# Every numerical semigroup is one over $d$ of infinitely many symmetric numerical semigroups

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**Abstract.** For every numerical semigroup $S$ and every positive integer $d > 1$ there exist infinitely many symmetric numerical semigroups $\overline{S}$ such that $S = \{n \in \mathbb{Z} : dn \in \overline{S}\}$. If $d \geq 3$, there exist infinitely many pseudo-symmetric numerical semigroups $\overline{S}$ such that $S = \{n \in \mathbb{Z} : dn \in \overline{S}\}$.

**Keywords.** Numerical semigroup, symmetric semigroup.

**AMS classification.** 20M14.

This note was motivated by the recent results of Rosales and García-Sánchez [5, 6] that for every numerical semigroup $S$ there exist infinitely many symmetric numerical semigroups $\overline{S}$ such that $S = \{n \in \mathbb{Z} : 2n \in \overline{S}\}$. The main result in this note, Theorem 5, is that 2 is not a special integer, bigger positive integers work as well. The Rosales–García-Sánchez construction for $d = 2$ gives all the possible $\overline{S}$, whereas the construction below does not.

Throughout, $S$ stands for a numerical semigroup, $\mathrm{F}(S)$ stands for its Frobenius number, $\mathrm{PF}(S)$ for the set of all pseudo-Frobenius numbers (i.e., all $n \in \mathbb{Z} \setminus S$ such that $n + (S \setminus \{0\}) \subseteq S$), and $d$ for a positive integer strictly bigger than 1. The notation $dS$ stands for the set $\{ds : s \in S\}$ and $\frac{S}{d}$ stands for $\{n \in \mathbb{N} : dn \in S\}$.

The goal is to construct infinitely many symmetric numerical semigroups $T$ such that $S = \frac{T}{d}$. Another goal is to construct, for $d \geq 3$, infinitely many pseudo-symmetric numerical semigroups $T$ such that $S = \frac{T}{d}$.

The ring-theoretic consequence, by a result of Kunz [1], is that for any affine domain of the form $A = k[t^{a_1}, \ldots, t^{a_m}]$, with $a_1, \ldots, a_m$ positive integers generating a numerical semigroup and with $k$ a field, there exist infinitely many (Gorenstein) affine extension domains $R$ of the same form such that any equation $X^d - a$ with $a \in A$ has a solution in $A$ if and only if it has a solution in $R$. Such rings $R$ are called $d$-closed.

It is clear that if $S = \frac{T}{a}$ and $T = \frac{U}{b}$, then $S = \frac{U}{ab}$. Thus it suffices to prove that for every $S$ and for every positive prime integer $d$ there exist infinitely many (symmetric) numerical semigroups $T$ for which $S = \frac{T}{d}$. The proofs below, however, will not assume that $d$ is a prime.

**Definition 1.** Let $d$ be a positive integer. A numerical semigroup $S$ is said to be **d-symmetric** if for all integers $n \in \mathbb{Z}$, whenever $d$ divides $n$, either $n$ or $\mathrm{F}(S) - n$ is in $S$.

Observe that a symmetric numerical semigroup is $d$-symmetric for all $d$, that a 1-symmetric numerical semigroup $S$ is symmetric, and that a 2-symmetric numerical semigroup $S$ is symmetric if $\mathrm{F}(S)$ is an odd integer.

**Proposition 2.** *Let $S \subseteq T$ be numerical semigroups such that $\mathrm{F}(S) = \mathrm{F}(T)$. If $S$ is $d$-symmetric, so is $T$, and $\frac{S}{d} = \frac{T}{d}$.*

*Proof.* Let $m \in \mathbb{Z}$ be a multiple of $d$. If $m \notin T$, then $m \notin S$, so by the $d$-symmetric assumption on $S$, $\mathrm{F}(T) - m = \mathrm{F}(S) - m \in S \subseteq T$. Thus $T$ is $d$-symmetric. It remains to prove that $\frac{T}{d} \subseteq \frac{S}{d}$. Let $m \in \frac{T}{d}$. Suppose that $m \notin \frac{S}{d}$. Then $dm \in T \setminus S$. Since $S$ is $d$-symmetric, $\mathrm{F}(S) - dm \in S \subseteq T$, whence $\mathrm{F}(T) = \mathrm{F}(S) = (\mathrm{F}(S) - dm) + dm \in T$, which is a contradiction. □

