Primary Decompositions of Powers of Ideals

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Abstract

Let R be a Noetherian ring and I an ideal. We prove that there exists an integer k such that for all $n \geq 1$ there exists an irredundant primary decomposition $I^n = q_1 \cap \cdots \cap q_l$ such that $\sqrt{q_i}^{nk} \subseteq q_i$ whenever $\mathrm{ht}\,(q_i/I) \leq 1$. In particular, if R is a local ring with maximal ideal m and I is a prime ideal of dimension 1, then $m^{kn} I^{(n)} \subseteq I^n$, where $I^{(n)}$ denotes the n'th symbolic power of I.

We study some asymptotic properties of primary decompositions of powers of ideals in a Noetherian ring. In particular, we consider the following question:

Let (R, m) be a regular local ring and P a prime ideal of dimension 1. Then for some $c_n \in N$, $m^{c_n} P^{(n)} \subseteq P^n$. How does this c_n depend on n? (cf. [2])

The main theorem 1 says that c_n is bounded linearly, i.e. there exists an integer k such that $m^{nk} P^{(n)} \subseteq P^n$ for all $n \geq 0$.

Note that if $P^n = P^{(n)} \cap J_n$ is a primary decomposition of P^n with $m \subseteq \sqrt{J_n}$ and $m^{c_n} \subseteq J_n$, then $m^{c_n} P^{(n)} \subseteq P^n$. So we tackle the question via selected irreducible primary components of powers of ideals. Hence we consider more generally: if I is an ideal in a Noetherian ring and $P \in \cup_{n=1}^{\infty} \mathrm{Ass}\,(R/I^n)$, does there exist an irredundant primary decomposition of $I^n = q_{n1} \cap \cdots \cap q_{nk_n}$ such that if $\sqrt{q_{ni}} = P$, then the least integer c_n for which $P^{c_n} \subseteq q_{ni}$ is bounded linearly with respect to n?

We would also like to know whether there are good primary decompositions of ideals of the form $(x_1^q, \ldots, x_n^q)$, q ranging over powers of the characteristic of R. A positive answer to this question would solve the open question whether tight closure commutes with localization, at least for ideals generated by elements $x_1, \ldots, x_n$ for which $\cup_q \mathrm{Ass}\,(R/(x_1^q, \ldots, x_n^q))$ is a finite set. There may be infinitely many primes associated to such ideals if q is allowed to vary over all positive integers. An example of Hochster is the following: let $R = Z[X, Y]$, a

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polynomial ring in two variables over the ring of integers. Let $x_1 = X$, $x_2 = Y$, and $x_3 = X + Y$. It is easy to see that for each prime integer p, (p, X, Y) is a prime ideal of R associated to (x_1^p, x_2^p, x_3^p). No such examples are known for local rings or for rings containing fields, so there is some hope for this method in proving that tight closure commutes with localization.

In this paper we give partial answers to the three stated lines of inquiry. The main theorem is the following:

THEOREM 1 *Let R be a Noetherian ring and I an ideal. Then there exists an integer k such that for all $n \geq 1$ there exists an irredundant primary decomposition $I^n = q_1 \cap \cdots \cap q_l$ such that $\sqrt{q_i}^{nk} \subseteq q_i$ whenever $ht(q_i/I) \leq 1$.*

Note that the theorem holds trivially if all of the associated primes of I are minimal over I since the minimal primary components are uniquely determined. Namely, if k is a positive integer such that P^k is contained in the P-primary component $IR_P \cap R$ of I for every minimal prime P over I (there are only finitely many of them as R is Noetherian), then this k works also for all higher powers of I. For let n be a positive integer. Then $P^{kn} \subseteq (IR_P \cap R)^n \subseteq I^n R_P \cap R$, which is the P-primary component of I^n.

If P is not minimal over I, P-primary component of I is not uniquely determined. In fact, it is not true that $\sqrt{q_i}^{kn} \subseteq q_i$ for *every* irreducible primary decomposition $I^n = q_1 \cap \cdots \cap q_l$. We show this in an example:

Let $R = k[X, Y]$, a polynomial ring in two variables X and Y over a field k. Let $I = (X^2, XY)$. It is easy to verify that $I = (X) \cap (X^2, XY, Y^m)$ for all positive integers m. Each one of these decompositions is an irredundant primary decomposition, but for any integer k there exists an integer m such that $(X, Y)^k \not\subseteq (X^2, XY, Y^m)$. So the theorem can only hold for *some* primary decompositions.

