PERMANENTAL IDEALS

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Abstract. We show that the (2×2)-subpermanents of a generic matrix generate an ideal whose height, unmixedness, primary decomposition, the number and structure of the minimal components, resolutions, radical, integral closure and Gröbner bases all depend on the characteristic of the underlying subfield: if the characteristic of the subfield is two, this ideal is the determinantal ideal for which all of these properties are already well-known. We show that as long as the characteristic of the subfield is not two, the results are in marked contrast with those for the determinantal ideals.

I. INTRODUCTION.

Permanents were introduced by Cauchy and Binet at the beginning of the nineteenth century as a special type of alternating symmetric function. Later they were studied by I. Schur as a special type of what are now known as Schur functions. Permanents have since found applications in combinatorics, probability theory, and, more recently, invariant theory [HK]. A good survey of the theory of permanents is [Mi].

In contrast to determinantal ideals, permanental ideals have not received much attention to date. For one thing, computation of permanents is an NP hard problem, which translates into a very difficult problem for permanental ideals as well. We are aware of only one other work on permanental ideals, namely M. Niermann's Ph.D. thesis [N], and both that and this work are about size 2×2 generic permanental ideals. M. Niermann calculates the radicals and real radicals. Niermann's motivation, as well as ours, came from work by D. Eisenbud and B. Sturmfels on binomial ideals [ES].

Permanental ideals behave very differently from the better known determinantal ideals. As this paper shows, permanental ideals give yet another example of ideals for which the primary decomposition structure, minimal primes, Gröbner bases, radicals, Cohen-Macaulayness and resolutions depend on the characteristic of the base subfield.

1991 *Mathematics Subject Classification.* Primary: 13C05, 13P10, Secondary: 13C14, 13C40, 15A15.

Key words and phrases. permanental ideal, primary decomposition, Gröbner basis.

The authors thank Bernd Sturmfels for drawing their attention to the subject of this paper, the Mathematical Sciences Research Institute, Berkeley, for its hospitality while part of this paper was being written, Eberhard Becker for drawing their attention to Michael Niermann's thesis, and Luchezar Avramov for helpful comments. The second author also thanks the NSF for partial support.

Typeset by $\mathcal{AMS}$-TEX

The *permanent* of an $(n \times n)$ square matrix $M = (a_{ij})$ is defined as

$$\mathrm{perm}(M) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Thus, the permanent differs from the determinant only in the lack of minus signs in the expansion.

In this paper we work with generic matrices M. More precisely, let F be a field, m, n, r positive integers, and x_{ij} variables over F with $1 \le i \le m$ and $1 \le j \le n$. Let $R = F[x_{ij} | 1 \le i \le m, 1 \le j \le n]$ be the polynomial ring, and let M be the $(m \times n)$-matrix whose (i, j)-entry is x_{ij}. Then let $P_r(M)$ be the ideal of R generated by all the $(r \times r)$-subpermanents of M. If the field F has characteristic 2, the permanental ideals equal determinantal ideals, which are relatively well-understood [BH, BV]. Thus for the rest of this paper we assume that the characteristic of F is different from 2.

In this paper we study the binomial ideals $P_2(M)$. Their properties turn out to be very different from those of the much better understood determinantal ideals. For example, we prove that the generating permanents of $P_2(M)$ are not a Gröbner basis of $P_2(M)$ in any diagonal order, whereas Narasimhan [Na], Caniglia, Guccione and Guccione [CGG], and Sturmfels [S] proved independently that the generating (2×2)-minors of M are a Gröbner basis of the ideal they generate in any diagonal order. Also, in contrast to determinantal ideals, we prove that if m and n are both at least 3, then $P_2(M)$ is not a radical ideal, is not Cohen-Macaulay, and there are minimal primes of distinct heights over it. Furthermore, we prove that the radical of $P_2(M)$ requires generators of degree higher than those of $P_2(M)$ itself. If $m, n \ge 4$, $P_2(M)$ is not integrally closed. We explicitly calculate a primary decomposition, the unmixed parts of $P_2(M)$ of all possible dimensions, and the radical. Radicals and real radicals in characteristic 0 were also calculated independently by Niermann [N].

