) L_2 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 + L_2' \right) \subseteq J_1.$$

But $H_1 \subseteq (s, f)$, so the intersection is contained in

$$(s, f) \left(\left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right) L_2 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 + L_2' \right).$$

As $sL_2 \in J$, $(s, f)L_2' \subseteq J_1$, $f(c_{01} - c_{04}), fc_{01} - sc_{02} \in J$, and $(s, f) = (s - fb_{01}^d, f)$, the ideal above is contained in $J_1 + \left(fc_{01}^3, (s - fb_{01}^d)c_{04}^3\right)$. But

$$(s - fb_{01}^d)c_{04}^3 = c_{04}^2(sc_{04} - fc_{04}b_{01}^d) \in J + fc_{04}^3(b_{04}^d - b_{01}^d) = J + fc_{01}^3(b_{04}^d - b_{01}^d),$$

so that $H_1 \cap J_2 \subseteq J_1 + H_1 \cap \left(c_{01}^3\right)$, which equals J_1 by the previous proposition. ∎

7. Sixteen embedded components of J and J_2

We'll show that for every subset $\Lambda \subseteq \{1, 2, 3, 4\}$,

$$P_{-5\Lambda} = (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01} - c_{03}b_{04}) + (c_{1i} | i \not\in \Lambda) + (b_{1i} - b_{1j} | i, j \in \Lambda)$$

is an associated prime of J and J_2, with its embedded component being

$$p_{-5\Lambda} = \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}\right) + (c_{1i} | i \not\in \Lambda) + (b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j} | i, j \in \Lambda).$$

It is clear that these ideals are prime, respectively primary, and that there are sixteen of each kind. Note that the height of $P_{-5\{\}}$ is 10, and if $\Lambda \neq \{\}$, the height of $P_{-5\Lambda}$ equals 9. By a calculation similar as for q_4,

$$p_{-5} = \bigcap_{\Lambda} p_{-5\Lambda} = \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}\right) + (c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}) | i, j = 1, \ldots, 4).$$

Now observe that

$$
\begin{aligned}
J_2 : (c_{02}(c_{02}-c_{03}))^\infty &= J_2 : c_{02}(c_{02}-c_{03}) \\
&= \left(s, c_{01}, c_{04}, b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), b_{01}b_{03}^d)\right) \\
&= (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01}-c_{03}b_{04}) \\
&\qquad \cap \left(s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}^d, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i}-b_{1j})\right) \\
&\qquad \cap \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j})\right) \\
&= p_{-3} \cap p_{-4} \cap p_{-5}.
\end{aligned}
$$

None of the components is redundant, which proves that J_2 has the specified embedded components $p_{-5\Lambda}$. Furthermore,

$$
\begin{aligned}
J &: (c_{02}(c_{02}-c_{03}))^\infty = \\
&= \left(s, fc_{01}, fc_{04}, fb_{02}^d, fc_{03}b_{03}^d, f(c_{02}b_{01}-c_{03}b_{04}), fc_{1i}(b_{02}-b_{1i}b_{03}), fb_{01}b_{03}^d\right) \\
&= (s, f) \cap \left(s, c_{01}, c_{04}, b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), b_{01}b_{03}^d\right) \\
&= (s, f) \cap (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01}-c_{03}b_{04}) \\
&\qquad \cap \left(s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}^d, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i}-b_{1j})\right) \\
&\qquad \cap \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j})\right) \\
&= p_{-1} \cap p_{-3} \cap p_{-4} \cap p_{-5},
\end{aligned}
$$

and here also the embedded components $p_{-5\Lambda}$ are not redundant.

Thus so far we have obtained that

$$
J = \left(\bigcap_{i=-1}^{n} p_i\right) \cap p_{-3} \cap p_{-4} \cap p_{-5} \cap J_3,
$$

where p_{-5} is the intersection of 16 embedded components and

$$
\begin{aligned}
J_3 &= J_2 + (c_{02}(c_{02}-c_{03}) \\
&= J_1 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right) L_2 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 + L_2' + (c_{02}(c_{02}-c_{03})) \\
&= J + \left(s, fb_{02}^d b_{03}^d, fb_{02}^d b_{04}\right)^3 + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)^3 + (c_{02}(c_{02}-c_{03})) \\
&\qquad + \left(s(s-fb_{01}^d), s(s-fb_{04}^d)\right) + \left(s-fb_{02}^d\right)\left(s, fb_{03}^d, fb_{04}\right) + \left(s-fb_{03}^d\right)\left(s, fb_{02}^d\right) \\
&\qquad + \left(s, c_{01}, c_{02}, c_{04}, b_{03}^d, b_{04}\right)\left(c_{1i}(b_{02}-b_{1i}b_{03}) | i = 1, \ldots, 4\right) \\
&\qquad + \left(c_{0i}(c_{02}b_{01}-c_{03}b_{04}), c_{0i}(c_{01}-c_{04}) | i = 1, \ldots, 4\right) \\
&\qquad + \left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d\right)\left(c_{02}-c_{03}, c_{01}-c_{02}b_{01}^d, c_{04}-c_{03}b_{04}^d\right).
\end{aligned}
$$

It remains to find a primary decomposition of J_3.

But $J_3 : (c_{02}-c_{03})^\infty = (s, c_{01}, c_{02}, c_{04}, b_{03}^d, b_{04}) = p_{-2} = J_3 : (c_{02}-c_{03})^2$, so that

Proposition 7.1:

$$J = \left(\bigcap_{i=-5}^{n} p_i \right) \cap \left(J_3 + (c_{02} - c_{03})^2 \right). \qquad \blacksquare$$

Thus it remains to find a primary decomposition of $J_3 + (c_{02} - c_{03})^2$.

New approach – not components, associated primes only

8. Finding associated primes, not components

Lemma 8.1: *Let I be an ideal in a ring R. Then for any $x \in R$,*

$$Ass\left(\frac{R}{I}\right) \subseteq Ass\left(\frac{R}{I:x}\right) \cup Ass\left(\frac{R}{I+(x)}\right),$$

and every associated prime of $\frac{R}{I:x}$ is an associated prime of $\frac{R}{I}$.

Proof: This follows from the short exact sequence

$$0 \longrightarrow \frac{R}{I:x} \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{I+(x)} \longrightarrow 0. \qquad \blacksquare$$

Lemma 8.2: *For all $i, j = 1, \ldots, 4$, $s^2 C_0 C_1^2 (b_{1i} - b_{1j}) + s^2 C_0^2 C_1 b_{03}^d (b_{1i} - b_{1j}) \subseteq J$.*

Proof: By Lemma 5.3, $s^2 C_0 C_1^2 (b_{1i} - b_{1j})$ is contained in $s^2 c_{02} c_{1i} c_{1j} (b_{1i} - b_{1j}) + J$. Modulo h_{15} and h_{03} this is contained in $sfc_{02}c_{1i}c_{1j}(b_{1i} - b_{1j})b_{03}^d + J$, but that is J by going modulo $h_{1,6+i}$ and $h_{1,6+j}$.

Similarly,

$$\begin{aligned} s^2 C_0^2 C_1 b_{03}^d (b_{1i} - b_{1j}) &\subseteq sfc_{01}c_{02}b_{03}^d (c_{1i}b_{1i} - c_{1j}b_{1j}) + J \\ &\subseteq sfc_{01}c_{02}b_{02}b_{03}^{d-1}(c_{1i} - c_{1j}) + J, \end{aligned}$$

and the latter equals J by Lemma 5.3. $\blacksquare$

Recall that $C_n = (0)$.

Proposition 8.3: *Temporarily set $c_{n2} = 0$. For $k = 0, \ldots, n-1$, define*

$$I_k = J + (s^2 c_{02} c_{12} \cdots c_{n-k,2} b_{03} b_{13}),$$
$$x_k = s^2 c_{02} c_{12} \cdots c_{n-k-1,2} b_{03} b_{13}.$$

(For example $I_0 = J$.) Then

$$I_k : x_k = p_{n-k} = \begin{cases} D_0 + \cdots + D_{n-1-k} + C_{n-k} + E + F + B_{n-1-k}, & \text{if } n-1 > k, \\ D_0 + C_1 + E, & \text{if } n-1 = k, \end{cases}$$

and $I_k : x_k$ has exactly d^2 associated primes, each of which is minimal over $I_k : x_k$ and over J.

Proof: Set $x' = s^2 c_{02} c_{12} \cdots c_{n-k-1,2}$. Lemma 5.3 proves that $D_0 + D_1 + \cdots + D_{n-k} \subseteq I_k : x'$, thus as $c_{n-k,2} \in I_k : x_k$, also $C_{n-k} \subseteq I_k : x_k$.

By Lemma 5.4, $E \subseteq I_k : x'$.

First assume that $k = n-1$. We have proved that $D_0 + C_1 + E \subseteq I_{n-1} : x_{n-1}$, and it remains to prove the other inclusion. So let $y \in I_{n-1} : x_{n-1}$. Then

$$x_{n-1} y = s^2 c_{02} y \in I_{n-1} \subseteq C_1 + D_0 + E,$$

whence $y \in C_1 + D_0 + E$, which proves that $D_0 + C_1 + E = I_{n-1} : x_{n-1}$.

Now assume that $k < n-1$. Remark 5.9 proves that for all r such that $3 \le r < n-k$, and for all $i = 1, \ldots, 4$, $1 - b_{ri} \in I_k : x'$. By Lemma 5.10, $\big(b_{02} - b_{1i} b_{03}, b_{1i}^d - 1\big) \subseteq I_k : x'$, so that for all $i, j = 1, \ldots, 4$, $b_{1i} - b_{1j} \in I_k : x' b_{03}$. Thus $F \subseteq I_k : x' b_{03}$. Finally, when $n - k - 1 > 1$, by Lemma 5.3, for all $k = 1, \ldots, 4$, $b_{12} - b_{2i} b_{13} \in I_k : x'$, so that $b_{13}(1 - b_{2i}) \in I_k : x' b_{03}$, whence $1 - b_{2i} \in I_k : x' b_{03} b_{13} = I_k : x_k$. This proves that $I_k : x_k$ contains the stated ideal.

Now let $y \in I_k : x_k$. Thus $s^2 c_{02} c_{12} \cdots c_{n-1-k,2} b_{03} b_{13} y \in I_k \subseteq p_{n-k}$, so $y \in p_{n-k}$. This confirms that $I_k : x_k$ does equal the stated ideal. ∎

Now with notation as in the Proposition, by Lemma 8.1, to find possible embedded primes of J, it suffices to search among the associated prime ideals of $J : x_{n-1}$ and $J + (x_{n-1})$. However, the Proposition, together with determination of the minimal primes not containing sf (see [S]), proves that the associated prime ideals of $p_1 = J : x_{n-1}$ are all minimal over J, so it suffices to search among the associated prime ideals of $I_{n-1} = J + (x_{n-1})$. By another application of Lemma 8.1, it now suffices to search among the associated prime ideals of $I_{n-1} : x_{n-2}$ and $I_{n-1} + (x_{n-2}) = J + (x_{n-2}) = I_{n-2}$. By the Proposition, it suffices to search among the associated prime ideals of I_{n-2} only. By continuing the applications of Lemma 8.1 and the Proposition, we conclude that in order to find the embedded primes of J, it suffices to consider the associated prime ideals of the ideal $J + (s^2 c_{02} b_{03} b_{13})$.

Lemma 8.4: *To find the embedded primes of J, it suffices to consider the associated prime ideals of $R/(J + (s^2 b_{03}))$.*

Proof: We know that it suffices to consider the associated prime ideals of the ideal $I' = J + (s^2 c_{02} b_{03} b_{13})$. Set $x' = s^2 b_{03} b_{13}$. It is easy to see that $I' : x' = C_0$, which is a prime ideal of height 4 and is minimal over J.

Thus by Lemma 8.1, it suffices to consider the associated prime ideals of $I = J + (s^2 b_{03} b_{13})$. Set $x = s^2 b_{03}$.

We claim that $I : x = (b_{13}) + D_0 + c_{02}(E + C_1)$. By Lemmas 5.3, 5.4, and 5.10, $D_0 + c_{02}(E + C_1(1 - b_{1i}^d))$ is contained in the colon ideal. Hence as b_{13} is in the colon ideal, so is $c_{02}C_1$, so that one inclusion of the claim follows.

To prove the other inclusion, let $y \in I : x$. Then $xy = s^2 b_{03} y \in I \subseteq C_0 + (b_{13})$, so that $y \in C_0 + (b_{13})$. Thus without loss of generality $y \in C_0$, and as $D_0 \subseteq I : x$, we may assume that $y \in (c_{02})$. But $xy = s^2 b_{03} y \in I \subseteq D_0 + E + C_1 + (b_{13})$, so that $y \in (D_0 + E + C_1 + (b_{13})) \cap (c_{02})$, which proves the other inclusion.

Now, $I : x$ decomposes as

$$\begin{aligned}
&((b_{13}, c_{02}) + D_0) \cap ((b_{13}) + D_0 + E + C_1) \\
&\qquad = ((b_{13}) + C_0) \cap ((b_{13}) + D_0 + E + C_1) \\
&\qquad = ((b_{13}) + C_0) \cap \bigcap_{\alpha,\beta} ((b_{13}) + D_0 + E + C_1 + (b_{01} - \alpha b_{02}, b_{02} - \beta b_{03})),
\end{aligned}$$

where α and β vary over dth roots of unity and each ideal is prime. Each of these primes is associated to $I : x$ so that by Lemma 8.1 these primes are candidates for the associated primes of J. They do not equal any of the minimal primes of J, and they do not contain s, so that by Proposition 5.11 they are also not embedded primes of J. Thus by Lemma 8.1 it suffices to look for embedded primes of J among the associated primes of $I + (x) = J + (s^2 b_{03})$. ∎

Proposition 8.5: *Temporarily set $c_{n2} = 0$. For $k = 0, \ldots, n-1$, define*

$$I_k = J + (s^2 b_{03}, s^2 (c_{02} c_{12} \cdots c_{n-k,2}) c_{12} b_{13}),$$
$$x_k = s^2 (c_{02} c_{12} \cdots c_{n-k-1,2}) c_{12} b_{13}.$$

(For example $I_0 = J + (s^2 b_{03})$.) Then

$$I_k : x_k = \begin{cases} p_1 + (b_{03}) = (s, c_{01}, c_{04}, c_{02} - c_{03}, b_{03}, b_{01} - b_{04}, b_{02}^d, b_{04}^d) + C_1, & \text{if } n-1 = k, \\[1ex] p_{n-k} + (b_{03}) = (s, c_{01}, c_{04}, c_{02} - c_{03}, b_{02}, b_{03}, b_{01} - b_{04}, b_{04}^d) & \\ \qquad + D_1 + \cdots + D_{n-k-1} + C_{n-k} + B_{n-k-1} & \\ \qquad + (b_{1i}^d - 1, b_{1j} - b_{1i} | i, j = 1, \ldots, 4), & \text{if } n-1 > k. \end{cases}$$

Proof: Lemma 5.3 proves that $D_0 + D_1 + \cdots + D_{n-k} \subseteq I_k : x_k$, thus as $c_{n-k,2} \in I_k : x_k$, also $C_{n-k} \subseteq I_k : x_k$.

By Lemma 5.4, $E \subseteq I_k : x_k$.

First assume that $k = n - 1$. We have proved that $(s, c_{01}, c_{04}, c_{02} - c_{03}, b_{03}, b_{01} - b_{04}, b_{02}^d, b_{04}^d) + C_1 = (b_{03}) + D_0 + C_1 + E \subseteq I_{n-1} : x_{n-1}$, and it remains to prove the other inclusion. So let $y \in I_{n-1} : x_{n-1}$. Then $x_{n-1} y = s^2 c_{02} y \in I_{n-1} \subseteq (s^2 b_{03}) + C_1 + D_0 + E$, whence $y \in (b_{03}) + C_1 + D_0 + E$, which proves this case.

Now assume that $k < n - 1$. Remark 5.9 proves that for all $r \in \{3, \ldots, n-k-1\}$ and all $i = 1, \ldots, 4$, $1 - b_{ri} \in I_k : x_k$. By Lemma 5.10, $\left(b_{02} - b_{1i} b_{03}, b_{1i}^d - 1\right) \subseteq I_k : x_k$, so that also $b_{02} \in I_k : x_k$. For all $i, j = 1, \ldots, 4$, by Lemmas 8.2 and 5.6, $b_{ri} - b_{rj} \in I_k : x_k$ whenever $1 \le r < n - k$. Actually, $b_{1i} - b_{1j} \in I_k : s^2 c_{02} c_{12}^2$, so that by using $h_{2,6+i}$, also

$1 - b_{2i} \in I_k : x_k$. This proves that in the cases $n-1 > k$, the stated ideal is contained in $I_k : x_k$.