By Theorem 1 and Proposition 2 in [4, Rosales and Branco], every numerical semigroup $S$ with odd $\mathrm{F}(S)$ can be embedded in a symmetric numerical semigroup $T$ such that $\mathrm{F}(S) = \mathrm{F}(T)$. There are only finitely many choices for such $T$, but they are in general not unique. For example, let $S = \langle 12, 16, 21, 22, 23 \rangle$. Then $S$ is 4-symmetric, $\mathrm{F}(S) = 41$, and the numbers $n$ for which $n, \mathrm{F}(S) - n$ are not in $S$ are $10, 11, 14, 15, 26, 27, 30, 31$. If one adds 10 to $S$, then $2 \cdot 10 + 21 = 41$ would be in the numerical semigroup, so the Frobenius number would not be preserved. Thus any symmetric numerical semigroup $T$ containing $S$ with $\mathrm{F}(S) = \mathrm{F}(T)$ needs to contain 31. However, there are symmetric (and 4-symmetric) $T$ that contain 11 and there are those that contain 30. All the possible symmetric $T$ containing $S$ with $\mathrm{F}(S) = \mathrm{F}(T)$ are as follows: $\langle 11, 12, 16, 21, 22, 23, 26, 31 \rangle$, $\langle 12, 14, 16, 21, 22, 23, 31 \rangle$, $\langle 12, 15, 16, 21, 22, 23, 31 \rangle$, $\langle 12, 16, 21, 22, 23, 26, 27, 30, 31 \rangle$.

**Proposition 3.** *Let $S$ be a numerical semigroup and $d, t$ and $e$ positive integers. Let $g_1, \ldots, g_t, h_1, \ldots, h_e$ be positive integers such that:*

*(1) For all distinct $i, j \in \{1, \ldots, e\}$, $d$ does not divide $h_i - h_j$, and does not divide $h_i$.*

*(2) $h_1 = \min\{h_1, \ldots, h_e\}$.*

*(3) For all $i = 1, \ldots, e$, $h_i - d\,\mathrm{F}(S) > \frac{1}{2} h_1$.*

*(4) $g_1, \ldots, g_t$ are not contained in $S$.*

*Set $T = dS + \langle h_i - dg_j : i = 1, \ldots, e; j = 1, \ldots, t \rangle + \langle h_1 + 1, h_1 + 2, \ldots, 2h_1 + 1 \rangle$. Then $T$ is a numerical semigroup, $\mathrm{F}(T) = h_1$, and $S = \frac{T}{d}$.*

*If $\mathrm{PF}(S) \subseteq \{g_1, \ldots, g_t\}$, then $T$ is $d$-symmetric.*

*Proof.* The set $\langle h_1 + 1, \ldots, 2h_1 + 1 \rangle$ is contained in $T$, and thus $T$ is a numerical semigroup with $\mathrm{F}(T) \le h_1$. Suppose that $h_1 \in T$. Then

$$h_1 = ds + \sum_{i,j} a_{ij}(h_i - dg_j)$$

for some $s \in S$ and some non-negative integers $a_{ij}$. Since $d$ does not divide $h_1$, at least one $a_{ij}$ is non-zero. By condition (3), at most one $a_{ij}$ is non-zero, and so it is necessarily 1. Then $h_1 - h_i = ds - dg_j$, so that by condition (1), $i = 1$ and $s = g_j \in S$, which is a contradiction. This proves that $h_1 \notin T$, whence $h_1 = \mathrm{F}(T)$.

By (3), $h_1 > 2d\,\mathrm{F}(S) > d\,\mathrm{F}(S)$.

Clearly $dS \subseteq T$, so $S \subseteq \frac{T}{d}$. Let $n \in \mathbb{Z}$ such that $dn \in T$. We want to prove that $n \in S$. If $dn \geq h_1$, by the previous paragraph $n > \mathrm{F}(S)$, whence $n \in S$. Now suppose that $dn < h_1$. Since $dn \in T$, write

$$dn = ds + \sum_{i,j} a_{ij}(h_i - dg_j),$$

for some $s \in S$ and some non-negative integers $a_{ij}$. As before, either $dn = ds + h_i - dg_j$ for some $i, j$, or $dn = ds$. The former case is impossible as $h_i$ is not a multiple of $d$, so necessarily $dn = ds$ and so $n = s \in S$. This proves that $S = \frac{T}{d}$.