Thus, the permanental ideals are another case of ideals for which the Gröbner bases, irreducibility, primary decompositions, Cohen-Macaulayness, and integral closedness depend on the characteristic of the underlying field: we have "good" properties in characteristic 2 versus very different properties in all other characteristics. Moreover, for all the properties that we study, the results are independent of the characteristic as long as the characteristic is not 2.

II. MONOMIALS IN $P_2(M)$.

In this section we prove that, unlike the determinantal ideals, the ideals $P_2(M)$ contain many monomials. The proof depends heavily on the characteristic of the field not being 2. There are two basic types of monomials in $P_2(M)$, treated in the two lemmas below.

Lemma 2.1. *The ideal $P_2(M)$ contains all products of three entries of M, taken from three distinct columns and two distinct rows, or from two distinct columns and three distinct rows.*

Proof. We only prove the first part. Without changing $P_2(M)$ we may permute rows and columns of M, and so we assume that the three entries are contained in a submatrix

$$\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}$$

of M. Then $P_2(M)$ contains the elements $ay+bx$ and $bz+cy$. Therefore,

$$c(ay+bx)-a(bz+cy)=cbx-abz=b(cx-az)\in P_2(M).$$

Since $cx+az \in P_2(M)$ and 2 is a unit in R, we obtain that bcx and abz are in $P_2(M)$. A similar argument shows that all other products as in the statement are in $P_2(M)$ as well. This proves the lemma.

Lemma 2.2. *If $m,n\geq 3$, $P_2(M)$ contains all monomials of the form $x_{i_1j_1}^{e_1}x_{i_2j_2}^{e_2}x_{i_3j_3}^{e_3}$ with distinct i_1,i_2,i_3, distinct j_1,j_2,j_3, and where e_1, e_2, and e_3 are positive integers which sum to* 4.

Proof. Permuting rows and columns as before, we consider the submatrix

$$\begin{pmatrix} a & b & c \\ x & y & z \\ u & v & w \end{pmatrix}.$$

It suffices to prove that ayw^2 lies in $P_2(M)$. But

$$ayw^2=aw(yw+vz)-awvz.$$

As $yw+vz\in P_2(M)$ by definition and $awvz\in P_2(M)$ by Lemma 2.1, we are done.

III. A GRÖBNER BASIS.

We compute a Gröbner basis for $P_2(M)$ with respect to any lexicographic diagonal ordering of monomials. Recall that a monomial order on the x_{ij} is *diagonal* if for any square submatrix of M, the leading term of the permanent (or of the determinant) of that submatrix is the product of the entries on the main diagonal. An example of such an order is the lexicographic order defined by:

$$x_{ij}<x_{kl} \text{ if and only if } l>j \text{ or } l=j \text{ and } k>i.$$

Throughout this section we use an arbitrary lexicographic diagonal ordering.

Theorem 3.1. *The following collection G of polynomials is a minimal reduced Gröbner basis for $P_2(M)$, with respect to any diagonal ordering:*

(1) *The subpermanents $x_{ij}x_{kl}+x_{kj}x_{il}, i<k, j<l$;*

(2) $x_{i_1j_1}x_{i_1j_2}x_{i_2j_3}, i_1 > i_2, j_1 < j_2 < j_3$;
(3) $x_{i_1j_1}x_{i_2j_2}x_{i_2j_3}, i_1 > i_2, j_1 < j_2 < j_3$;
(4) $x_{i_1j_1}x_{i_2j_1}x_{i_3j_2}, i_1 < i_2 < i_3, j_1 > j_2$;
(5) $x_{i_1j_1}x_{i_2j_2}x_{i_3j_2}, i_1 < i_2 < i_3, j_1 > j_2$;
(6) $x_{i_1j_1}^{e_1}x_{i_2j_2}^{e_2}x_{i_3j_3}^{e_3}, i_1 < i_2 < i_3, j_1 > j_2 > j_3, e_1e_2e_3 = 2$.