Now we prove the other inclusion. Let $y \in I_k : x_k$. Thus $s^2(c_{02}c_{12}\cdots c_{n-k-1,2})c_{12}b_{13}y$ is contained in $I_k \subseteq p_{n-k} + (s^2 b_{03})$. By taking advantage of elements of p_{n-k} which are known to be in $I_k : x_k$, we may subtract their multiples to assume that the following variables do not appear in y:

$$s, c_{n-k,i}, b_{01}, b_{02}, b_{11}, b_{12}, b_{14}, c_{r1}, c_{r3}, c_{r4}, b_{r'i},\ r = 0, \ldots, n-k-1,\ r' = 2, \ldots, n-k-1.$$

Thus after rewriting s modulo $s - f b_{03}^d$, $x_k y \in p_{n-k} + (s^2 b_{03})$ means that

$$f^2 b_{03}^{2d}(c_{02}c_{12}\cdots c_{n-k-1,2})c_{12}b_{13}y \in \left(f^2 b_{03}^{2d+1}, b_{03}^d - b_{04}^d, b_{13}^d - 1\right),$$

so that $y \in \left(b_{03}, b_{04}^d, b_{13}^d - 1\right)$, which proves the proposition. ∎

With notation as in the previous lemma, define

$$\begin{aligned} Q &= (I_{n-1} : x_{n-1}) + (b_{02}, b_{04}), \\ &= (s, c_{01}, c_{04}, c_{02} - c_{03}, b_{01}, b_{02}, b_{03}, b_{04}) + C_1, \end{aligned}$$

and for $k = 0, \ldots, n-2$, define

$$\begin{aligned} Q_{k\alpha} &= (s, c_{01}, c_{04}, c_{02} - c_{03}, b_{02}, b_{03}, b_{01} - b_{04}, b_{04}) \\ &\qquad + D_1 + \cdots + D_{n-k-1} + C_{n-k} + B_{n-k-1} \\ &\qquad + (b_{11} - \alpha, b_{1j} - b_{1i} | i, j = 1, \ldots, 4), \\ &= (I_k : x_k) + (b_{04}, b_{11} - \alpha), \end{aligned}$$

where α varies over dth roots of unity. Note that Q is the radical of $(b_{03}) + p_1$ and that $Q_{k\alpha}$ is the radical of each minimal prime $P_{n-k,\alpha\beta}$ over J (as β varies over dth roots of unity, see all the minimal primes not containing sf in [S]). The heights of these primes are easily seen to be as follows:

$$\begin{aligned} \text{ht } Q &= 11, \\ \text{ht } Q_{k\alpha} &= 7(n-k) - 4, \text{ if } k = 0. \\ \text{ht } Q_{k\alpha} &= 7(n-k), \text{ if } k > 0. \end{aligned}$$

Note that there is a total of $1 + (n-1)d$ primes.

Also define the following prime ideal of height 12 (radical of $p_1 + (b_{03}, b_{13})$):

$$Q' = (s, c_{01}, c_{04}, c_{02} - c_{03}, b_{03}, b_{01} - b_{04}, b_{02}, b_{04}, b_{13}) + C_1.$$

Lemma 8.6: *The set of embedded primes of J is contained in the set*

$$\{Q, Q_{k\alpha}, Q'\} \cup Ass\ \left(R/(J + (s^2))\right).$$

Proof: By Lemmas 8.4 and 8.1, we know that the set of embedded primes of J is contained in the set of associated primes of $J + (s^2 b_{03})$. By Proposition 8.5, this set is contained in

$$\{Q, Q_{k\alpha}\} \cup \text{Ass}\ \left(R/(J + s^2(b_{03}, c_{02}c_{12}b_{13}))\right).$$

Set $I = J + (s^2 b_{03}, s^2 c_{02}c_{12}b_{13})$ and $x = s^2 c_{02} b_{13}$.

Certainly $I : x$ contains b_{03}, D_0, C_1 and E. If $y \in I : x$, then $s^2 c_{02} b_{13} y \in I \subseteq (s^2 b_{03}) + E + D_0 + C_0$, so that $s^2 y \in I \subseteq (s^2 b_{03}) + E + D_0 + C_0$, whence $y \in I \subseteq (b_{03}) + E + D_0 + C_0$, so that $I : x$ equals $(b_{03}) + E + D_0 + C_0$. The latter ideal is primary to Q. Thus by Lemma 8.1, to find the embedded primes of J it suffices to look among $\{Q, Q_{k\alpha}\}$ and the associated primes of $I' = I + (x) = J + (s^2 b_{03}, s^2 c_{02} b_{13})$.

Set $x' = s^2 b_{13}$. It is easy to see that $I' : x' = (b_{03}) + C_0$. Note that this prime ideal is not associated to J as it is not minimal and it does not contain s (cf. Proposition 5.11). Thus it suffices to look among the associated primes of $I'' = I' + (x') = J + (s^2 b_{03}, s^2 b_{13})$.

Further, we claim that $I'' : s^2 c_{02} = (b_{03}, b_{13}) + D_0 + E + C_1$. Certainly $(b_{03}, b_{13}) + D_0 + E + D_1$ is contained in $I'' : s^2 c_{02}$. By Lemma 5.10, $C_1(1 - b_{13}^d)$ is in the colon ideal, whence C_1 is in it, proving that $I'' : s^2 c_{02}$ contains $(b_{03}, b_{13}) + D_0 + E + C_1$. If instead y is in the colon ideal, $s^2 c_{02} y \in I'' \subseteq s^2(b_{03}, b_{13}) + D_0 + E + C_1$, so that by similar arguments as before, $y \in (b_{03}, b_{13}) + D_0 + E + C_1$. This proves the claim. It is now easy to see that $(b_{03}, b_{13}) + D_0 + E + C_1$ is a Q'-primary ideal, so that by Lemma 8.1 the embedded primes of J are contained in the union of $\{Q, Q_{k\alpha}, Q'\}$ and the set of associated primes of $I''' = I'' + (s^2 c_{02}) = J + s^2(b_{03}, b_{13}, c_{02})$.

Finally, $I''' : s^2 = (b_{03}, b_{13}) + C_0$, which is a prime ideal which by Proposition 5.11 is not associated to J as it does not contain s. Thus by Lemma 8.1, it remains to find the associated primes of $I''' + (s^2) = J + (s^2)$. ∎

Lemma 8.7: *The set of embedded primes of J is contained in the set*

$$\{Q, Q_{k\alpha}, Q'\} \cup \mathit{Ass}\ \left(R/(J + (s^2, sfc_{02}b_{02}))\right).$$

Proof: By the previous lemma we have to find the associated primes of $J + (s^2)$. First note that $(J + (s^2)) : sfc_{02}b_{02}b_{03}^{d-1}$ equals the ideal $I = (s, b_{01}^d, b_{01} - b_{04}, b_{02}^{d-1}, b_{03}, c_{01}, c_{04}, c_{02} - c_{03}) + C_1$. For one direction, note that

$$\begin{aligned} sfc_{02}b_{01}^d &\equiv f^2 c_{01} b_{01}^d \equiv sfc_{01} \equiv s^2 c_{02} \in J + (s^2), \\ sfc_{02}(b_{01} - b_{04}) &\equiv sf(c_{02}b_{01} - c_{03}b_{04}) \in J, \\ sfc_{02}b_{02}b_{03}^{d-1}c_{1i} &\equiv sfc_{02}b_{03}^d c_{1i} b_{1i} \equiv sfc_{03}b_{03}^d c_{1i} b_{1i} \equiv s^2 c_{03} c_{1i} b_{1i} \in J + (s^2), \end{aligned}$$

and certainly $s, b_{02}^{d-1}, b_{03}, c_{01}, c_{04}, c_{02} - c_{03}$ are contained in the colon ideal. Conversely, if y is in the colon ideal, then $sfc_{02}b_{02}b_{03}^{d-1} y \in J + (s^2) \subseteq (s^2) + E + D_0 + C_1$, and then easily one proves that y is in I. This proves that $(J + (s^2)) : sfc_{02}b_{02}b_{03}^{d-1}$ equals I, which is Q-primary. Thus it remains to find the associated primes of $I_1 = (J + (s^2, sfc_{02}b_{02}b_{03}^{d-1})$.

For $k = 1, \ldots, d$, define $I_k = (J + (s^2, sfc_{02}b_{02}b_{03}^{d-k})$, and $x_k = sfc_{02}b_{02}b_{03}^{d-k-1}$. It is straightforward to see that $I_k : x_k$ equals $(s, b_{01}^d, b_{01} - b_{04}, b_{02}^{d-1}, b_{03}, c_{01}, c_{04}, c_{02} - c_{03}) + C_1$, which is primary to Q. Thus this reduces the proof to finding the associated primes of $I_d = (J + (s^2, sfc_{02}b_{02})$. ■

Lemma 8.8: *The set of embedded primes of J is contained in the set*

$$\{Q, Q_{k\alpha}, Q'\} \cup Ass\ \left(R/(J + (s^2, sf))\right).$$

Proof: By the previous lemma we have to find the associated primes of $J+(s^2, sfc_{02}b_{02})$. It is easy to see that $(J + (s^2, sfc_{02}b_{02})) : sfb_{02}^d = (s) + C_0$. This is a prime ideal, associated to $J + (s^2, sfc_{02}b_{02})$, but as it does not contain $fb_{02}b_{03}$ it follows by Proposition 5.12 that this prime is not associated to J. Thus it remains to find the associated primes of $I_0 = J + (s^2, sfc_{02}b_{02}, sfb_{02}^d)$.

For $k = 0, \ldots, d-1$, define $I_k = (J + (s^2, sfc_{02}b_{02}, sfb_{02}^{d-k})$, and $x_k = sfb_{02}^{d-k-1}$. It is straightforward to see that $I_k : x_k$ equals the prime ideal $(s, b_{02}) + C_0$. However, this prime ideal does not contain $b_{04}c_{1i}(b_{02} - b_{1i}b_{03})$, so that by Proposition 7.1, $(s, b_{02}) + C_0$ is not associated to J.

Thus this reduces to finding the associated primes of $I_{d-1} = (J + (s^2, sf)$. ■

9. Reduction to $J + (s^2, sf, sc_{02})$

Whereas $J : s$ is practically incalculable, $(J+(s^2, sf)) : s$ is calculable. This is the aim of the first lemma below. By Lemma 8.1, Ass $(R/(J+(s^2, sf))) \subseteq$ Ass $(R/((J+(s^2, sf)) : s)) \cup$Ass $(R/(J+(s)))$, where $J+(s)$ is a much simpler ideal whose associated primes would not be too hard to establish (compare with HERE). However, the associated primes of the ideal $(J + (s^2, sf)) : s$ are not so easy to establish. In this section our aim is calculating these associated primes by calculating various $(J + (s^2, sf) + I) : x_I$ for various ideals I and elements x_I. In the process, knowing the ideal $(J + (s^2, sf)) : s$ is very useful.

Lemma 9.1: *Let JJ be the ideal in R generated by all the h_{rj}/s, $r \geq 2$. (Note that all these h_{rj} are multiples of s.) Then $(J + (s^2, sf)) : s$ equals*

$$\begin{aligned}
&= (s, f, c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d) + JJ \\
&\quad + c_{02}\left(c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d, c_{01} - c_{04}, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d\right) \\
&\quad + c_{02}\left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(c_{03} - b_{1i}^d c_{02}), c_{1i}c_{1j}(b_{1i} - b_{1j})\right) \\
&= (s, f, c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d) + JJ \\
&\quad + c_{02}^2\left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) + c_{02}\left(b_{01}b_{03}^d - b_{04}b_{02}^d\right) \\
&\quad + c_{02}\left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(c_{03} - b_{1i}^d c_{02}), c_{1i}c_{1j}(b_{1i} - b_{1j})\right),
\end{aligned}$$

where the indices i and j vary from 1 to 4.

Proof: In the course of this proof we will use several times the easy fact that for any ideals I and I', and any element x, $(I + xI') : x = (I : x) + I'$.

First observe that

$$J = s\left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d\right) + JJ + K,$$

where K is the ideal $fK' + (fc_{01} - sc_{02})$, with

$$K' = \left(c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d, c_{01} - c_{04}, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03})\right).$$

Let $x \in K : s$. Write $xs = k'f + a(fc_{01} - sc_{02})$ for some $k' \in K'$ and $a \in R$. By modifying x by a multiple of $fc_{01} - sc_{02}$, without loss of generality no s appears in a. From $xs = k'f + a(fc_{01} - sc_{02})$ it follows that

$$a \in (K' + (s)) : fc_{01} = (s) + (K' : c_{01}).$$

We claim that $K' : c_{01}$ equals

$$K' + \left(b_{01}b_{03}^d - b_{04}b_{02}^d, c_{04} - c_{03}b_{03}^d, c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03})\right).$$

Here is a proof. First of all, clearly $K' : c_{01} = (c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d) + (K'' : c_{02}b_{02}^d)$, where

$$K'' = \left(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03})\right).$$

Then it is easy to see that

$$\begin{aligned} K'' : c_{02} &= \left(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}\right) : c_{02} + (c_{1i}(b_{02} - b_{1i}b_{03})) \\ &= \left(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03})\right). \end{aligned}$$

By the same proof as in the computation of p_{-4}, the decomposition of $I = K'' : c_{02}$ is easily seen to be

$$I = \cap_\Lambda \left(I + (c_{1i} | i \notin \Lambda) + (b_{02} - b_{1i}b_{03} | i \in \Lambda)\right),$$

so that $K'' : c_{02}b_{02}^d = I : b_{02}^d$ equals

$$I : b_{02}^d = \cap_\Lambda \left(I + (c_{1i} | i \notin \Lambda) + (b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j}, b_{01} - b_{1i}^d b_{04}, c_{03} - b_{1i}^d c_{02} | i, j \in \Lambda)\right),$$

and this intersection is (as the computation of p_{-4}):

$$I : b_{02}^d = I + \left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(c_{03} - b_{1i}^d c_{02}) | i, j\right).$$

This finishes the verification of $K' : c_{01}$.

Now we go back to the computation of $K : s$. Recall that $x \in K : s$ and $sx = k'f + a(fc_{01} - sc_{02})$. As no s appears in a, $xs \in (f, ac_{02})$ implies that $x \in (f) + c_{02}((K' + (s)) : fc_{01})$, as was to be proved. ■

From now on, to simplify the notation, we set

$$c_{ni} = b_{ni} = 1.$$

Note that this makes the generator h_{n7} a special case of all the $h_{r,6+i}$. Also, the symbol $\equiv$ stands for congruence modulo J.

Lemma 9.2: *For all $r = 1, \ldots, n-1$ and all $i = 1, \ldots, 4$,*

$$sc_{01} \cdots c_{r-2,1} \left(c_{r-1,1} - c_{r-1,4} b_{ri}^{d^{2^r}} \right) c_{ri} \in J,$$

when $r = 0$, $sc_{0i} - fc_{0i} b_{0i}^d$ is in J, and when $r = n$,

$$sc_{01} \cdots c_{n-2,1} \left(c_{n-1,1} - c_{n-1,4} \right) \in J.$$

Or, with the new notation as above, for all $r = 1, \ldots, n$ and all $i = 1, \ldots, 4$,

$$sc_{01} \cdots c_{r-2,1} \left(c_{r-1,1} - c_{r-1,4} b_{ri}^{d^{2^r}} \right) c_{ri} \in J.$$

Proof: The case $r = 0$ holds by definition. When $r = 1$, first note that

$$\begin{aligned} fc_{02}c_{1i}b_{01}b_{02}^d &\equiv fc_{02}c_{1i}b_{01}b_{03}^d b_{1i}^d \equiv fc_{03}c_{1i}b_{04}b_{03}^d b_{1i}^d \equiv sc_{03}c_{1i}b_{04}b_{1i}^d \\ &\equiv sc_{02}c_{1i}b_{04}b_{1i}^d \equiv fc_{02}c_{1i}b_{04}b_{02}^d b_{1i}^d, \end{aligned}$$

so that

$$\begin{aligned} sc_{01}c_{1i} &\equiv fc_{01}c_{1i}b_{01}^d \equiv sc_{02}c_{1i}b_{01}^d \equiv fc_{02}c_{1i}b_{01}^d b_{02}^d \\ &\equiv fc_{02}c_{1i}b_{04}^d b_{02}^d b_{1i}^{d^2} \quad \text{by above} \\ &\equiv sc_{02}c_{1i}b_{04}^d b_{1i}^{d^2} \equiv fc_{04}c_{1i}b_{04}^d b_{1i}^{d^2} \equiv sc_{04}c_{1i}b_{1i}^{d^2}, \end{aligned}$$

which proves the case $r = 1$. Now let $r > 1$. To simplify the notation, set $A = sc_{01} \cdots c_{r-3,1}$, where we of course interpret this to be s if $r < 3$. First observe that

$$\begin{aligned} Ac_{r-2,1}c_{r-1,2}c_{ri}b_{r-1,1} &\equiv Ac_{r-2,4}c_{r-1,2}c_{ri}b_{r-1,1}b_{r-1,2}^{d^{2^{r-1}}} \quad \text{by induction} \\ &\equiv Ac_{r-2,4}c_{r-1,2}c_{ri}b_{r-1,1}b_{r-1,3}^{d^{2^{r-1}}} b_{ri}^{d^{2^{r-1}}} \quad \text{modulo } h_{r,6+i} \\ &\equiv Ac_{r-2,4}c_{r-1,3}c_{ri}b_{r-1,4}b_{r-1,3}^{d^{2^{r-1}}} b_{ri}^{d^{2^{r-1}}} \quad \text{modulo } h_{r6} \\ &\equiv Ac_{r-2,1}c_{r-1,3}c_{ri}b_{r-1,4}b_{ri}^{d^{2^{r-1}}} \quad \text{by induction} \\ &\equiv Ac_{r-2,1}c_{r-1,2}c_{ri}b_{r-1,4}b_{ri}^{d^{2^{r-1}}} \quad \text{modulo } h_{r5}, \end{aligned}$$

so that

$$\begin{aligned} sc_{01} \cdots c_{r-2,1}c_{r-1,1}c_{ri} &\equiv Ac_{r-2,1}c_{r-1,1}c_{ri} \equiv Ac_{r-2,4}c_{r-1,1}c_{ri}b_{r-1,1}^{d^{2^{r-1}}} \quad \text{by induction} \\ &\equiv Ac_{r-2,1}c_{r-1,2}c_{ri}b_{r-1,1}^{d^{2^{r-1}}} \quad \text{modulo } h_{r3} \\ &\equiv Ac_{r-2,1}c_{r-1,2}c_{ri}b_{r-1,4}^{d^{2^{r-1}}} (b_{ri}^{d^{2^{r-1}}})^{d^{2^{r-1}}} \quad \text{by above} \\ &\equiv Ac_{r-2,1}c_{r-1,2}c_{ri}b_{r-1,4}^{d^{2^{r-1}}} b_{ri}^{d^{2^r}} \end{aligned}$$

$$\equiv Ac_{r-2,1}c_{r-1,3}c_{ri}b_{r-1,4}^{d^{2^{r-1}}}b_{ri}^{d^{2^r}} \quad \text{modulo } h_{r5}$$
$$\equiv Ac_{r-2,4}c_{r-1,4}c_{ri}b_{r-1,4}^{d^{2^{r-1}}}b_{ri}^{d^{2^r}} \quad \text{modulo } h_{r4}$$
$$\equiv Ac_{r-2,1}c_{r-1,4}c_{ri}b_{ri}^{d^{2^r}} \quad \text{by induction.} \qquad \blacksquare$$