It remains to prove that $T$ is $d$-symmetric if $\mathrm{PF}(S) \subseteq \{g_1, \ldots, g_t\}$. Let $n \in \mathbb{Z}$ with $n = dm$ for some $m \in \mathbb{Z}$. If $n \notin T$, then $m \notin S$, and by [3, Proposition 12] there exists $g_i \in \mathrm{PF}(S)$ such that $g_i - m \in S$. Then $h_1 - n = h_1 - dm = (h_1 - dg_i) + d(g_i - m) \in T$. Thus $T$ is $d$-symmetric. □

**Corollary 4.** *(Rosales–García-Sánchez [6]) Every numerical semigroup is one half of infinitely many symmetric numerical semigroups.*

*Proof.* Let $\mathrm{PF}(S) = \{g_1, \ldots, g_t\}$ and let $h_1$ be an arbitrary odd integer bigger than $4\,\mathrm{F}(S)$. Then by Proposition 3, there exists a 2-symmetric numerical semigroup $T$ such that $\frac{T}{2} = S$ and such that $\mathrm{F}(T) = h_1$. We already observed that a 2-symmetric numerical semigroup with an odd Frobenius number is symmetric. Since there are infinitely many choices for $h_1$, we are done. □

In general, the construction in the proof of Proposition 3 does not necessarily give a symmetric numerical group $T$. Say $S = \langle 3, 4 \rangle$ and $d = 4$. Then $\mathrm{F}(S) = 5$, $\mathrm{PF}(S) = \{5\}$. The maximal possible $e$ is $d - 1 = 3$, so if we take $h_1 = 41$, $h_2 = 42$, $h_3 = 43$, the hypotheses of the theorem are satisfied, and the construction gives $T = \langle 12, 16, 21, 22, 23 \rangle$. By the theorem, $\mathrm{F}(T) = 41$ and $S = \frac{T}{4}$, but $T$ is not symmetric as neither 10 nor 31 are in $T$. One can find a symmetric numerical semigroup $U$ such that $S = \frac{U}{4}$ by using the Rosales–García-Sánchez result above twice (with $d = 2$), or one can apply the following main theorem of this paper:

**Theorem 5.** *Let $S$ be a numerical semigroup and let $d$ be an integer greater than or equal to 2. Then there exist infinitely many symmetric numerical semigroups $T$ such that $S = \frac{T}{d}$.*

*Proof.* By choosing large odd integers $h_1$ that are not multiplies of $d$, applying Proposition 3 with $e = 1$ and $\{g_1, \ldots, g_t\} = \mathrm{PF}(S)$ gives a $d$-symmetric numerical semigroup $T$ such that $S = \frac{T}{d}$ and $\mathrm{F}(T) = h_1$. But then by Theorem 1 in [4] there exists a symmetric numerical semigroup $U$ containing $T$ such that $\mathrm{F}(U) = \mathrm{F}(T)$. By Proposition 2, $\frac{T}{d} = \frac{U}{d}$. Thus there exists a symmetric numerical semigroup $U$ such that $S = \frac{U}{d}$ and $\mathrm{F}(U) = h_1$. Since there are infinitely many choices of $h_1$, we are done. □

Recall that a numerical semigroup $S$ is **pseudo-symmetric** if $\mathrm{F}(S)$ is even and if for all $n \in \mathbb{Z} \setminus \{\mathrm{F}(S)/2\}$, either $n$ or $\mathrm{F}(S) - n$ is in $S$. The following is a modification of the main theorem for pseudo-symmetric semigroups:

**Theorem 6.** *Let $S$ be a numerical semigroup and let $d$ be an integer greater than or equal to* 3*. Then there exist infinitely many pseudo-symmetric numerical semigroups $T$ such that $S = \frac{T}{d}$.*

*Proof.* By choosing large even integers $h_1$ that are not multiples of $d$, applying Proposition 3 with $e = 1$ and $\{g_1, \dots, g_t\} = \mathrm{PF}(S)$ gives a $d$-symmetric numerical semigroup $T$ such that $S = \frac{T}{d}$ and $\mathrm{F}(T) = h_1$. Similar to Theorem 1 in [4], there exists a pseudo-symmetric numerical semigroup $U$ containing $T$ such that $\mathrm{F}(U) = \mathrm{F}(T)$, say $U = T \cup \{n \in \mathbb{N} \mid h_1/2 < n < h_1, h_1 - n \notin T\}$. By Proposition 2, $\frac{T}{d} = \frac{U}{d}$. Thus $U$ is a pseudo-symmetric numerical semigroup such that $S = \frac{U}{d}$ and $\mathrm{F}(U) = h_1$. Since there are infinitely many choices of $h_1$, we are done. □

The integer $d = 2$ has to be excluded from the theorem above: if $T$ is pseudo-symmetric with even Frobenius number $\mathrm{F}(T)$ and $S = \frac{T}{2}$, then necessarily $\mathrm{F}(T) \leq 2\,\mathrm{F}(S)$. But there are only finitely many such $T$.

A related result is in Rosales [2]: every numerical semigroup is of the form $\frac{T}{4}$ for some pseudo-symmetric numerical semigroup. Also, Rosales [2] proves that a numerical semigroup is irreducible if and only if it is one half of a pseudo-symmetric numerical semigroup.

**Acknowledgments.** I thank the referee for helping strengthen the original Proposition 3 and Theorem 6.

## References

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