Before we prove the theorem we state two results needed in the proof:

THEOREM 2 *(Ratliff [5]) Let I be an ideal in a Noetherian ring R. Then $\cup_{n \geq 1} Ass(R/I^n)$ is a finite set.*

However, in general $\operatorname{Ass}(R/I^n) \neq \operatorname{Ass}(R/I)$. Brodmann showed in [1] that for large n, $\operatorname{Ass}(R/I^n)$ stabilizes.

THEOREM 3 *(Katz-McAdam [4, (1.5)]) Let R be a Noetherian ring. For any ideal I there exists an integer l such that $I^n : J^{nl} = I^n : J^{nl+1}$ for all ideals J and all $n \geq 0$.*

Actually, Katz and McAdam prove existence of such an integer l depending on I and J, but their argument can be easily extended to show that there is an l independent of J:

Proof: Let $S = R[It, t^{-1}]$, where t is an indeterminate over R. It is clear that $I^n :_R J = (t^{-n}S :_S J) \cap R$ for all n and all ideals J of R. Thus it suffices to show that there exists an integer l such that $t^{-n}S :_S J^{nl} = t^{-n}S :_S J^{nl+1}$ for all n and all ideals J of S. So by replacing R by S we may assume without loss of generality that I is a principal ideal generated by a regular element a.

Now let $(a) = q_1 \cap \cdots \cap q_s$ be a primary decomposition of $I = (a)$. Let l be such that $(\sqrt{q_i})^l \subseteq q_i$ for all $i = 1, \ldots, s$.

We prove by induction on n that this l works for all ideals J. First assume that $n = 1$. If $J \subseteq \sqrt{q_i}$ then by the choice of l we have $J^l \subseteq q_i$. If, however, $J \not\subseteq \sqrt{q_i}$, then $q_i : J^l = q_i : J^{l+1} = q_i$ as q_i is primary. Thus $I : J^l = (q_1 : J^l) \cap \cdots \cap (q_s : J^l) = (q_1 : J^{l+1}) \cap \cdots \cap (q_s : J^{l+1}) = I : J^{l+1}$, so we are done.

Now let $n \geq 1$. It is enough to show that $I^n : J^{nl+1} \subseteq I^n : J^{nl}$. Let x be an element of $I^n : J^{nl+1}$. Then $x \in I^{n-1} : J^{nl+1}$ and by induction assumption x lies in $I^{n-1} : J^{(n-1)l}$. Let y be an element of $J^{(n-1)l}$ and z an element of J^{l+1}. Then xy lies in I^{n-1}, so $xy = ba^{n-1}$ for some b. As yz is in J^{ln+1}, we get that xyz lies in I^n, so $xyz = ca^n$ for some c. These two equations say that $ca^n = bza^{n-1}$, hence that $ca = bz$. Since z is an arbitrary element of J^{l+1}, this says that $b \in (a) : J^{l+1}$. So by induction assumption for $n = 1$ we get that $b \in (a) : J^l$. Now let w be an element of J^l. Then $xyw = ba^{n-1}w \in (a^n)$. This holds for arbitrary $y \in J^{(n-1)l}$ and arbitrary $w \in J^l$, so $x \in I^n : J^{nl}$. ∎

Author's original motivation for studying asymptotic properties of primary decompositions of powers of an ideal was the question whether tight closure commutes with localization. Part of this problem is determining asymptotic properties of primary components of ideals of the form $(x_1^{n_1}, \ldots, x_s^{n_s})$. If the x_i form a regular sequence, a similar argument as above, together with an argument from [3, Theorem 4.5], gives:

PROPOSITION 4 *Let R be a Noetherian ring and $x_1, \ldots, x_s$ a regular sequence in R. Let l be as in Theorem 3 for $I = (x_1, \ldots, x_s)$. Then for any $n_1, \ldots, n_s \geq 1$ and any ideal J,*

$$(x_1^{n_1}, \ldots, x_s^{n_s}) : J^{lN} = (x_1^{n_1}, \ldots, x_s^{n_s}) : J^{lN+1},$$

where $N = (\sum n_i) - s + 1$. ∎

Now we are ready to prove Theorem 1:

Proof: By the remark immediately after the statement of the theorem there exists an integer k' which works for all minimal associated primes. By Theorem 2 there are only finitely many prime ideals $P_1, \ldots, P_t$ which are associated to some power of I and are of height 1 over I. By prime avoidance we can choose an element b contained in each one of these primes P_i but not contained in any minimal prime over I. Let l be as in Theorem 3.

Claim 1: $I^n = (I^n : b^{ln}) \cap (I^n + (b^{ln}))$.

Proof of claim 1: If $a + rb^{ln} \in I^n : b^{ln}$ for some $a \in I^n$, then $ab^{ln} + rb^{2ln} \in I^n$. Since $a \in I^n$ then $rb^{2ln} \in I^n$, so $r \in I^n : b^{2ln} = I^n : b^{ln}$. Hence $rb^{ln} \in I^n$. The other inclusion is easy.

Claim 2: $\operatorname{Ass}(R/(I^n : b^{ln})) = \{P \in \operatorname{Ass}(R/I^n) | b \notin P\}$.

Proof of claim 2: If $P \in \mathrm{Ass}\,(R/(I^n : b^{ln}))$, then $P = (I^n : b^{ln}) : c$ for some $c \in R$. Then $P = I^n : b^{ln}c$, so $P \in \mathrm{Ass}\,(R/I^n)$. If $b \in P = I^n : cb^{ln}$, then $c \in I^n : b^{ln+1} = I^n : b^{ln}$, so $P = I^n : cb^{ln} = R$, which is impossible. Conversely, if $P \in \mathrm{Ass}\,(R/I^n)$ and b is not in P, then $P = I^n : c$ for some $c \in R$. As b is not in P, $P = P : b^{ln} = I^n : cb^{ln}$, so $P \in \mathrm{Ass}\,(R/(I^n : b^{ln}))$. This proves the claim.

It follows that $\{P \in \mathrm{Ass}\,(R/I^n) | b \in P\} \subseteq \mathrm{Ass}\,(R/I^n + (b^{ln}))$.

By the choice of b, each P_i is minimal over $I + (b^l)$. Let k'' be such that $P_i^{k''} \subseteq (I + (b^l))R_{P_i} \cap R$ for all i. Then

$$\begin{aligned} P_i^{2nk''} &\subseteq (I + (b^l))^{2n} R_{P_i} \cap R \\ &\subseteq (I^n + (b^{ln}))R_{P_i} \cap R \end{aligned}$$

which is the P_i-primary component of $I^n + (b^{ln})$ and by the claims also a (possibly redundant) P_i-primary component of I^n.

Finally, set $k = \max\{k', 2k''\}$. ∎

The question remains whether there are good primary decompositions with similarly bounded properties for primary components of height greater than 1 over the chosen ideal.

Mark Johnson observed that the argument above shows that if (R, m) is a regular local ring and P is a prime ideal of dimension 1, then k is bounded below by $2\left((ed!)^{1/d} - d + 1\right)$, where $d = \dim(R)$ and $e = e(R/P)$.

Proof: (Due to Johnson) We use notation from above. As $m^{k''} \subseteq P + (b^l)$,

$$\binom{d + k'' - 1}{d} = \lambda\left(R/m^{k''}\right) \geq \lambda\left(R/(P + (b^l))\right).$$

But R/P is one-dimensional Cohen-Macaulay, so

$$\lambda\left(R/(P + (b^l))\right) = e_m\left(R/(P + (b^l))\right) \geq e_m\left(R/P\right).$$

Thus $\binom{d+k''-1}{d} \geq e$ and

$$(d + k'' - 1)^d \geq d!\binom{d + k'' - 1}{d} \geq ed!,$$

from which we get

$$k = \max\{1, 2k''\} \geq 2\left((ed!)^{1/d} - d + 1\right). \blacksquare$$

Note that Theorem 1 may hold for smaller k, it is only the k from the proof which has this lower bound.

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