Lemma 9.3: *For $i = 2, 3$, $k = 0, \ldots, n-2$ and for all $r = k+1, \ldots, n$,*

$$sc_{k3}(c_{01}c_{11}\cdots c_{r1}) \equiv sc_{k3}(c_{01}c_{11}\cdots c_{r1})b_{ri}^{d^{2^k}} \quad \textit{modulo } J.$$

Proof: When $k = 0$ and $r = 1$, by using Lemma 9.2:

$$\begin{aligned} sc_{01}c_{03}c_{11} &\equiv sc_{01}c_{02}c_{11} \equiv sc_{04}c_{02}c_{11}b_{11}^{d^2} \equiv sc_{01}c_{02}c_{1i}b_{11}^{d^2} \quad \text{modulo } h_{13}, h_{15} \\ &\equiv fc_{01}c_{02}c_{1i}b_{11}^{d^2}b_{02}^{d} \equiv fc_{01}c_{02}c_{1i}b_{11}^{d^2}b_{03}^{d}b_{1i}^{d} \\ &\equiv fc_{01}c_{03}c_{1i}b_{11}^{d^2}b_{03}^{d}b_{1i}^{d} \equiv sc_{01}c_{03}c_{1i}b_{11}^{d^2}b_{1i}^{d} \\ &\equiv sc_{04}c_{03}c_{11}b_{11}^{d^2}b_{1i}^{d} \equiv sc_{01}c_{03}c_{11}b_{1i}^{d}. \end{aligned}$$

Now for the case $k > 0$ and $r = k+1$, to simplify notation, set $A = s(c_{01}c_{11}\cdots c_{k-2,1})$. Then

$$\begin{aligned} Ac_{k-1,1}c_{k1}c_{k3}c_{k+1,1} &\equiv Ac_{k-1,1}c_{k1}c_{k2}c_{k+1,1} \quad \text{modulo } h_{k+1,5} \\ &\equiv Ac_{k-1,1}c_{k4}c_{k2}c_{k+1,1}b_{k+1}^{d^{2^{k+1}}} \quad \text{by Lemma 9.2} \\ &\equiv Ac_{k-1,1}c_{k1}c_{k2}c_{k+1,i}b_{k+1}^{d^{2^{k+1}}} \quad \text{modulo } h_{k+2,3}, h_{k+2,5} \\ &\equiv Ac_{k-1,4}c_{k1}c_{k2}c_{k+1,i}b_{k+1}^{d^{2^{k+1}}}\ b_{k2}^{d^{2^k}} \quad \text{by Lemma 9.2} \\ &\equiv Ac_{k-1,4}c_{k1}c_{k2}c_{k+1,i}b_{k+1}^{d^{2^{k+1}}}\ b_{k3}^{d^{2^k}}\ b_{k+1,i}^{d^{2^k}} \quad \text{modulo } h_{k+1,6+i} \\ &\equiv Ac_{k-1,1}c_{k2}c_{k2}c_{k+1,i}b_{k+1}^{d^{2^{k+1}}}\ b_{k3}^{d^{2^k}}\ b_{k+1,i}^{d^{2^k}} \quad \text{modulo } h_{k+1,3} \\ &\equiv Ac_{k-1,1}c_{k2}c_{k3}c_{k+1,i}b_{k+1}^{d^{2^{k+1}}}\ b_{k3}^{d^{2^k}}\ b_{k+1,i}^{d^{2^k}} \quad \text{modulo } h_{k+1,5} \\ &\equiv Ac_{k-1,4}c_{k1}c_{k3}c_{k+1,i}b_{k+1}^{d^{2^{k+1}}}\ b_{k3}^{d^{2^k}}\ b_{k+1,i}^{d^{2^k}} \quad \text{modulo } h_{k+1,3} \\ &\equiv Ac_{k-1,1}c_{k1}c_{k3}c_{k+1,i}b_{k+1}^{d^{2^{k+1}}}\ b_{k+1,i}^{d^{2^k}} \quad \text{by Lemma 9.2} \\ &\equiv Ac_{k-1,1}c_{k4}c_{k3}c_{k+1,1}b_{k+1}^{d^{2^{k+1}}}\ b_{k+1,i}^{d^{2^k}} \quad \text{modulo } h_{k+2,3}, h_{k+2,5} \\ &\equiv Ac_{k-1,1}c_{k1}c_{k3}c_{k+1,1}b_{k+1,i}^{d^{2^k}}, \end{aligned}$$

which proves the case $r = k+1$. Now for the case $r > k+1$, to simplify notation, set $A = s(c_{01}c_{11}\cdots c_{r-3,1})c_{k1}$. Then

$$Ac_{r-2,1}c_{r-1,1}c_{r1} \equiv Ac_{r-2,1}c_{r-1,4}c_{r1}b_{r1}^{d^{2^r}} \quad \text{by Lemma 9.2}$$

$$\equiv Ac_{r-2,1}c_{r-1,1}c_{ri}b_{r1}^{d^{2^r}} \quad \text{modulo } h_{r+1,3}, h_{r+1,5}$$
$$\equiv Ac_{r-2,1}c_{r-1,1}c_{ri}b_{r1}^{d^{2^r}} b_{r-1,2}^{d^{2^k}} \quad \text{by induction on } r$$
$$\equiv Ac_{r-2,4}c_{r-1,1}c_{ri}b_{r1}^{d^{2^r}} b_{r-1,2}^{d^{2^k}} b_{r-1,1}^{d^{2^{r-1}}} \quad \text{by Lemma 9.2}$$
$$\equiv Ac_{r-2,1}c_{r-1,2}c_{ri}b_{r1}^{d^{2^r}} b_{r-1,2}^{d^{2^k}} b_{r-1,1}^{d^{2^{r-1}}} \quad \text{modulo } h_{r3}$$
$$\equiv Ac_{r-2,4}c_{r-1,2}c_{ri}b_{r1}^{d^{2^r}} b_{r-1,2}^{d^{2^k}} b_{r-1,1}^{d^{2^{r-1}}} b_{r-1,2}^{d^{2^{r-1}}} \quad \text{by Lemma 9.2}$$
$$\equiv Ac_{r-2,4}c_{r-1,2}c_{ri}b_{r1}^{d^{2^r}} b_{r-1,3}^{d^{2^k}} b_{ri}^{d^{2^k}} b_{r-1,1}^{d^{2^{r-1}}} b_{r-1,2}^{d^{2^{r-1}}} \quad \text{modulo } h_{r,6+i}$$
$$\equiv Ac_{r-2,1}c_{r-1,2}c_{ri}b_{r1}^{d^{2^r}} b_{r-1,3}^{d^{2^k}} b_{ri}^{d^{2^k}} b_{r-1,1}^{d^{2^{r-1}}} \quad \text{by Lemma 9.2}$$
$$\equiv Ac_{r-2,4}c_{r-1,1}c_{ri}b_{r1}^{d^{2^r}} b_{r-1,3}^{d^{2^k}} b_{ri}^{d^{2^k}} b_{r-1,1}^{d^{2^{r-1}}} \quad \text{modulo } h_{r3}$$
$$\equiv Ac_{r-2,1}c_{r-1,1}c_{ri}b_{r1}^{d^{2^r}} b_{r-1,3}^{d^{2^k}} b_{ri}^{d^{2^k}} \quad \text{by Lemma 9.2}$$
$$\equiv Ac_{r-2,1}c_{r-1,4}c_{r1}b_{r1}^{d^{2^r}} b_{r-1,3}^{d^{2^k}} b_{ri}^{d^{2^k}} \quad \text{modulo } h_{r+1,3}, h_{r+1,5}$$
$$\equiv Ac_{r-2,1}c_{r-1,1}c_{r1}b_{r-1,3}^{d^{2^k}} b_{ri}^{d^{2^k}} \quad \text{by Lemma 9.2}$$
$$\equiv Ac_{r-2,1}c_{r-1,1}c_{r1}b_{ri}^{d^{2^k}} \quad \text{by induction.} \qquad \blacksquare$$

Recall that $c_{ni} = 1$. We start tabulating the elements of $(J + (s^2, sf)) : sc_{01} \cdots c_{r1}$, $r = 0, \ldots, n$:

	Elements	Justification
1	s, f	clearly
2	$c_{k2} - c_{k3}$ $k = 0, \ldots, r+1$	by using $h_{k+1,5}$
3	$c_{k1} - c_{k4}b_{k+1,1}^{d^{2^{k+1}}}$ $k = 1, \ldots, r-1$	by Lemma 9.2
4	$c_{k1} - c_{k3}b_{k1}^{d^{2^k}}$ $k = 0, \ldots, r+1$	when $k > 0$: $sc_{01} \cdots c_{k1} \equiv sc_{01} \cdots c_{k-2,1}c_{k-1,4}c_{k1}b_{k1}^{d^{2^k}}$ $\equiv sc_{01} \cdots c_{k-1,1}c_{k3}b_{k1}^{d^{2^k}}$ when $k = 0$: trivial
5	$c_{01} - c_{04}$	$sc_{01}(c_{01} - c_{04}) \equiv fc_{01}(c_{01} - c_{04})b_{01}^d \in J$ by using h_{13}, h_{14}, h_{15}
6	$c_{k1} - c_{k4}$	$sc_{01} \cdots c_{k1}(c_{k1} - c_{k4})$

	$k = 1, \ldots, r$	$\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k1}(c_{k1}-c_{k4})b_{k1}^{d^{2^k}}$ which is in J by using $h_{k+1,3}$, $h_{k+1,4}$, $h_{k+1,5}$
7	$c_{k3}(1-b_{k+1,1}^{d^{2^k}})$ $k = 0, \ldots, r-1$	$k=0$: $sc_{01}c_{03}c_{11}b_{11}^{d} \equiv fc_{01}c_{03}c_{11}b_{11}^{d}b_{03}^{d}$ $\equiv sc_{02}c_{03}c_{11}b_{11}^{d}b_{03}^{d} \equiv fc_{02}c_{03}c_{11}b_{11}^{d}b_{02}^{d}b_{03}^{d}$ $\equiv fc_{02}c_{03}c_{11}b_{02}^{2d} \equiv sc_{02}c_{03}c_{11}b_{02}^{d}$ $\equiv sc_{02}^{2}c_{11}b_{02}^{d} \equiv fc_{01}c_{02}c_{11}b_{02}^{d} \equiv sc_{01}c_{02}c_{11}$ $k>0$: $s(c_{01}\cdots c_{k+1,1})c_{k3}b_{k+1,1}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k1}c_{k+1,1}c_{k3}b_{k+1,1}^{d^{2^k}}b_{k3}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,1}c_{k2}c_{k+1,1}c_{k3}b_{k+1,1}^{d^{2^k}}b_{k3}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k2}c_{k+1,1}c_{k3}b_{k+1,1}^{d^{2^k}}b_{k3}^{2d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k2}c_{k+1,1}c_{k3}b_{k2}^{d^{2^k}}b_{k3}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,1}c_{k2}c_{k+1,1}c_{k3}b_{k3}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k1}c_{k+1,1}c_{k3}b_{k3}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,1}c_{k1}c_{k+1,1}c_{k3}$
8	$c_{k4}-c_{k3}b_{k4}^{d^{2^k}}$ $k = 0, \ldots, r+1$	when $k>0$: $sc_{01}\cdots c_{k-1,1}(c_{k4}-c_{k3}b_{k4}^{d^{2^k}})$ $\equiv sc_{01}\cdots c_{k-2,1}(c_{k-1,4}c_{k4}-c_{k-1,1}c_{k3})b_{k4}^{d^{2^k}}$ when $k=0$: $s(c_{04}-c_{03}b_{04}^{d}) \in J$
9	$c_{k3}(c_{k4}-c_{k1})$ $k = 0, \ldots, r-1$	$k=0$: $sc_{03}c_{04} \equiv fc_{03}b_{03}^{d}c_{04}$ $\equiv fc_{03}b_{03}^{d}c_{01} \equiv sc_{03}c_{01}$ $k>0$: $sc_{01}\cdots c_{k-1,1}c_{k3}c_{k4} \equiv sc_{01}\cdots c_{k-1,4}c_{k3}c_{k4}b_{k3}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-1,4}c_{k3}c_{k1}b_{k3}^{d^{2^k}} \equiv sc_{01}\cdots c_{k-1,1}c_{k3}c_{k1}$
10	$c_{k3}(c_{l4}-B\cdot c_{l1})$ $k = 0, \ldots, r$ $l = k+1, \ldots, r+1$ $B = b_{k+1,4}^{d^{2^{k+1}}}\cdots b_{l4}^{d^{2^l}}$	$l=k+1$: $sc_{01}\ldots c_{k1}c_{k3}c_{k+1,4}$ $\equiv sc_{01}\ldots c_{k4}c_{k3}c_{k+1,4}b_{k+1}^{d^{k+1}}$ $\equiv sc_{01}\ldots c_{k4}c_{k3}c_{k+1,1}b_{k+1}^{d^{k+1}}$ $\equiv sc_{01}\ldots c_{k1}c_{k3}c_{k+1,1}b_{k+1}^{d^{k+1}}$ by 9 $l>k+1$: $sc_{01}\ldots c_{l-1,1}c_{k3}c_{l4}$ $\equiv sc_{01}\ldots c_{l-2,1}c_{l-1,4}c_{k3}c_{l4}b_{l4}^{d^{2^l}}$

		now proceed by induction
11	$c_{k3}(c_{l3} - B \cdot c_{l1})$ $k = 0, \dots, r$ $l = k+1, \dots, r+1$ $B = b_{k+1,4}^{d^{2^{k+1}}} \cdots b_{l-1,4}^{d^{2^{l-1}}}$	$l = k+1$: $sc_{01} \cdots c_{k1} c_{k3} c_{k+1,3}$ $\equiv sc_{01} \cdots c_{k-1,1} c_{k4} c_{k3} c_{k+1,1}$ $\equiv sc_{01} \cdots c_{k-1,1} c_{k1} c_{k3} c_{k+1,1}$ by 9 $l > k+1$: $sc_{01} \cdots c_{l-1,1} c_{k3} c_{l3}$ $\equiv sc_{01} \cdots c_{l-2,1} c_{l-1,4} c_{k3} c_{l1}$ now use 10
12	$b_{k-1,2} - b_{k1} b_{k-1,3}$ $k = 1, \dots, r$	$k = 1$: $sc_{01} c_{11}(b_{02} - b_{11} b_{03}) \equiv f c_{02} c_{11}(b_{02} - b_{11} b_{03}) b_{01}^d b_{02}^d$, $k > 1$: $sc_{01} \cdots c_{k1}(b_{k-1,2} - b_{k1} b_{k-1,3})$ $\equiv sc_{01} \cdots c_{k-3,1} c_{k-2,4} c_{k-1,1} c_{k1}$ $\equiv (b_{k-1,2} - b_{k1} b_{k-1,3}) b_{k-1,1}^{d^{2^{k-1}}}$ $\equiv sc_{01} \cdots c_{k-2,1} c_{k-1,2} c_{k1}(b_{k-1,2} - b_{k1} b_{k-1,3}) b_{k-1,1}^{d^{2^{k-1}}}$ $\in (sc_{01} \cdots c_{k-3,1} c_{k-2,4} c_{k-1,2} c_{k1}(b_{k-1,2} - b_{k1} b_{k-1,3})) + J$
13	$b_{k-1,2} - b_{k2} b_{k-1,3}$ $k = 1, \dots, r$	$sc_{01} \cdots c_{k1}(b_{k-1,2} - b_{k2} b_{k-1,3})$ $\equiv sc_{01} \cdots c_{k-2,1} c_{k-1,4} c_{k1}(b_{k-1,2} - b_{k2} b_{k-1,3}) b_{k1}^{d^{2^k}}$ $\equiv sc_{01} \cdots c_{k-1,1} c_{k2}(b_{k-1,2} - b_{k2} b_{k-1,3}) b_{k1}^{d^{2^k}}$ $\equiv sc_{01} \cdots c_{k-3,1} c_{k-2,4} c_{k-1,1} c_{k2} \cdot$ $\cdot (b_{k-1,2} - b_{k2} b_{k-1,3}) b_{k1}^{d^{2^k}} b_{k-1,1}^{d^{2^{k-1}}}$ $\equiv sc_{01} \cdots c_{k-2,1} c_{k-1,2} c_{k2}(b_{k-1,2} - b_{k2} b_{k-1,3}) b_{k1}^{d^{2^k}} b_{k-1,1}^{d^{2^{k-1}}}$ $\equiv sc_{01} \cdots c_{k-3,1} c_{k-2,4} c_{k-1,2} c_{k2} \cdot$ $\cdot (b_{k-1,2} - b_{k2} b_{k-1,3}) b_{k1}^{d^{2^k}} b_{k-1,1}^{d^{2^{k-1}}} b_{k-1,2}^{d^{2^{k-1}}}$
14	$b_{k-1,2} - b_{k3} b_{k-1,3}$ $k = 1, \dots, r$	$sc_{01} \cdots c_{k1}(b_{k-1,2} - b_{k3} b_{k-1,3})$ $\equiv sc_{01} \cdots c_{k-2,1} c_{k-1,4} c_{k1}(b_{k-1,2} - b_{k3} b_{k-1,3}) b_{k1}^{d^{2^k}}$ $\equiv sc_{01} \cdots c_{k-1,1} c_{k2}(b_{k-1,2} - b_{k3} b_{k-1,3}) b_{k1}^{d^{2^k}}$ $\equiv sc_{01} \cdots c_{k-1,1} c_{k3}(b_{k-1,2} - b_{k3} b_{k-1,3}) b_{k1}^{d^{2^k}}$ continue as in 13
15	$b_{k-1,2} - b_{k4} b_{k-1,3}$ $k = 1, \dots, r$	$k = 1$: $sc_{01} c_{11} b_{02} \equiv sc_{04} c_{11} b_{02} b_{11}^{d^2}$ $\equiv sc_{04} c_{14} b_{02} b_{11}^{d^2} \equiv f c_{04} c_{14} b_{02} b_{11}^{d^2} b_{04}^d$

		$\equiv sc_{02}c_{14}b_{02}b_{11}^{d^2}b_{04}^{d} \equiv fc_{02}c_{14}b_{02}b_{11}^{d^2}b_{04}^{d}b_{02}^{d}$ $\equiv fc_{02}c_{14}b_{14}b_{03}b_{11}^{d^2}b_{04}^{d}b_{02}^{d} \equiv sc_{02}c_{14}b_{14}b_{03}b_{11}^{d^2}b_{04}^{d}$ $\equiv fc_{04}c_{14}b_{14}b_{03}b_{11}^{d^2}b_{04}^{d} \equiv sc_{04}c_{14}b_{14}b_{03}b_{11}^{d^2}$ $\equiv sc_{04}c_{11}b_{14}b_{03}b_{11}^{d^2} \equiv sc_{01}c_{11}b_{14}b_{03}$ $k > 1$: $sc_{01}\cdots c_{k1}b_{k-1,2} \equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k1}b_{k-1,2}b_{k1}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k4}b_{k-1,2}b_{k1}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-3,1}c_{k-2,4}c_{k-1,4}c_{k4}b_{k-1,2}b_{k1}^{d^{2^k}}\, b_{k-1,4}^{d^{2^{k-1}}}$ $\equiv sc_{01}\cdots c_{k-3,1}c_{k-2,1}c_{k-1,2}c_{k4}b_{k-1,2}b_{k1}^{d^{2^k}}\, b_{k-1,4}^{d^{2^{k-1}}}$ $\equiv sc_{01}\cdots c_{k-3,1}c_{k-2,4}c_{k-1,2}c_{k4}b_{k-1,2}b_{k1}^{d^{2^k}}\, b_{k-1,4}^{d^{2^{k-1}}}b_{k-1,2}^{d^{2^{k-1}}}$ $\equiv sc_{01}\cdots c_{k-3,1}c_{k-2,4}c_{k-1,2}c_{k4}b_{k4}b_{k-1,3}b_{k1}^{d^{2^k}}\, b_{k-1,4}^{d^{2^{k-1}}}b_{k-1,2}^{d^{2^{k-1}}}$ $\equiv sc_{01}\cdots c_{k-3,1}c_{k-2,1}c_{k-1,2}c_{k4}b_{k4}b_{k-1,3}b_{k1}^{d^{2^k}}\, b_{k-1,4}^{d^{2^{k-1}}}$ $\equiv sc_{01}\cdots c_{k-3,1}c_{k-2,4}c_{k-1,4}c_{k4}b_{k4}b_{k-1,3}b_{k1}^{d^{2^k}}\, b_{k-1,4}^{d^{2^{k-1}}}$ $\equiv sc_{01}\cdots c_{k-3,1}c_{k-2,1}c_{k-1,4}c_{k4}b_{k4}b_{k-1,3}b_{k1}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-3,1}c_{k-2,1}c_{k-1,4}c_{k1}b_{k4}b_{k-1,3}b_{k1}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-3,1}c_{k-2,1}c_{k-1,1}c_{k1}b_{k4}b_{k-1,3}$
16	$b_{k-1,3}(b_{ki} - b_{kj})$ $k = 1, \ldots, r$ $i, j = 1, \ldots, 4$	by 12, 13, 14, 15
17	$c_{ki}(b_{k1} - b_{k4})$ $k = 0, \ldots, r$ $i = 1, \ldots, 4$	$k = 0$: $sc_{01}C_0(b_{01} - b_{04}) \equiv fc_{01}C_0(b_{01} - b_{04})b_{01}^{d}$ $\equiv sc_{02}C_0(b_{01} - b_{04})b_{01}^{d} \equiv sC_0(c_{02}b_{01} - c_{03}b_{04})b_{01}^{d}$, now use that $sc_{0i} \in J + (f)$ $k > 0$: by 2, 5, 6, and 4 it suffices to assume that $i = 2$ $sc_{01}\cdots c_{k-1,1}c_{k1}c_{k2}(b_{k1} - b_{k4})$ $\equiv sc_{01}\cdots c_{k-1,1}c_{k1}(c_{k2}b_{k1} - c_{k3}b_{k4})$ $\equiv sc_{01}\cdots c_{k-1,4}c_{k1}(c_{k2}b_{k1} - c_{k3}b_{k4})b_{k1}^{d^{2^k}} \in J$
18	$b_{k1} - b_{k4}b_{k+1,1}^{d^{2^k}}$ $k = 0, \ldots, r-1$	$sc_{01}\cdots c_{k+1,1}b_{k4}b_{k+1,1}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k1}c_{k+1,1}b_{k4}b_{k+1,1}^{d^{2^k}}b_{k1}^{d^{2^k}}$

		$\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,1}c_{k3}c_{k+1,1}b_{k4}b_{k+1,1}^{d^{2^k}}b_{k1}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k3}c_{k+1,1}b_{k4}b_{k+1,1}^{d^{2^k}}b_{k1}^{d^{2^k}}b_{k3}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k2}c_{k+1,1}b_{k+1,1}^{d^{2^k}}b_{k1}^{d^{2^k}+1}b_{k3}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k2}c_{k+1,1}b_{k1}^{d^{2^k}+1}b_{k2}^{d^{2^k}}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,1}c_{k2}c_{k+1,1}b_{k1}^{d^{2^k}+1}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k1}c_{k+1,1}b_{k1}^{d^{2^k}+1}$ $\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,1}c_{k1}c_{k+1,1}b_{k1}$
19	$c_{l3}(1-b_{ki}^{d^{2^l}})$ $l=0,\ldots,r-1$ $k=l+1,\ldots,r$ $i=2,3$	by Lemma 9.3
20	$c_{k3}(b_{k1}^{d^{2^k}}-b_{k3}^{d^{2^k}})$ $k=0,\ldots,r$	$k=0$: $sc_{01}c_{03}b_{01}^{d}\equiv sc_{01}c_{02}b_{01}^{2d}\equiv fc_{01}c_{02}b_{01}^{2d}$ $\equiv fc_{01}c_{03}b_{01}^{2d-1}b_{04}\equiv sc_{02}c_{03}b_{01}^{2d-1}b_{04}$ $\equiv sc_{02}^{2}b_{01}^{2d-1}b_{04}\equiv fc_{01}c_{02}b_{01}^{2d-1}b_{04}$, and by repetition of the last 4 steps, $\equiv fc_{01}c_{02}b_{01}^{d}b_{04}^{d}\equiv fc_{04}c_{02}b_{01}^{d}b_{04}^{d}\equiv sc_{04}c_{03}b_{01}^{d}$ $\equiv sc_{04}c_{03}b_{01}^{d}\equiv fc_{04}c_{03}b_{01}^{d}b_{03}^{d}$ $\equiv fc_{01}c_{03}b_{01}^{d}b_{03}^{d}\equiv sc_{01}c_{03}b_{03}^{d}$, when $k>0$: $s(c_{01}\cdots c_{k1})c_{k3}b_{k1}^{d^{2^k}}\equiv s(c_{01}\cdots c_{k-2,1})c_{k-1,4}c_{k1}c_{k3}b_{k1}^{2d^{2^k}}$ $\equiv s(c_{01}\cdots c_{k1})c_{k3}b_{k1}^{d^{2^k}}b_{k3}^{d^{2^k}}$ $\equiv s(c_{01}\cdots c_{k1})c_{k3}b_{k3}^{d^{2^k}}$
21	$c_{k3}(b_{k1}^{d^{2^k}}-b_{k2}^{d^{2^k}})$ $k=0,\ldots,r$	proof as for 20
22	$c_{l3}(b_{ki}-b_{kj})$ $l=0,\ldots,r-1$ $k=l+2,\ldots,r$	by Lemma 9.3, 16

	$i,j=1,\dots,4$	
23	$c_{k3}b_{k3}(1-b_{k+2,i})$	$s(c_{01}\cdots c_{r1})c_{k3}b_{k3}(1-b_{k+2,i})$
	$k=0,\dots,r-2$	$\equiv s(c_{01}\cdots c_{r1})c_{k3}b_{k+1,3}^{d^{2^k}}b_{k3}(1-b_{k+2,i})$
	$i=1,\dots,4$	$\equiv s(c_{01}\cdots c_{r1})c_{k3}b_{k+1,3}^{d^{2^k}-1}b_{k3}(b_{k+1,3}-b_{k+1,3}b_{k+2,i})$
		$\equiv s(c_{01}\cdots c_{r1})c_{k3}b_{k+1,3}^{d^{2^k}-1}b_{k3}(b_{k+1,3}-b_{k+1,2})$
		by 12, 13, 14, 15
24	$c_{l3}(1-b_{ki})$	by 16, 19, 22
	$l=0,\dots,r-3$	
	$k=l+3,\dots,r$	
	$i=1,\dots,4$	
25	$c_{k1}-c_{k3}b_{ki}^{d^{2^k}}$	$k=0$: $sc_{01}c_{03}b_{03}^d\equiv fc_{01}c_{03}b_{01}^d b_{03}^d$
	$k=0,\dots,r$	$\equiv sc_{01}c_{03}b_{01}^d\equiv fc_{01}^2b_{01}^d\equiv sc_{01}^2$
	$i=2,3$	$k>0:$ $sc_{01}\cdots c_{k1}c_{k3}b_{ki}^{d^{2^k}}$
		$\equiv sc_{01}\cdots c_{k1}c_{ki}b_{ki}^{d^{2^k}}$
		$\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k1}c_{ki}b_{ki}^{d^{2^k}}b_{k1}^{d^{2^k}}$
		$\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,1}c_{k1}c_{ki}b_{k1}^{d^{2^k}}$
		$\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,4}c_{k1}^2b_{k1}^{d^{2^k}}$
		$\equiv sc_{01}\cdots c_{k-2,1}c_{k-1,1}c_{k1}^2$
26	$c_{01}(b_{1i}-b_{11})$	if $i=1$: trivial
	$r\geq 1$	if $i=2,3$: $sc_{01}^2c_{11}(b_{1i}-b_{11})\equiv sc_{01}c_{04}c_{11}(b_{1i}-b_{11})b_{11}^{d^2}$
	$i=1,\dots,4$	$\equiv sc_{01}^2c_{11}c_{1i}(b_{1i}-b_{11})b_{11}^{d^2}\equiv fc_{01}^2c_{1i}(b_{1i}-b_{11})b_{11}^{d^2}b_{01}^d$
		$\equiv sc_{01}c_{03}c_{1i}(b_{1i}-b_{11})b_{11}^{d^2}b_{01}^d$
		$\equiv fc_{01}c_{03}c_{1i}(b_{1i}-b_{11})b_{11}^{d^2}b_{01}^d b_{03}^d$
		$\equiv fc_{02}c_{03}c_{1i}(b_{1i}-b_{11})b_{11}^{d^2}b_{01}^d b_{03}^d b_{02}^d$
		$\equiv fc_{02}c_{03}c_{1i}(b_{02}-b_{11}b_{03})b_{11}^{d^2}b_{01}^d b_{03}^{d-1}b_{02}^d$
		$\equiv sc_{02}c_{03}c_{1i}(b_{02}-b_{11}b_{03})b_{11}^{d^2}b_{01}^d b_{03}^{d-1}$
		$\equiv sc_{01}c_{02}c_{1i}(b_{02}-b_{11}b_{03})b_{11}^{d^2}b_{03}^{d-1}$

		$\equiv sc_{04}c_{02}c_{11}(b_{02}-b_{11}b_{03})b_{11}^{d^2}b_{03}^{d-1}$ $\equiv sc_{01}c_{02}c_{11}(b_{02}-b_{11}b_{03})b_{03}^{d-1}$ $\equiv fc_{01}c_{02}c_{11}(b_{02}-b_{11}b_{03})b_{03}^{d-1}b_{02}^{d} \in J$ if $i=4$: $sc_{01}^2c_{11}(b_{14}-b_{11}) \equiv sc_{01}^2c_{14}(b_{14}-b_{11})$, now continue as for $i=2,3$
27	$c_{k1}(b_{k+1,i}-b_{k+1,1})$ $k=0,\dots,r-1$ $i=1,\dots,4$	essentially the same proof as for 26
28	$c_{k3}c_{k+1,1}(b_{k+1,i}-b_{k+1,1})$ $k=0,\dots,r-1$ $i=1,\dots,4$	$i=1$: trivial, $i=4$: by 17 $i=2,3$: $k=0$: $sc_{01}c_{03}c_{11}^2b_{1i} \equiv sc_{04}c_{03}c_{11}^2b_{1i}b_{11}^{d^2}$ $\equiv sc_{01}c_{03}c_{11}c_{1i}b_{1i}b_{11}^{d^2} \equiv fc_{01}c_{03}c_{11}c_{1i}b_{1i}b_{11}^{d^2}b_{03}^{d}$ $\equiv fc_{02}c_{03}c_{11}c_{1i}b_{1i}b_{11}^{d^2}b_{03}^{d}b_{02}^{d}$ $\equiv fc_{02}c_{03}c_{11}c_{1i}b_{11}^{d^2}b_{03}^{d-1}b_{02}^{d+1}$ $\equiv fc_{02}c_{03}c_{11}c_{1i}b_{11}^{d^2+1}b_{03}^{d}b_{02}^{d} \equiv fc_{01}c_{03}c_{11}c_{1i}b_{11}^{d^2+1}b_{03}^{d}$ $\equiv sc_{01}c_{03}c_{11}c_{1i}b_{11}^{d^2+1} \equiv sc_{04}c_{03}c_{11}^2b_{11}^{d^2+1}$ $\equiv sc_{01}c_{03}c_{11}^2b_{11}$ $k>0$: set $A=s(c_{01}\cdots c_{k-2,1})$: $s(c_{01}\cdots c_{k-1,1})c_{k4}c_{k3}c_{k+1,1}^2$ $= Ac_{k-1,1}c_{k1}c_{k3}c_{k+1,1}^2b_{k+1,i}$ $\equiv Ac_{k-1,1}c_{k4}c_{k3}c_{k+1,1}^2b_{k+1,i}b_{k+1,1}^{d^{k+1}}$ $\equiv Ac_{k-1,1}c_{k1}c_{k3}c_{k+1,1}c_{k+1,i}b_{k+1,i}b_{k+1,1}^{d^{k+1}}$ $\equiv Ac_{k-1,4}c_{k1}c_{k3}c_{k+1,1}c_{k+1,i}b_{k+1,i}b_{k+1,1}^{d^{k+1}}b_{k3}^{d^{2^k}}$ $\equiv Ac_{k-1,1}c_{k2}c_{k3}c_{k+1,1}c_{k+1,i}b_{k+1,i}b_{k+1,1}^{d^{k+1}}b_{k3}^{d^{2^k}}$ $\equiv Ac_{k-1,4}c_{k2}c_{k3}c_{k+1,1}c_{k+1,i}b_{k+1,i}b_{k+1,1}^{d^{k+1}}b_{k3}^{d^{2^k}}b_{k2}^{d^{2^k}}$ $\equiv Ac_{k-1,4}c_{k2}c_{k3}c_{k+1,1}c_{k+1,i}b_{k+1,1}^{d^{k+1}}b_{k3}^{d^{2^k}-1}b_{k2}^{d^{2^k}+1}$ $\equiv Ac_{k-1,4}c_{k2}c_{k3}c_{k+1,1}c_{k+1,i}b_{k+1,1}^{d^{k+1}+1}b_{k3}^{d^{2^k}}b_{k2}^{d^{2^k}}$

		$\equiv Ac_{k-1,1}c_{k2}c_{k3}c_{k+1,1}c_{k+1,i}b_{k+1,1}^{d^{k+1}+1}b_{k3}^{d^{2^k}}$ $\equiv Ac_{k-1,4}c_{k1}c_{k3}c_{k+1,1}c_{k+1,i}b_{k+1,1}^{d^{k+1}+1}b_{k3}^{d^{2^k}}$ $\equiv Ac_{k-1,1}c_{k1}c_{k3}c_{k+1,1}c_{k+1,i}b_{k+1,1}^{d^{k+1}+1}$ $\equiv Ac_{k-1,1}c_{k4}c_{k3}c_{k+1,1}^2b_{k+1,1}^{d^{k+1}+1}$ $\equiv Ac_{k-1,1}c_{k1}c_{k3}c_{k+1,1}^2b_{k+1,1}$
29	$c_{ki}(b_{ki}-b_{k1})$ $k=1,\ldots,r$ $i=1,\ldots,4$	$k=1$: $sc_{01}c_{11}c_{1i}b_{1i}\equiv fc_{02}c_{11}c_{1i}b_{1i}b_{01}^d b_{02}^d$ $\equiv fc_{02}c_{11}c_{1i}b_{11}b_{1i}b_{01}^d b_{02}^{d-1}b_{03}$ $\equiv fc_{02}c_{11}c_{1i}b_{11}b_{01}^d b_{02}^d$ $k>1$: set $A=sc_{01}\cdots c_{k-3,1}$, then $sc_{01}\cdots c_{k1}c_{ki}b_{ki}=Ac_{k-2,1}c_{k-1,1}c_{k1}c_{ki}b_{ki}$ $\equiv Ac_{k-2,1}c_{k-1,4}c_{k1}c_{ki}b_{ki}b_{k1}^{d^{2^k}}$ $\equiv Ac_{k-2,4}c_{k-1,4}c_{k1}c_{ki}b_{ki}b_{k1}^{d^{2^k}}b_{k-1,4}^{d^{2^{k-1}}}$ $\equiv Ac_{k-2,1}c_{k-1,3}c_{k1}c_{ki}b_{ki}b_{k1}^{d^{2^k}}b_{k-1,4}^{d^{2^{k-1}}}$ $\equiv Ac_{k-2,4}c_{k-1,3}c_{k1}c_{ki}b_{ki}b_{k1}^{d^{2^k}}b_{k-1,4}^{d^{2^{k-1}}}b_{k-1,3}^{d^{2^{k-1}}}$ $\equiv Ac_{k-2,4}c_{k-1,2}c_{k1}c_{ki}b_{ki}b_{k1}^{d^{2^k}}b_{k-1,4}^{d^{2^{k-1}}-1}b_{k-1,1}b_{k-1,3}^{d^{2^{k-1}}}$ $\equiv Ac_{k-2,4}c_{k-1,2}c_{k1}c_{ki}b_{k1}^{d^{2^k}}b_{k-1,4}^{d^{2^{k-1}}-1}b_{k-1,1}b_{k-1,3}^{d^{2^{k-1}}-1}b_{k-1,2}$ $\equiv Ac_{k-2,4}c_{k-1,2}c_{k1}c_{ki}b_{k1}^{d^{2^k}+1}b_{k-1,4}^{d^{2^{k-1}}-1}b_{k-1,1}b_{k-1,3}^{d^{2^{k-1}}}$ $\equiv Ac_{k-2,4}c_{k-1,3}c_{k1}c_{ki}b_{k1}^{d^{2^k}+1}b_{k-1,4}^{d^{2^{k-1}}}b_{k-1,3}^{d^{2^{k-1}}}$ $\equiv Ac_{k-2,1}c_{k-1,3}c_{k1}c_{ki}b_{k1}^{d^{2^k}+1}b_{k-1,4}^{d^{2^{k-1}}}$ $\equiv Ac_{k-2,4}c_{k-1,4}c_{k1}c_{ki}b_{k1}^{d^{2^k}+1}b_{k-1,4}^{d^{2^{k-1}}}$ $\equiv Ac_{k-2,1}c_{k-1,4}c_{k1}c_{ki}b_{k1}^{d^{2^k}+1}$ $\equiv Ac_{k-2,1}c_{k-1,1}c_{k1}c_{ki}b_{k1}$
30	$c_{k1}(b_{ki}-b_{k1})$ $k=1,\ldots,r$ $i=1,\ldots,4$	by 4, 5, 6, 2, and 29
31	$c_{r1}c_{r+1,i}(b_{r+1,i}-1)$ $r\geq 1$	set $A=sc_{01}\cdots c_{r-2,1}$, then $sc_{01}\cdots c_{r1}c_{r1}c_{r+1,i}b_{r+1,i}$

	$i=1,\ldots,4$	$= Ac_{r-1,1}c_{r1}c_{r1}c_{r+1,i}b_{r+1,i}$ $\equiv Ac_{r-1,1}c_{r1}c_{r4}c_{r+1,i}b_{r+1,i}^{d^{2^{r+1}}+1}$ $\equiv Ac_{r-1,4}c_{r1}c_{r4}c_{r+1,i}b_{r+1,i}^{d^{2^{r+1}}+1}b_{r4}^{d^{2^r}}$ $\equiv Ac_{r-1,1}c_{r3}c_{r4}c_{r+1,i}b_{r+1,i}^{d^{2^{r+1}}+1}b_{r4}^{d^{2^r}}$ $\equiv Ac_{r-1,4}c_{r3}c_{r4}c_{r+1,i}b_{r+1,i}^{d^{2^{r+1}}+1}b_{r4}^{d^{2^r}}b_{r3}^{d^{2^r}}$ $\equiv Ac_{r-1,4}c_{r2}c_{r4}c_{r+1,i}b_{r+1,i}^{d^{2^{r+1}}+1}b_{r4}^{d^{2^r}-1}b_{r1}b_{r3}^{d^{2^r}}$ $\equiv Ac_{r-1,4}c_{r2}c_{r4}c_{r+1,i}b_{r+1,i}^{d^{2^{r+1}}}b_{r4}^{d^{2^r}-1}b_{r1}b_{r3}^{d^{2^r}-1}b_{r2}$ $\equiv Ac_{r-1,4}c_{r3}c_{r4}c_{r+1,i}b_{r+1,i}^{d^{2^{r+1}}}b_{r4}^{d^{2^r}}b_{r3}^{d^{2^r}-1}b_{r2}$ $\equiv Ac_{r-1,1}c_{r3}c_{r4}c_{r+1,i}b_{r+1,i}^{d^{2^{r+1}}}b_{r3}^{d^{2^r}-1}b_{r2}$ $\equiv Ac_{r-1,1}c_{r3}c_{r1}c_{r+1,i}b_{r3}^{d^{2^r}-1}b_{r2}$ $\equiv Ac_{r-1,1}c_{r3}c_{r1}c_{r+1,i}b_{r1}^{d^{2^r}-1}b_{r2}$ by 29, $\equiv Ac_{r-1,1}c_{r2}c_{r1}c_{r+1,i}b_{r1}^{d^{2^r}-1}b_{r2}$ $\equiv Ac_{r-1,1}c_{r2}c_{r1}c_{r+1,i}b_{r1}^{d^{2^r}}$ by 29, $\equiv Ac_{r-1,4}c_{r1}c_{r1}c_{r+1,i}b_{r1}^{d^{2^r}}$ $\equiv Ac_{r-1,1}c_{r1}c_{r1}c_{r+1,i}$
32	$c_{r1}c_{r+1,1}(b_{r+1,i}-1)$ $r\geq 1$ $i=1,\ldots,4$	by 4, 5, 6, 2, and 31

Lemma 9.4:

$$\begin{aligned}(J+(s^2,sf)) :s(c_{01}\cdots c_{n-1,1})c_{03} = {} & (s,f)+D_0+D_1+\cdots+D_{n-1}+B_{3,n-1}\\ & +(b_{2i}-b_{21},c_{01}(b_{1i}-b_{11}),1-b_{1i}^d,1-b_{2i}^d)\\ & +(b_{01}^d-b_{03}^d,b_{01}^d-b_{02}^d)\\ & +(b_{k2}-b_{k+1,i}b_{k3},b_{k1}-b_{k4}\mid k=0,\ldots,n-1),\end{aligned}$$

where all the indices i vary from 1 to 4.

Proof: Let $I=(J+(s^2,sf)):s(c_{01}\cdots c_{n-1,1})c_{03}$. $B_{3,n-1}$ is contained in I by 24 from the table. By 19, I also contains $1-b_{k2}^d, 1-b_{k3}^d$ for all $k=1,\ldots,n-1$. Thus by 16, all the differences $b_{2i}-b_{21}$ are in I, and hence all $1-b_{2i}^d\in I$. By 7, $1-b_{11}^d\in I$.

By 12-15, all $b_{k2}-b_{k+1,i}b_{k3}$ are in I, and by 18 also all $b_{k1}-b_{k+1,1}^{d^{2^k}}b_{k4}$ are in I. As all $1-b_{k+1,1}^{d^{2^k}}$ are also in I, it follows that all $b_{k1}-b_{k4}$ are in I. In particular, also all $1-b_{1i}^d$

are in I.

Entries 2, 4, 5, and 6 prove that $D_0 + \cdots + D_{n-1}$ is in I. Finally, $c_{01}(b_{1i} - b_{11})$ is in I by 26. This proves one inclusion.

Now let $y \in I$. Then $s(c_{01} \cdots c_{n-1,1})c_{03}y \in J + (s^2, sf)$, so that by Lemma 9.1, $(c_{01} \cdots c_{n-1,1})c_{03}y$ is in

$$
\begin{aligned}
&(s, f, c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d) + JJ \\
&\quad + c_{02}^2 \left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) + c_{02}\left(b_{01}b_{03}^d - b_{04}b_{02}^d\right) \\
&\quad + c_{02}\left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(c_{03} - b_{1i}^d c_{02}), c_{1i}c_{1j}(b_{1i} - b_{1j})\right) \\
&= (s, f, c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d) \\
&\quad + c_{02}^2 \left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) + c_{02}\left(b_{01}b_{03}^d - b_{04}b_{02}^d\right) \\
&\quad + c_{02}\left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(c_{03} - b_{1i}^d c_{02}), c_{1i}c_{1j}(b_{1i} - b_{1j})\right) \\
&\quad + B_{3,n-1} + D_2 + \cdots + D_{n-1} + (b_{2i} - b_{21}) + \frac{1}{s}(h_{2j}, h_{3j}, h_{4j} | \text{ all appropriate } j \\
&= (s, f, c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d) \\
&\quad + c_{02}^2 \left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) + c_{02}\left(b_{01}b_{03}^d - b_{04}b_{02}^d\right) \\
&\quad + c_{02}\left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(c_{03} - b_{1i}^d c_{02}), c_{1i}c_{1j}(b_{1i} - b_{1j})\right) \\
&\quad + B_{3,n-1} + D_2 + \cdots + D_{n-1} + (b_{2i} - b_{21}) \\
&\quad + (c_{04}c_{11} - c_{01}c_{12}, c_{04}c_{14} - c_{01}c_{13}, c_{01}(c_{12} - c_{13})) \\
&\quad + (c_{04}(c_{12}b_{11} - c_{13}b_{14}), c_{04}c_{12}c_{21}(b_{12} - b_{21}b_{13}), c_{01}(c_{14} - c_{11})c_{21}).
\end{aligned}
$$

Thus by going modulo $(s, f) + D_2 + \cdots + D_{n-1} + B_{3,n-1} + (b_{2i} - b_{21}) + (c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d)$, this says that $c_{02}^2 b_{01}^d (c_{11} \cdots c_{n-1,1})y$ is contained in

$$
\begin{aligned}
&c_{02}^2 \left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) + c_{02}\left(b_{01}b_{03}^d - b_{04}b_{02}^d\right) \\
&\quad + c_{02}\left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}c_{02}(1 - b_{1i}^d), c_{1i}c_{1j}(b_{1i} - b_{1j})\right) \\
&\quad + (c_{02}(b_{04}^d c_{11} - b_{01}^d c_{12}), (c_{02}(b_{04}^d c_{14} - b_{01}^d c_{13}), c_{02}b_{01}^d(c_{12} - c_{13})) \\
&\quad + (c_{02}b_{04}^d(c_{12}b_{11} - c_{13}b_{14}), c_{02}b_{04}^d c_{12}c_{21}(b_{12} - b_{21}b_{13}), c_{02}b_{01}^d(c_{14} - c_{11})c_{21}).
\end{aligned}
$$

By coloning out with c_{02}^d, it follows that $b_{01}^d(c_{11} \cdots c_{n-1,1})y$ is contained in

$$
\begin{aligned}
&\left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d\right) \\
&\quad + \left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(1 - b_{1i}^d), c_{1i}c_{1j}(b_{1i} - b_{1j})\right) \\
&\quad + (b_{04}^d c_{11} - b_{01}^d c_{12}, b_{04}^d c_{14} - b_{01}^d c_{13}, b_{01}^d(c_{12} - c_{13})) \\
&\quad + (b_{04}^d(c_{12}b_{11} - c_{13}b_{14}), b_{04}^d c_{12}c_{21}(b_{12} - b_{21}b_{13}), b_{01}^d(c_{14} - c_{11})c_{21}).
\end{aligned}
$$

Next we may go modulo $b_{04} - b_{01}$ so that

$$
\begin{aligned}
&\left(b_{01}^d - b_{02}^d, b_{01}^d - b_{03}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}(1 - b_{1i}^d), c_{1i}c_{1j}(b_{1i} - b_{1j})\right) \\
&\quad + (b_{01}^d(c_{11} - c_{12}), b_{01}^d(c_{14} - c_{13}), b_{01}^d(c_{12} - c_{13})) \\
&\quad + (b_{01}^d(c_{12}b_{11} - c_{13}b_{14}), b_{01}^d c_{12}c_{21}(b_{12} - b_{21}b_{13}), b_{01}^d(c_{14} - c_{11})c_{21}),
\end{aligned}
$$

and then colon with b_{01}^d to get that $(c_{11}\cdots c_{n-1,1})y$ is contained in

$$\begin{aligned}&\big(b_{01}^d-b_{02}^d, b_{01}^d-b_{03}^d, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}(1-b_{1i}^d), c_{1i}c_{1j}(b_{1i}-b_{1j})\big) : b_{01}^d\\ &\quad+(c_{11}-c_{12}, c_{14}-c_{13}, c_{12}-c_{13}, c_{12}b_{11}-c_{13}b_{14}, c_{12}c_{21}(b_{12}-b_{21}b_{13}), (c_{14}-c_{11})c_{21})\\ &=\big(b_{01}^d-b_{02}^d, b_{01}^d-b_{03}^d, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}(1-b_{1i}^d), c_{1i}c_{1j}(b_{1i}-b_{1j})\big)\\ &\quad+(c_{11}-c_{12}, c_{14}-c_{13}, c_{12}-c_{13}, c_{12}b_{11}-c_{13}b_{14}, c_{12}c_{21}(b_{12}-b_{21}b_{13}), (c_{14}-c_{11})c_{21}).\end{aligned}$$

Next, by going modulo D_1,

$$\begin{aligned}c_{11}\cdots c_{n-1,1}y\in{}&\big(b_{01}^d-b_{02}^d, b_{01}^d-b_{03}^d, c_{11}(b_{02}-b_{1i}b_{03}), c_{11}(1-b_{1i}^d), c_{11}^2(b_{1i}-b_{1j})\big)\\ &+(c_{11}(b_{11}-b_{14}), c_{11}c_{21}(b_{12}-b_{21}b_{13})),\end{aligned}$$

whence by coloning with $c_{11}\cdots c_{n-1,1}$

$$y\in\big(b_{01}^d-b_{02}^d, b_{01}^d-b_{03}^d, b_{02}-b_{1i}b_{03}, 1-b_{1i}^d, c_{11}(b_{1i}-b_{1j}), b_{11}-b_{14}, b_{12}-b_{21}b_{13}\big).\qquad\blacksquare$$

Proposition 9.5: *The set of embedded primes of J is contained in*

$$\{Q, Q_{k\alpha}, Q'\}\cup Ass\ \big(R/(J+(s^2, sf, sc_{01}^2c_{11}\cdots c_{n-1,1}))\big).$$

Proof: Let $x=sc_{01}^2c_{11}\cdots c_{n-1,1}$. We already know that the set of embedded primes of J is contained in $\{Q, Q_{k\alpha}, Q'\}\cup \mathrm{Ass}\ \big(R/(J+(s^2, sf))\big)$. By Lemma 9.2,

$$\mathrm{Ass}\ \big(R/(J+(s^2,sf))\big)\subseteq \mathrm{Ass}\ \big(R/(J+(s^2,sf,x))\big)\cup \mathrm{Ass}\ \big(R/((J+(s^2,sf)):x)\big),$$

so that it suffices to prove that none of the associated primes of $(J+(s^2,sf)):x$ is associated to J.

Clearly, $(s,f)\subseteq (J+(s^2,sf)):x$. By numbers 12, 16, 19, and 26 from the table, $F\subseteq (J+(s^2,sf)):x$, and by 17, 20, and 21, $E\subseteq (J+(s^2,sf)):x$. As $b_{12}-b_{13}, 1-b_{13}^d\in (J+(s^2,sf)):x$, by 12-15 then for all $i=1,\ldots,4$, $1-b_{2i}\in (J+(s^2,sf)):x$. Thus by 24, $B_{n-1}\subseteq (J+(s^2,sf)):x$. It follows that by 2, 5, 6, 4, and 19, $D_0+\cdots+D_{n-1}\subseteq (J+(s^2,sf)):x$. Thus $p_n+(s,f)$ is contained in $(J+(s^2,sf)):x$.

If $y\in (J+(s^2,sf)):x$, then

$$xy=sc_{01}^2c_{11}\cdots c_{n-1,1}y\in p_n+(s^2,sf).$$

Modulo p_n, we may rewrite s as fb_{01}^d, c_{03} as c_{02}, c_{04} as c_{01}, c_{01} as $c_{02}b_{01}^d$, b_{02} as $b_{11}b_{03}$, b_{04} as b_{01}, all b_{1i} as b_{11}, for $r>0$, we can rewrite each c_{ri} as c_{r1}, and for $r>1$, we can write each b_{ri} as 1. Thus the displayed equation implies that

$$fc_{02}^2b_{01}^{3d}c_{11}\cdots c_{n-1,1}y\in (f^2b_{01}^d, b_{01}^d-b_{03}^d, 1-b_{11}^d),$$

whence $y\in (f, b_{01}^d-b_{03}^d, 1-b_{11}^d)$, so that $(J+(s^2,sf)):x=p_n+(s,f)$. But now, $p_n+(s,f)$ is the intersection of d^2 prime ideals none of which contains c_{01} so that by Proposition 6.8, none of these prime ideals is associated to J. $\blacksquare$

Lemma 9.6: *For all $k \leq n-2$ and $r \geq k+1$,*

$$s(c_{01}\cdots c_{r-1,1})c_{k1}(c_{r1},c_{r3}) \subseteq J + (s(c_{01}\cdots c_{r1})c_{k1}).$$

Proof: Certainly $s(c_{01}\cdots c_{r-1,1})c_{k1}c_{r1}$ is in the ideal. Modulo J,

$$s(c_{01}\cdots c_{r-1,1})c_{k1}c_{r3} \equiv s(c_{01}\cdots c_{r-2,1})c_{k1}c_{r-1,4}c_{r1} \equiv s(c_{01}\cdots c_{r-2,1})c_{k1}c_{r-1,1}c_{r1},$$

where the last equivalence is by in 5 and 6 in the table. This proves the lemma. ∎

Lemma 9.7: *For $k = 1, \ldots, n-1$, define*

$$I_k = J + (s^2, sf, sc_{01}^2 c_{11}\cdots c_{n-k,1}),$$
$$x_k = sc_{01}^2 c_{11}\cdots c_{n-k-1,1}.$$

Then $I_k : x_k = p_{n-k} + (s,f)$, which is the intersection of d^2 primes, none of which contains c_{01}.

Proof: The same proof as in Proposition 9.5 shows that $(s,f) + E + D_0 + D_{11} + \cdots + D_{n-k-1,n-k-1} + B_{n-k-1}$ is contained in $I_k : x_k$. Furthermore, if $k \leq n-2$, also $F \subseteq I_k : x_k$. By the previous lemma, $c_{n-k,1}, c_{n-k,3} \in I_k : x_k$, so that by 2 and 8, $C_{n-k} \subseteq I_k : x_k$. This proves that $p_{n-k} + (s,f) \subseteq I_k : x_k$.

Now let $y \in I_k : x_k$. Then $yx_k \in I_k \subseteq p_{n-k} + (s^2, sf, sc_{01}^2 c_{11}\cdots c_{n-k,1})$, and similar arguments as before show that $y \in p_{n-k} + (s,f)$. ∎

Proposition 9.8: *The set of embedded primes of J is contained in the set*

$$\{Q, Q_{k\alpha}, Q'\} \cup Ass\ \left(R/(J + (s^2, sf, sc_{01}^2))\right).$$

Proof: By Proposition 9.5, it suffices to prove that each element of

$$\text{Ass}\ \left(R/(J + (s^2, sf, sc_{01}^2 c_{11}\cdots c_{n-1,1}))\right)$$

is either associated to $R/(J + (s^2, sf, sc_{01}^2))$ or is not associated to J. We use the notation from the previous lemma, Lemma 9.7. Define for each $k = 1, \ldots, n-1$ and each of the dth roots of unity $\alpha, \beta \in F$, the ideals

$$Q_{k\alpha\beta} = p_k + (s, f, b_{01} - \alpha b_{02}, b_{02} - \beta b_{03}, (1 - \delta_{k1})(\beta - b_{11})),$$

where δ is the Kronecker delta function. Each $Q_{k\alpha\beta}$ is a prime ideal, minimal over $p_k + (s,f)$, and furthermore,

$$p_k + (s,f) = \bigcap_{\alpha,\beta} Q_{k\alpha\beta}.$$

Lemma 9.7 proves that

$$\text{Ass}\ \left(R/(J + (s^2, sf, sc_{01}^2 c_{11}\cdots c_{n-1,1}))\right) = \text{Ass}\ (R/I_1))$$

$$
\begin{aligned}
&\subseteq \{Q_{n-1,\alpha\beta}|\alpha,\beta\} \cup \text{Ass } (R/I_2)\\
&\subseteq \{Q_{n-1,\alpha\beta}, Q_{n-2,\alpha\beta}|\alpha,\beta\} \cup \text{Ass } (R/I_3)\\
&\subseteq \cdots\\
&\subseteq \{Q_{n-1,\alpha\beta}, Q_{n-2,\alpha\beta}, \ldots, Q_{2\alpha\beta}|\alpha,\beta\} \cup \text{Ass } (R/I_{n-1})\\
&\subseteq \{Q_{k\alpha\beta}|k;\alpha,\beta\} \cup \text{Ass } \left(R/(s^2, sf, sc_{01}^2)\right).
\end{aligned}
$$

However, none of the prime ideals $Q_{k\alpha\beta}$ contains c_{01}, so that by Proposition 6.8, none of them is associated to J. This proves the proposition. ■

Our continuation of the search for the embedded primes of R/J is leading us to consider ideals of the following form:

$$
L_r = J + \left(s^2, sf\right) + s\left(c_{k1}\left(\prod_{i=0}^{k} c_{i1}\right)\left(\prod_{i=0}^{k} b_{i3}\right)\middle| k = 0, \ldots, r-1\right),
$$

where $r = 1, \ldots, n$. We have reduced the problem of finding the rest of the embedded primes of J to finding the associated primes of R/L_1. We will use the ideals $L_r : l_r$, where l_r is the element

$$
l_r = s(c_{01} \cdots c_{n-1,1})c_{r1}(b_{03} \cdots b_{r3}).
$$

To simplify notation, we define the following ideals in R:

$$
\begin{aligned}
D_{rr} &= (c_{r2} - c_{r3}, c_{r1} - c_{r4}, c_{r1} - c_{r2}b_{r1}^{d^{2^r}}),\\
V_r &= (b_{02} - b_{11}b_{03}, b_{01} - b_{04}b_{11}^d, b_{ki} - b_{kj}\big|k = 1, \ldots, r; i, j = 1, \ldots, 4)\\
U_r &= c_{03}(1 - b_{11}^d, b_{01}^d, b_{03}^d) + (c_{k1}, c_{k4}, c_{k2} - c_{k3}, c_{k3}b_{k1}^{d^{2^k}}|k = 0, \ldots, r).
\end{aligned}
$$

Note that $D_{00} = D_0$.

Proposition 9.9: *For all $r = 1, \ldots, n-1$,*

$$
L_r : l_r = (s, f) + V_r + B_{n-1} + U_{r-1} + D_{rr} + D_{r+1} + \cdots + D_{n-1}.
$$

Note that when $r \geq 2$, the same ideal is generated when D_{rr} above is replaced by D_r.

Definition 9.10: *It is easy to see that $C_2 + \cdots + C_{r-1} \subseteq U_{r-1} + B_{n-1}$, so that when $r \geq 2$,*

$$
\begin{aligned}
L_r : l_r = &((s, f) + V_r + B_{n-1} + C_0 + \cdots + C_{r-1} + D_r + D_{r+1} + \cdots + D_{n-1})\\
&\cap \left((s, f, c_{01}, c_{04}, c_{02} - c_{03}, b_{01}^d, b_{03}^d, 1 - b_{11}^d)\right.\\
&\quad \left. + V_r + B_{n-1} + C_1 + \cdots + C_{r-1} + D_r + D_{r+1} + \cdots + D_{n-1}\right)\\
&\cap \Big((s, f, c_{11}, c_{14}, c_{12} - c_{13}, b_{11}^{d^2})\\
&\quad + V_r + B_{n-1} + C_0 + C_2 + \cdots + C_{r-1} + D_r + D_{r+1} + \cdots + D_{n-1}\Big).
\end{aligned}
$$

Clearly the first ideal Q''_{r0} in the decomposition above is prime; the second ideal is the intersection of the d primary ideals

$$= (s, f, c_{01}, c_{04}, c_{02} - c_{03}, b_{01}^d, b_{03}^d, \alpha - b_{11}) \\ + V_r + B_{n-1} + C_1 + \cdots + C_{r-1} + D_r + D_{r+1} + \cdots + D_{n-1},$$

where α ranges over the dth roots of unity, and the third ideal q'''_{r0} is primary. Thus the associated prime ideals of $L_r : l_r$ for $r \geq 2$ are

$$\begin{aligned} Q''_{r0} &= (s, f) + V_r + B_{n-1} + C_0 + \cdots + C_{r-1} + D_r + D_{r+1} + \cdots + D_{n-1} \\ Q''_{r0\alpha} &= (s, f, c_{01}, c_{04}, c_{02} - c_{03}, b_{01}, b_{03}, \alpha - b_{11}) \\ &\qquad V_r + B_{n-1} + C_1 + \cdots + C_{r-1} + D_r + \cdots + D_{n-1}, \\ Q'''_{r0} &= (s, f, c_{11}, c_{14}, c_{12} - c_{13}, b_{11}) \\ &\qquad + V_r + B_{n-1} + C_0 + C_2 + \cdots + C_{r-1} + D_r + D_{r+1} + \cdots + D_{n-1}. \end{aligned}$$

When $r = 1$, the associated prime ideals of $L_1 : l_1$ are

$$\begin{aligned} Q''_{10} &= (s, f) + V_1 + B_{n-1} + C_0 + D_{11} + D_2 + \cdots + D_{n-1} \\ Q''_{10\alpha} &= (s, f, c_{01}, c_{04}, c_{02} - c_{03}, b_{01}, b_{03}, \alpha - b_{11}) \\ &\qquad + V_1 + B_{n-1} + C_1 + D_2 + \cdots + D_{n-1}. \end{aligned}$$

Proof of the proposition: Certainly $s, f \in L_r : l_r$. By 12-15 and 18, $V_r \subseteq L_r : l_r$. By 24 and 23, for all $k = r+2, \ldots, n-1$ and all $i = 1, \ldots, 4$, $1 - b_{ki} \in L_r : l_r$. Also, for all $k = 1, \ldots, r+1$ and $i = 1, \ldots, 4$, by 16, $b_{ki} - b_{k3} \in (L_r : sc_{01} \cdots c_{n-1,1} b_{k-1,3})$, so that by 12-15, $b_{k3}(1 - b_{k+1,i}) \in (L_r : sc_{01} \cdots c_{n-1} b_{k-1,3})$. This implies that for all $k = 2, \ldots, r+1$ and all $i = 1, \ldots, 4$, $1 - b_{ki} \in L_r : l_r$, and hence that $B_{n-1} \subseteq L_r : l_r$. Then by 2, 4, 5, 6, 8 and 19, $D_0 + D_{11} + D_2 + \cdots + D_{r-1} + D_{rr} + D_{r+1} + \cdots + D_{n-1} \subseteq L_r : l_r$. For $k = 0, \ldots, r-1$, certainly $c_{k1} \in L_r : s(c_{01} \cdots c_{k1}) c_{k1} (b_{03} \cdots b_{k3})$. Thus by 4 from the table, $c_{k3} b_{k1}^{d^{2^k}} \in L_r : l_r$, and then by 2, 5 and 6, $U_{r-1} \subseteq L_r : l_r$. Also, by 25 and 7, $c_{03} b_{03}^d, c_{03}(1 - b_{11}^d) \in L_r : l_r$. This proves one inclusion.

Now let $y \in L_r : l_r$. Then

$$\begin{aligned} y l_r &= y s (c_{01} \cdots c_{n-1,1}) c_{r1} (b_{03} \cdots b_{r3}) \in L_r \\ &\subseteq (s^2, sf, sc_{01}^2, \ldots, sc_{r-1,1}^2) + (b_{1i} - b_{1j} | i, j = 1, \ldots, 4) \\ &\qquad + B_{n-1} + D_2 + \cdots + D_{r-1} + D_{rr} + D_{r+1} + \cdots + D_{n-1} \\ &\qquad + (h_{0j}, h_{1j}, h_{2j}, h_{3j} | \text{ all appropriate } j) \\ &= (s^2, sf, sc_{01}^2, \ldots, sc_{r-1,1}^2) + (b_{1i} - b_{1j} | i, j = 1, \ldots, 4) \\ &\qquad + B_{n-1} + D_2 + \cdots + D_{r-1} + D_{rr} + D_{r+1} + \cdots + D_{n-1} \\ &\qquad + (s(c_{01} - c_{02} b_{01}^d), f(c_{01} - c_{02} b_{02}^d), f(c_{04} - c_{03} b_{03}^d), s(c_{04} - c_{03} b_{04}^d), fc_{01} - sc_{02}, \\ &\qquad\quad f(c_{01} - c_{04}), s(c_{02} - c_{03}), f(c_{02} b_{01} - c_{03} b_{04}), fc_{02} c_{1i} (b_{02} - b_{11} b_{03}), \\ &\qquad\quad s(c_{04} c_{11} - c_{01} c_{12}), s(c_{04} c_{14} - c_{01} c_{13}), sc_{01}(c_{12} - c_{13}), \\ &\qquad\quad sc_{04}(c_{12} - c_{13}) b_{11}, sc_{01}(c_{14} - c_{11}) c_{21}). \end{aligned}$$

(Above, as well as below, the index i varies from 1 to 4.) First assume that $r > 1$. Thus D_{rr} may be replaced by D_r. It follows that by coloning with $b_{23} \cdots b_{r3}$ and reducing modulo the established elements of $L_r : l_r$, such as $b_{1i} - b_{1j}$, generators of B_{n-1} and the D_i, $ys(c_{01} \cdots c_{n-1,1})c_{r1}b_{03}b_{11}$ is contained in

$$\begin{aligned}
&(s^2, sf, sc_{01}^2, \ldots, sc_{r-1,1}^2, s(c_{01} - c_{02}b_{01}^d), f(c_{01} - c_{02}b_{02}^d), f(c_{04} - c_{03}b_{03}^d),\\
&\quad s(c_{04} - c_{03}b_{04}^d), fc_{01} - sc_{02}, f(c_{01} - c_{04}), s(c_{02} - c_{03}), f(c_{02}b_{01} - c_{03}b_{04}),\\
&\quad fc_{02}c_{1i}(b_{02} - b_{11}b_{03}), s(c_{04}c_{11} - c_{01}c_{12}), s(c_{04}c_{14} - c_{01}c_{13}), sc_{01}(c_{12} - c_{13}),\\
&\quad sc_{04}(c_{12} - c_{13})b_{11}, sc_{01}(c_{14} - c_{11})c_{21}).
\end{aligned}$$

By taking the colon with $c_{21} \cdots c_{r-1,1}c_{r1}^2 c_{r+1,1} \cdots c_{r-1,1}$, it follows that

$$\begin{aligned}
ysc_{01}c_{11}b_{03}b_{11} \in {}&(s^2, sf, sc_{01}^2, sc_{11}^2, c_{21}, \ldots, c_{r-1,1}, s(c_{01} - c_{02}b_{01}^d), f(c_{01} - c_{02}b_{02}^d),\\
&f(c_{04} - c_{03}b_{03}^d), s(c_{04} - c_{03}b_{04}^d), fc_{01} - sc_{02}, f(c_{01} - c_{04}), s(c_{02} - c_{03}),\\
&f(c_{02}b_{01} - c_{03}b_{04}), fc_{02}c_{1i}(b_{02} - b_{11}b_{03}), s(c_{04}c_{11} - c_{01}c_{12}), s(c_{04}c_{14} - c_{01}c_{13}),\\
&sc_{01}(c_{12} - c_{13}), sc_{04}(c_{12} - c_{13})b_{11}, sc_{01}(c_{14} - c_{11})).
\end{aligned}$$

and thus by reducing modulo $c_{21}, \ldots, c_{r-1,1} \in L_r : l_r$, without loss of generality

$$\begin{aligned}
ysc_{01}c_{11}b_{03}b_{11} \in {}&(s^2, sf, sc_{01}^2, sc_{11}^2, s(c_{01} - c_{02}b_{01}^d), f(c_{01} - c_{02}b_{02}^d),\\
&f(c_{04} - c_{03}b_{03}^d), s(c_{04} - c_{03}b_{04}^d), fc_{01} - sc_{02}, f(c_{01} - c_{04}), s(c_{02} - c_{03}),\\
&f(c_{02}b_{01} - c_{03}b_{04}), fc_{02}c_{1i}(b_{02} - b_{11}b_{03}), s(c_{04}c_{11} - c_{01}c_{12}),\\
&s(c_{04}c_{14} - c_{01}c_{13}), sc_{01}(c_{12} - c_{13}), sc_{04}(c_{12} - c_{13})b_{11}, sc_{01}(c_{14} - c_{11})).
\end{aligned}$$

We take the colon with s:

$$\begin{aligned}
yc_{01}c_{11}b_{03}b_{11} \in {}&(s, f, c_{01}^2, c_{11}^2, c_{01} - c_{02}b_{01}^d, c_{04} - c_{03}b_{04}^d, c_{02} - c_{03},\\
&c_{04}c_{11} - c_{01}c_{12}, c_{04}c_{14} - c_{01}c_{13}, c_{01}(c_{12} - c_{13}), c_{04}(c_{12} - c_{13})b_{11}, c_{01}(c_{14} - c_{11}))\\
&+ (K : s),
\end{aligned}$$

where K is the ideal $fK' + (fc_{01} - sc_{02})$, with

$$K' = \left(c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d, c_{01} - c_{04}, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{11}b_{03}) | i = 1, \ldots, 4\right).$$

Let $x \in K : s$. Write $xs = k'f + a(fc_{01} - sc_{02})$ for some $k' \in K'$ and $a \in R$. By modifying x by a multiple of $fc_{01} - sc_{02}$, without loss of generality no s appears in a. From $xs = k'f + a(fc_{01} - sc_{02})$ it follows that

$$a \in (K' + (s)) : fc_{01} = (s) + (K' : c_{01}),$$

and that can be verified to be

$$\begin{aligned}
&(s, c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d, c_{01} - c_{04}, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d,\\
&\quad c_{1i}(b_{02} - b_{11}b_{03}), c_{1i}(b_{01} - b_{11}^d b_{04}), c_{1i}(c_{03} - b_{11}^d c_{02}) | i = 1, \ldots, 4).
\end{aligned}$$

As no s appears in a, $xs \in (f, ac_{02})$ implies that $x \in (f) + c_{02}((K' + (s)) : fc_{01})$, and so after rewriting c_{01} as $c_{02}b_{02}^d$, c_{03} as c_{02}, and c_{04} as $c_{02}b_{03}^d$ (note that all the differences are contained in $L_r : l_r$),

$$\begin{aligned}yc_{02}c_{11}b_{03}b_{02}^d b_{11} \in (&s, f, c_{02}^2 b_{02}^{2d}, c_{11}^2, c_{02}(c_{11}b_{03}^d - c_{12}b_{02}^d), c_{02}(c_{14}b_{03}^d - c_{13}b_{02}^d),\\ &c_{02}b_{02}^d(c_{12} - c_{13}), c_{02}b_{03}^d(c_{12} - c_{13})b_{11}, c_{02}b_{02}^d(c_{14} - c_{11}),\\ &c_{02}^2(b_{01} - b_{04}), c_{02}(b_{01}b_{03}^d - b_{04}b_{02}^d),\\ &c_{02}c_{1i}(b_{02} - b_{11}b_{03}), c_{02}c_{1i}(b_{01} - b_{11}^d b_{04}), c_{02}^2 c_{1i}(1 - b_{11}^d) | i = 1, \ldots, 4).\end{aligned}$$

Then going modulo $s, f, b_{02} - b_{11}b_{03}, b_{01} - b_{11}^d b_{04} \in L_r : l_r$, and coloning with c_{02} gives

$$\begin{aligned}yc_{11}b_{03}b_{04}^d b_{11}^{2d^2+1} \in (&c_{02}, c_{11}^2, b_{03}^d(c_{11} - c_{12}b_{11}^{d^2}), b_{03}^d(c_{14} - c_{13}b_{11}^{d^2}),\\ &b_{03}^d b_{11}^{d^2}(c_{12} - c_{13}), b_{03}^d(c_{12} - c_{13})b_{11}, b_{03}^d b_{11}^{d^2}(c_{14} - c_{11})).\end{aligned}$$

Now by going modulo $(c_{02}) + D_{11} \subseteq L_r : l_r$ and taking the colon with $b_{03}^d b_{04}^d$,

$$yc_{12}b_{11}^{3d^2+1} \in (c_{12}^2 b_{11}^{2d^2}),$$

whence finally $y \in (c_{12})$ modulo the established elements of $L_r : l_r$, as was to be proved.

The proof of the case $r = 1$ is almost the same. It is left for the reader. ∎

Lemma 9.11: *For each $r = 1, \ldots, n-1$ and each $k = 1, \ldots, n-r$, define*

$$L_{rk} = L_r + (s(c_{01} \cdots c_{n-k,1})c_{r1}(b_{03} \cdots b_{r3}))\,,$$
$$l_{rk} = s(c_{01} \cdots c_{n-k-1,1})c_{r1}(b_{03} \cdots b_{r3}).$$

Then for $r \le n-2$ and $k = 1, \ldots, n-r-2$,

$$L_{rk} : l_{rk} = (s, f) + V_r + B_{n-k-1} + U_{r-1} + D_{rr} + D_{r+1} + \cdots + D_{n-k-1} + C_{n-k}.$$

Proof: By Lemma 9.6, $c_{n-k,3} \in L_{rk} : l_{rk}$. With this, the tables prove just as in the proof of Proposition 9.9 that the stated ideal is contained in $L_{rk} : l_{rk}$. Now let $y \in L_{rk} : l_{rk}$. Then

$$\begin{aligned}yl_{rk} &= ys(c_{01} \cdots c_{n-k-1,1})c_{r1}(b_{03} \cdots b_{r3})\\ &\in (s^2, sf, sc_{01}^2, \ldots, sc_{r-1,1}^2) + C_{n-k}\\ &\qquad + (b_{1i} - b_{13}) + B_{n-k-1} + D_2 + \cdots + D_{r-1} + D_{rr} + D_{r+1} + \cdots + D_{n-k-1}\\ &\qquad + (h_{0j}, h_{1j}, h_{2j}, h_{3j} | \text{ all appropriate } j)\\ &= (s^2, sf, sc_{01}^2, \ldots, sc_{r-1,1}^2, b_{1i} - b_{13}) + C_{n-k}\\ &\qquad + B_{n-k-1} + D_2 + \cdots + D_{r-1} + D_{rr} + D_{r+1} + \cdots + D_{n-k-1}\\ &\qquad + (s(c_{01} - c_{02}b_{01}^d), f(c_{01} - c_{02}b_{02}^d), f(c_{04} - c_{03}b_{03}^d), s(c_{04} - c_{03}b_{04}^d), fc_{01} - sc_{02},\\ &\qquad\quad f(c_{01} - c_{04}), s(c_{02} - c_{03}), f(c_{02}b_{01} - c_{03}b_{04}), fc_{02}c_{1i}(b_{02} - b_{13}b_{03}),\\ &\qquad\quad s(c_{04}c_{11} - c_{01}c_{12}), s(c_{04}c_{14} - c_{01}c_{13}), sc_{01}(c_{12} - c_{13}),\\ &\qquad\quad sc_{04}(c_{12} - c_{13})b_{13}, sc_{01}(c_{14} - c_{11})c_{21}).\end{aligned}$$

First we assume that $r \geq 2$. Then D_{rr} may be replaced by D_r above. Note that $n-k > 2$. It follows that by coloning with $s(c_{21} \cdots c_{n-k-1,1})c_{r1}b_{23} \cdots b_{r3}$ and reducing modulo the established elements of $L_{rk} : l_{rk}$, such as $b_{1i} - b_{1j}$, generators of C_{n-k}, B_{n-k-1}, the D_i and $C_2, \ldots, C_{r-1}$,

$$\begin{aligned} ysc_{01}c_{11}b_{03}b_{01}^d b_{11} \in (&s^2, sf, sc_{01}^2, sc_{11}^2, s(c_{01} - c_{02}b_{01}^d), f(c_{01} - c_{02}b_{02}^d), \\ &f(c_{04} - c_{03}b_{03}^d), s(c_{04} - c_{03}b_{04}^d), fc_{01} - sc_{02}, f(c_{01} - c_{04}), s(c_{02} - c_{03}), \\ &f(c_{02}b_{01} - c_{03}b_{04}), fc_{02}c_{1i}(b_{02} - b_{11}b_{03}), s(c_{04}c_{11} - c_{01}c_{12}), s(c_{04}c_{14} - c_{01}c_{13}), \\ &sc_{01}(c_{12} - c_{13}), sc_{04}(c_{12} - c_{13})b_{11}, sc_{01}(c_{14} - c_{11})). \end{aligned}$$

The rest of the proof goes as in the proof of Proposition 9.9. The same applies to the case $r = 1$. ■

Definition 9.12: *It is easy to see that the the associated prime ideals of $L_{rk} : l_{rk}$ for $r \geq 2$ are*

$$\begin{aligned} Q''_{rk} &= (s, f) + V_r + B_{n-k-1} + C_0 + \cdots + C_{r-1} + D_r + D_{r+1} + \cdots + D_{n-k-1} + C_{n-k}, \\ Q''_{rk\alpha} &= (s, f, c_{01}, c_{04}, c_{02} - c_{03}, b_{01}, b_{03}, \alpha - b_{11}) \\ &\qquad V_r + B_{n-k-1} + C_1 + \cdots + C_{r-1} + D_r + \cdots + D_{n-k-1} + C_{n-k}, \\ Q'''_{rk} &= (s, f, c_{11}, c_{14}, c_{12} - c_{13}, b_{11}) + V_r + B_{n-k-1} \\ &\qquad + C_0 + C_2 + \cdots + C_{r-1} + D_r + D_{r+1} + \cdots + D_{n-k-1} + C_{n-k}, \end{aligned}$$

and when $r = 1$, the associated prime ideals of $L_1 : l_1$ are

$$\begin{aligned} Q''_1 &= (s, f) + V_1 + B_{n-k-1} + C_0 + D_{11} + D_2 + \cdots + D_{n-k-1} + C_{n-k}, \\ Q''_{1\alpha} &= (s, f, c_{01}, c_{04}, c_{02} - c_{03}, b_{01}, b_{03}, \alpha - b_{11}) \\ &\qquad + V_1 + B_{n-k-1} + C_1 + D_2 + \cdots + D_{n-k-1} + C_{n-k}. \end{aligned}$$

Thus by Proposition 9.8, repeated applications of Lemma 8.1, and the last two propositions,

$$\begin{aligned} \text{Ass }(R/J) \setminus \text{Min }(R/J) &\subseteq \{Q, Q_{k\alpha}, Q'\} \cup \text{Ass }(R/L_1) \\ &\subseteq \{Q, Q_{k\alpha}, Q'\} \cup \text{Ass }(R/(L_1 : l_1)) \cup \text{Ass }(R/(L_1 + (l_1))) \\ &= \{Q, Q_{k\alpha}, Q', Q''_{10}, Q''_{10\alpha}\} \cup \text{Ass }(R/L_{11}) \\ &\subseteq \{Q, Q_{k\alpha}, Q', Q''_{10}, Q''_{10\alpha}\} \cup \text{Ass }(R/(L_{11} : l_{11})) \cup \text{Ass }(R/(L_{11} + (l_{11}))) \\ &= \{Q, Q_{k\alpha}, Q', Q''_{10}, Q''_{10\alpha}, Q''_{11}, Q''_{11\alpha}\} \cup \text{Ass }(R/L_{12})) \\ &\subseteq \{Q, Q_{k\alpha}, Q', Q''_{10}, Q''_{10\alpha}, Q''_{11}, Q''_{11\alpha}\} \\ &\qquad \cup \text{Ass }(R/(L_{12} : l_{12})) \cup \text{Ass }(R/(L_{12} + (l_{12}))) \\ &= \{Q, Q_{k\alpha}, Q', Q''_{10}, Q''_{10\alpha}, Q''_{11}, Q''_{11\alpha}, Q''_{12}, Q''_{12\alpha}\} \cup \text{Ass }(R/L_{13})) \\ &\subseteq \{Q, Q_{k\alpha}, Q', Q''_{1k}, Q''_{1k\alpha}\} \cup \text{Ass }(R/L_{1,n-1})) \\ &= \{Q, Q_{k\alpha}, Q', Q''_{1k}, Q''_{1k\alpha}\} \cup \text{Ass }(R/L_2)) \\ &\subseteq \cdots \end{aligned}$$

$$
\begin{aligned}
&\subseteq \{Q, Q_{k\alpha}, Q', Q''_{1k}, Q''_{1k\alpha}, Q''_{2l}, Q''_{2l\alpha}, Q'''_{2l} | l = 0, \ldots, n-3\} \\
&\qquad \cup \text{Ass } (R/L_3)) \\
&\subseteq \cdots \\
&\subseteq \{Q, Q_{k\alpha}, Q', Q''_{1k}, Q''_{1k\alpha}\} \\
&\qquad \cup \{Q''_{rl}, Q''_{rl\alpha}, Q'''_{rl} | r = 2, \ldots, n-2; l = 0, \ldots, n-r-1\} \cup \text{Ass } (R/L_{n-1})) \\
&\subseteq \cdots \\
&\subseteq \{Q, Q_{k\alpha}, Q', Q''_{1k}, Q''_{1k\alpha}\} \\
&\qquad \cup \{Q''_{rl}, Q''_{rl\alpha}, Q'''_{rl} | r = 2, \ldots, n-2; l = 0, \ldots, n-r-1\} \\
&\qquad \cup \{Q''_{n-1,0}, Q''_{n-1,0\alpha}, Q'''_{n-1,0}\} \cup \text{Ass } (R/(L_{n-1} + (l_{n-1})))
\end{aligned}
$$

(Above, the index k always varies from 0 to $n-2$.) Thus it remains to find the associated prime ideals of

$$
L_n = L_{n-1} + (l_{n-1}) = J + \left(s^2, sf\right) + s\left(c_{k1}\left(\prod_{i=0}^{k} c_{i1}\right)\left(\prod_{i=0}^{k} b_{i3}\right)\middle| k = 0, \ldots, n-1\right).
$$

Trying to eliminate $Q_{5r\Lambda}$, Q_{6r}

Lemma 9.13: *We would like to prove that for all $r = 0, \ldots, n-2$, none of $Q_{5r\Lambda}$, Q_{6r} is associated to $J(n,d)$.*

THIS IS UNFINISHED, but has a reasonable start.

Proof: The primes $Q_{5r\Lambda}$, Q_{6r}, $Q'_{5r\Lambda}$, Q'_{6r} are defined on page 31.

The case $r = 0$ holds by [S, Theorem 4.1]. For the rest, by the recursive construction in Section 7, it suffices to prove that for all $r = 0, \ldots, n-3$, none of $Q'_{5r+1\Lambda}$, Q'_{6r+1} is associated to $K_r + E_r + F_r$. Consider the element $x = (c_{r+3,2}b_{r+3,1} - c_{r+3,3}b_{r+3,4})(c_{r+3,2} - c_{r+3,3})c_{r+3,2}c_{r+3,3}(b_{r+3,2} - b_{r+3,3})$, and the ideal

$$
\begin{aligned}
G = b_{r1}^{d^{2^r}} & c_{r+1,1}(c_{r+2,1}, c_{r+2,4}, c_{r+2,2} - c_{r+2,3}) \\
&+ b_{r1}^{d^{2^r}}(c_{r+1,4}c_{r+2,1} - c_{r+1,1}c_{r+2,2}, c_{r+1,4}c_{r+2,4} - c_{r+1,1}c_{r+2,3}) \\
&+ b_{r1}^{d^{2^r}} c_{r+1,4}(c_{r+2,2}b_{r+2,1} - c_{r+2,3}b_{r+2,4}, c_{r+2,2}(b_{r+2,2} - b_{r+2,3}), c_{r+2,2}b_{r+2,3}\delta_{r+3<n}) \\
&+ \left(b_{r4}^{d^{2^r}} c_{r+1,1} - b_{r1}^{d^{2^r}} c_{r+1,2}, b_{r4}^{d^{2^r}}(c_{r+1,4} - c_{r+1,1}), b_{r1}^{d^{2^r}}(c_{r+1,3} - c_{r+1,2})\right) \\
&+ b_{r4}^{d^{2^r}}(c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4}, c_{r+1,2}c_{r+2,i}(b_{r+1,2} - b_{r+2,i}b_{r+1,3})) \\
&+ (b_{r1}b_{r3}^{d^{2^r}} - b_{r4}b_{r2}^{d^{2^r}}) + c_{r+1,i}(b_{r2} - b_{r+1,i}b_{r3}, c_{r+1,j}(b_{r+1,i} - b_{r+1,j}), b_{r1} - b_{r+1,i}^{d^{2^r}}b_{r4}).
\end{aligned}
$$

Clearly $xG \subseteq K_r + E_r + F_r \subseteq G$, and x is a non-zerodivisor modulo G. Thus $G = (K_r + E_r + F_r) : x$. As x is not an element of $Q'_{5r+1\Lambda}$, Q'_{6r+1}, it suffices to prove that none of these prime ideals is associated to G.

Modulo the last 4 generators, G can be rewritten as

$$
\begin{aligned}
G &= b_{r4}^{d^{2^r}} b_{r+1,1}^{d^{2^{r+1}}} c_{r+1,1}(c_{r+2,1}, c_{r+2,4}, c_{r+2,2} - c_{r+2,3}) \\
&+ b_{r4}^{d^{2^r}} (b_{r+1,4}^{d^{2^{r+1}}} c_{r+1,4} c_{r+2,1} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+1,1} c_{r+2,2}, b_{r+1,4}^{d^{2^{r+1}}} c_{r+1,4} c_{r+2,4} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+1,1} c_{r+2,3}) \\
&+ b_{r4}^{d^{2^r}} b_{r+1,4}^{d^{2^{r+1}}} c_{r+1,4}(c_{r+2,2} b_{r+2,1} - c_{r+2,3} b_{r+2,4}, c_{r+2,2}(b_{r+2,2} - b_{r+2,3}), c_{r+2,2} b_{r+2,3} \delta_{r+3<n}) \\
&+ b_{r4}^{d^{2^r}} \left(c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, b_{r+1,3}^{d^{2^{r+1}}} c_{r+1,3} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2} \right) \\
&+ b_{r4}^{d^{2^r}} (c_{r+1,2} b_{r+1,1} - c_{r+1,3} b_{r+1,4}, c_{r+1,2} c_{r+2,i}(b_{r+1,2} - b_{r+2,i} b_{r+1,3})) \\
&+ (b_{r1} b_{r3}^{d^{2^r}} - b_{r4} b_{r2}^{d^{2^r}}) + c_{r+1,i}(b_{r2} - b_{r+1,i} b_{r3}, c_{r+1,j}(b_{r+1,i} - b_{r+1,j}), b_{r1} - b_{r+1,i}^{d^{2^r}} b_{r4}).
\end{aligned}
$$

Compute $G : b_{r4}^{d^{2^r}}$:

$$
\begin{aligned}
G : b_{r4}^{d^{2^r}} &= b_{r+1,1}^{d^{2^{r+1}}} c_{r+1,1}(c_{r+2,1}, c_{r+2,4}, c_{r+2,2} - c_{r+2,3}) \\
&+ (b_{r+1,4}^{d^{2^{r+1}}} c_{r+1,4} c_{r+2,1} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+1,1} c_{r+2,2}, b_{r+1,4}^{d^{2^{r+1}}} c_{r+1,4} c_{r+2,4} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+1,1} c_{r+2,3}) \\
&+ b_{r+1,4}^{d^{2^{r+1}}} c_{r+1,4}(c_{r+2,2} b_{r+2,1} - c_{r+2,3} b_{r+2,4}, c_{r+2,2}(b_{r+2,2} - b_{r+2,3}), c_{r+2,2} b_{r+2,3} \delta_{r+3<n}) \\
&+ \left(c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, b_{r+1,3}^{d^{2^{r+1}}} c_{r+1,3} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2} \right) \\
&+ (c_{r+1,2} b_{r+1,1} - c_{r+1,3} b_{r+1,4}, c_{r+1,2} c_{r+2,i}(b_{r+1,2} - b_{r+2,i} b_{r+1,3})) \\
&+ (b_{r1} b_{r3}^{d^{2^r}} - b_{r4} b_{r2}^{d^{2^r}}) + c_{r+1,i}(b_{r2} - b_{r+1,i} b_{r3}, c_{r+1,j}(b_{r+1,i} - b_{r+1,j}), b_{r1} - b_{r+1,i}^{d^{2^r}} b_{r4}) \\
&= (c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}) + b_{r+1,1}^{d^{2^{r+1}}} b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}(c_{r+2,1}, c_{r+2,4}, c_{r+2,2} - c_{r+2,3}) \\
&+ b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}(b_{r+1,4}^{d^{2^{r+1}}} c_{r+2,1} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+2,2}, b_{r+1,4}^{d^{2^{r+1}}} c_{r+2,4} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+2,3}) \\
&+ b_{r+1,4}^{d^{2^{r+1}}} b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}(c_{r+2,2} b_{r+2,1} - c_{r+2,3} b_{r+2,4}, c_{r+2,2}(b_{r+2,2} - b_{r+2,3}), c_{r+2,2} b_{r+2,3} \delta_{r+3<n}) \\
&+ (b_{r+1,3}^{d^{2^{r+1}}} c_{r+1,3} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,2} b_{r+1,1} - c_{r+1,3} b_{r+1,4}, c_{r+1,2} c_{r+2,i}(b_{r+1,2} - b_{r+2,i} b_{r+1,3})) \\
&+ (b_{r1} b_{r3}^{d^{2^r}} - b_{r4} b_{r2}^{d^{2^r}}) + c_{r+1,i}(b_{r2} - b_{r+1,i} b_{r3}, c_{r+1,j}(b_{r+1,i} - b_{r+1,j}), b_{r1} - b_{r+1,i}^{d^{2^r}} b_{r4}).
\end{aligned}
$$

Consider the ideal

$$
\begin{aligned}
G' &= (c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}) + b_{r+1,2}^{2d^{2^{r+1}}} C_{r+2} \\
&+ (c_{r+2,i}(b_{r+1,2} - b_{r+2,i} b_{r+1,3}), b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}) \\
&+ b_{r+1,2}^{d^{2^{r+1}}}(b_{r+1,i} - b_{r+1,2}, c_{r+1,3} - c_{r+1,2}, c_{r+2,i}(1 - b_{r+2,i})) \\
&+ (c_{r+1,3}(b_{r+1,2} - b_{r+1,3}), c_{r+1,2} b_{r+1,1} - c_{r+1,3} b_{r+1,4}, b_{r+1,1}(b_{r+1,2} - b_{r+1,3})).
\end{aligned}
$$

It is straightforward to verify that $c_{r+1,2}^2 G' \subseteq G : b_{r4}^{d^{2^r}} \subseteq G'$. We'll find a primary decomposition of G', verify that $c_{r+1,2}$ is a non-zerodivisor modulo G'. This would then establish that $G' = G : b_{r4}^{d^{2^r}} c_{r+1,2}^2$. Now,

$$\begin{aligned} G' : b_{r+1,2}^{2d^{2^{r+1}}} &= (c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, c_{r+1,3} - c_{r+1,2}) + C_{r+2} \\ &\quad + (b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}, b_{r+1,i} - b_{r+1,2}), \end{aligned}$$

on which $c_{r+1,2}, b_{r+1,2}$ are non-zerodivisors. Thus

$$G' = (G' : b_{r+1,2}^{2d^{2^{r+1}}}) \cap (G' + (b_{r+1,2}^{2d^{2^{r+1}}})).$$

So we analyze $G' + (b_{r+1,2}^{2d^{2^{r+1}}})$:

$$\begin{aligned} G' + (b_{r+1,2}^{2d^{2^{r+1}}}) &= (c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, b_{r+1,2}^{2d^{2^{r+1}}}) \\ &\quad + (c_{r+2,i}(b_{r+1,2} - b_{r+2,i} b_{r+1,3}), b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}) \\ &\quad + b_{r+1,2}^{d^{2^{r+1}}} (b_{r+1,i} - b_{r+1,2}, c_{r+1,3} - c_{r+1,2}, c_{r+2,i}(1 - b_{r+2,i})) \\ &\quad + (c_{r+1,3}(b_{r+1,2} - b_{r+1,3}), c_{r+1,2} b_{r+1,1} - c_{r+1,3} b_{r+1,4}, b_{r+1,1}(b_{r+1,2} - b_{r+1,3})). \end{aligned}$$

This ideal decomposes as follows:

$$\begin{aligned} &= \Big((c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, b_{r+1,2}^{2d^{2^{r+1}}}, b_{r+1,i} - b_{r+1,2}, c_{r+1,3} - c_{r+1,2}) \\ &\qquad + (c_{r+2,i}(1 - b_{r+2,i}), b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}) \Big) \\ &\quad \cap \Big((c_{r+1,1}, c_{r+1,4}, b_{r+1,2}^{d^{2^{r+1}}}, b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}, c_{r+2,i}(b_{r+1,2} - b_{r+2,i} b_{r+1,3})) \\ &\qquad + (c_{r+1,3}(b_{r+1,2} - b_{r+1,3}), c_{r+1,2} b_{r+1,1} - c_{r+1,3} b_{r+1,4}, b_{r+1,1}(b_{r+1,2} - b_{r+1,3})) \Big) \\ &= \Big((c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, b_{r+1,2}^{2d^{2^{r+1}}}, b_{r+1,i} - b_{r+1,2}, c_{r+1,3} - c_{r+1,2}) \\ &\qquad + (c_{r+2,i}(1 - b_{r+2,i}), b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}) \Big) \\ &\quad \cap \Big((c_{r+1,1}, c_{r+1,4}, b_{r+1,2}^{d^{2^{r+1}}}, b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}, c_{r+2,i} b_{r+1,2}(1 - b_{r+2,i})) \\ &\qquad + (b_{r+1,2} - b_{r+1,3}, c_{r+1,2} b_{r+1,1} - c_{r+1,3} b_{r+1,4}) \Big) \\ &\quad \cap \Big((c_{r+1,1}, c_{r+1,4}, b_{r+1,2}^{d^{2^{r+1}}}, b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}, c_{r+2,i}(b_{r+1,2} - b_{r+2,i} b_{r+1,3})) \\ &\qquad + (c_{r+1,3}, b_{r+1,1}) \Big) \\ &= \Big((c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, b_{r+1,2}^{2d^{2^{r+1}}}, b_{r+1,i} - b_{r+1,2}, c_{r+1,3} - c_{r+1,2}) \\ &\qquad + (c_{r+2,i}(1 - b_{r+2,i}), b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}) \Big) \\ &\quad \cap \Big((c_{r+1,1}, c_{r+1,4}, b_{r+1,2}^{d^{2^{r+1}}}, b_{r2} - b_{r+1,2} b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}, c_{r+2,i}(1 - b_{r+2,i})) \end{aligned}$$

$$
\begin{aligned}
&\quad + (b_{r+1,2} - b_{r+1,3}, c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4})\Big)\\
&\cap \Big(c_{r+1,1}, c_{r+1,4}, b_{r1}, b_{r2}, b_{r+1,2}, b_{r+1,3}, c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4}\Big)\\
&\cap \Big((c_{r+1,1}, c_{r+1,4}, b_{r+1,2}^{d^{2^{r+1}}}, b_{r2} - b_{r+1,2}b_{r3}, b_{r1} - b_{r+1,2}^{d^{2^r}}b_{r4}, c_{r+2,i}(b_{r+1,2} - b_{r+2,i}b_{r+1,3}))\\
&\quad + (c_{r+1,3}, b_{r+1,1})\Big).
\end{aligned}
$$

Now it is clear that $c_{r+1,2}$ is a non-zerodivisor modulo G', which proves that $G' = G : b_{r4}^{d^{2^r}} c_{r+1,2}^2$. The decomposition of G' shows that b_{r4} and $c_{r+1,2}$ are non-zerodivisors modulo G', so that by Fact 1.5,

$$
G = G' \cap (G + (b_{r4}^{d^{2^r}} c_{r+1,2}^2)).
$$

$$
\begin{aligned}
G + (b_{r4}^{d^{2^r}} c_{r+1,2}^2) &= b_{r4}^{d^{2^r}} b_{r+1,1}^{d^{2^{r+1}}} c_{r+1,1}(c_{r+2,1}, c_{r+2,4}, c_{r+2,2} - c_{r+2,3})\\
&+ b_{r4}^{d^{2^r}} (b_{r+1,4}^{d^{2^{r+1}}} c_{r+1,4}c_{r+2,1} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+1,1}c_{r+2,2}, b_{r+1,4}^{d^{2^{r+1}}} c_{r+1,4}c_{r+2,4} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+1,1}c_{r+2,3})\\
&+ b_{r4}^{d^{2^r}} b_{r+1,4}^{d^{2^{r+1}}} c_{r+1,4}(c_{r+2,2}b_{r+2,1} - c_{r+2,3}b_{r+2,4}, c_{r+2,2}(b_{r+2,2} - b_{r+2,3}), c_{r+2,2}b_{r+2,3}\delta_{r+3<n})\\
&+ b_{r4}^{d^{2^r}} \left(c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, b_{r+1,3}^{d^{2^{r+1}}} c_{r+1,3} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}\right)\\
&+ b_{r4}^{d^{2^r}} \left(c_{r+1,2}^2, c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4}, c_{r+1,2}c_{r+2,i}(b_{r+1,2} - b_{r+2,i}b_{r+1,3})\right)\\
&+ (b_{r1}b_{r3}^{d^{2^r}} - b_{r4}b_{r2}^{d^{2^r}}) + c_{r+1,i}(b_{r2} - b_{r+1,i}b_{r3}, c_{r+1,j}(b_{r+1,i} - b_{r+1,j}), b_{r1} - b_{r+1,i}^{d^{2^r}} b_{r4}).
\end{aligned}
$$

Then

$$
\begin{aligned}
(G + (b_{r4}^{d^{2^r}} c_{r+1,2}^2)) : b_{r4}^{d^{2^r}} &= (c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, c_{r+1,2}^2)\\
&+ b_{r+1,1}^{d^{2^{r+1}}} b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}(c_{r+2,1}, c_{r+2,4}, c_{r+2,2} - c_{r+2,3})\\
&+ b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}(b_{r+1,4}^{d^{2^{r+1}}} c_{r+2,1} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+2,2}, b_{r+1,4}^{d^{2^{r+1}}} c_{r+2,4} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+2,3})\\
&+ b_{r+1,4}^{d^{2^{r+1}}} b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}(c_{r+2,2}b_{r+2,1} - c_{r+2,3}b_{r+2,4}, c_{r+2,2}(b_{r+2,2} - b_{r+2,3}), c_{r+2,2}b_{r+2,3}\delta_{r+3<n})\\
&+ (b_{r+1,3}^{d^{2^{r+1}}} c_{r+1,3} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4}, c_{r+1,2}c_{r+2,i}(b_{r+1,2} - b_{r+2,i}b_{r+1,3}))\\
&+ (b_{r1}b_{r3}^{d^{2^r}} - b_{r4}b_{r2}^{d^{2^r}}) + c_{r+1,i}(b_{r2} - b_{r+1,i}b_{r3}, c_{r+1,j}(b_{r+1,i} - b_{r+1,j}), b_{r1} - b_{r+1,i}^{d^{2^r}} b_{r4})\\
&= \Big((c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, c_{r+1,2}^2)\\
&+ b_{r+1,1}^{d^{2^{r+1}}} b_{r+1,2}^{d^{2^{r+1}}} (c_{r+2,1}, c_{r+2,4}, c_{r+2,2} - c_{r+2,3})\\
&+ b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}(b_{r+1,4}^{d^{2^{r+1}}} c_{r+2,1} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+2,2}, b_{r+1,4}^{d^{2^{r+1}}} c_{r+2,4} - b_{r+1,1}^{d^{2^{r+1}}} c_{r+2,3})\\
&+ b_{r+1,4}^{d^{2^{r+1}}} b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}(c_{r+2,2}b_{r+2,1} - c_{r+2,3}b_{r+2,4}, c_{r+2,2}(b_{r+2,2} - b_{r+2,3}), c_{r+2,2}b_{r+2,3}\delta_{r+3<n})\\
&+ (b_{r+1,3}^{d^{2^{r+1}}} c_{r+1,3} - b_{r+1,2}^{d^{2^{r+1}}} c_{r+1,2}, c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4}, c_{r+1,2}c_{r+2,i}(b_{r+1,2} - b_{r+2,i}b_{r+1,3}))
\end{aligned}
$$

$$
\begin{aligned}
&+ (b_{r1}b_{r3}^{d^{2^r}} - b_{r4}b_{r2}^{d^{2^r}}) + c_{r+1,i}(b_{r2} - b_{r+1,i}b_{r3}, c_{r+1,j}(b_{r+1,i} - b_{r+1,j}), b_{r1} - b_{r+1,i}^{d^{2^r}}b_{r4})\Big) \\
\cap \Big(&(c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}}c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, c_{r+1,2}) \\
&+ (b_{r+1,3}^{d^{2^{r+1}}}c_{r+1,3} - b_{r+1,2}^{d^{2^{r+1}}}c_{r+1,2}, c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4}) \\
&+ (b_{r1}b_{r3}^{d^{2^r}} - b_{r4}b_{r2}^{d^{2^r}}) + c_{r+1,i}(b_{r2} - b_{r+1,i}b_{r3}, c_{r+1,j}(b_{r+1,i} - b_{r+1,j}), b_{r1} - b_{r+1,i}^{d^{2^r}}b_{r4})\Big)
\end{aligned}
$$

Consider the ideal

$$
\begin{aligned}
G' = {} & (c_{r+1,1} - b_{r+1,2}^{d^{2^{r+1}}}c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, c_{r+1,2}^2) \\
& + b_{r+1,1}^{d^{2^{r+1}}}b_{r+1,2}^{d^{2^{r+1}}}(c_{r+1,2}, b_{r+1,3}^{d^{2^{r+1}}}, b_{r+1,4})(c_{r+2,1}, c_{r+2,4}, c_{r+2,2} - c_{r+2,3}) \\
& + b_{r+1,2}^{d^{2^{r+1}}}(c_{r+1,2}, b_{r+1,3}^{d^{2^{r+1}}}, b_{r+1,4})(b_{r+1,4}^{d^{2^{r+1}}}c_{r+2,1} - b_{r+1,1}^{d^{2^{r+1}}}c_{r+2,2}, b_{r+1,4}^{d^{2^{r+1}}}c_{r+2,4} - b_{r+1,1}^{d^{2^{r+1}}}c_{r+2,3}) \\
& + b_{r+1,4}^{d^{2^{r+1}}}b_{r+1,2}^{d^{2^{r+1}}}(c_{r+1,2}, b_{r+1,3}^{d^{2^{r+1}}}, b_{r+1,4}) \\
& \qquad \cdot (c_{r+2,2}b_{r+2,1} - c_{r+2,3}b_{r+2,4}, c_{r+2,2}(b_{r+2,2} - b_{r+2,3}), c_{r+2,2}b_{r+2,3}\delta_{r+3<n}) \\
& + (b_{r+1,3}^{d^{2^{r+1}}}c_{r+1,3} - b_{r+1,2}^{d^{2^{r+1}}}c_{r+1,2}, c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4}) \\
& + (c_{r+1,2}, b_{r+1,3}^{d^{2^{r+1}}}, b_{r+1,4})c_{r+2,i}(b_{r+1,2} - b_{r+2,i}b_{r+1,3}) \\
& + (b_{r2} - b_{r+1,3}b_{r3}, c_{r+1,j}(b_{r+1,3} - b_{r+1,j}), b_{r1} - b_{r+1,3}^{d^{2^r}}b_{r4}) \\
& + c_{r+1,i}(b_{r2} - b_{r+1,i}b_{r3}, c_{r+1,j}(b_{r+1,i} - b_{r+1,j}), b_{r1} - b_{r+1,i}^{d^{2^r}}b_{r4})
\end{aligned}
$$

Clearly $c_{r+1,3}G' \subseteq (G + (b_{r4}^{d^{2^r}} c_{r+1,2}^2)) : b_{r4}^{d^{2^r}} \subseteq G'$. We will next decompose G' and prove that $c_{r+1,3}$ is a non-zerodivisor modulo G', which would then prove that $(G+(b_{r4}^{d^{2^r}} c_{r+1,2}^2)) : b_{r4}^{d^{2^r}} c_{r+1,3} = G'$.

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