These monomials can also be described as follows. Monomials of type (6) are in one-to-three correspondence with the 3 by 3 submatrices of M: for each 3 by 3 submatrix of M, a monomial of type (6) is the product of the entries on its anti-diagonal, with one of the entries taken to the second power. Pictorially, in a given 3 by 3 submatrix of M, a monomial of type (6) is the product of the following entries marked $\circ$:

$$\begin{pmatrix} & & \circ^{\cdot} \\ & \circ^{\cdot} & \\ \circ^{\cdot} & & \end{pmatrix}$$

(The superscripts $\cdot$ are a reminder that one of the entries is raised to the second power.) Similarly, the monomials of types (2) through (5) are products of the following entries, marked $\circ$, of appropriately sized submatrices:

$$\begin{pmatrix} & & \circ \\ \circ & \circ & \end{pmatrix} \qquad \begin{pmatrix} & \circ & \circ \\ \circ & & \end{pmatrix} \qquad \begin{pmatrix} & \circ \\ & \circ \\ \circ & \end{pmatrix} \qquad \begin{pmatrix} & \circ \\ \circ & \\ \circ & \end{pmatrix}$$

type (2) type (3) type (4) type (5)

With this pictorial representation of the monomials in $P_2(M)$ it is easy to count the number of elements in the minimal Gröbner basis in the theorem, which is

$$\binom{m}{2}\binom{n}{2} + 2\binom{m}{2}\binom{n}{3} + 2\binom{n}{2}\binom{m}{3} + 3\binom{m}{3}\binom{n}{3}.$$

Proof of Theorem 3.1. By Lemmas 2.1 and 2.2, G is contained in $P_2(M)$. As all elements of the form (1) generate $P_2(M)$, G certainly is a generating set. It is easy to see that the set is reduced and minimal. Thus it is sufficient to show that all the S-polynomials $S(F, G)$ of pairs of elements of G reduce to zero with respect to G. As the S-polynomials of two monomials always reduce to zero, it suffices to prove that $S(F, G)$ reduces to zero whenever F is of type (1).

Note that the general case is now done if we can prove the theorem for the special cases where the matrix M is of one of the following sizes: 2×2, 2×3, 2×4, 3×2, 3×3, 3×4, 4×2, 4×3 and 4×4. Thus by symmetry, and by omitting the trivial case of the 2×2 matrix, it suffices to consider only the matrices of sizes 2×3, 2×4, 3×3, 3×4, and 4×4. But these finite cases can be verified by hand or by any of the symbolic computer algebra packages.

We leave the complete verification to the motivated reader, but here is an illustration what is involved in proving that the S-polynomials reduce to 0 when both F and G are of type (1). In this case the general case follows from the case of a 2×3 or 3×3 matrix. If

$$M = \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix},$$

the conclusion follows from the equalities

$$\begin{aligned} S(ay+bx, az+cx) &= z(ay+bx) - y(az+cx) = x(bz+cy) - 2ycx, \\ S(az+cx, bz+cy) &= b(az+cx) - a(bz+cy) = 2bxc - c(ay+bx), \\ S(ay+bx, bz+cy) &= bz(ay+bx) - ay(bz+cy) = b^2zx - acy^2 \\ &= bx(bz+cy) - cy(ay+bx), \end{aligned}$$

while if

$$M = \begin{pmatrix} a & b & c \\ x & y & z \\ u & v & w \end{pmatrix},$$

the conclusion follows from

$$\begin{aligned} w(ay+bx) - y(aw+cu) &= x(bw+cv) - c(xv+yu), \\ y(aw+cu) - a(yw+zv) &= u(bz+cy) - z(av+bu), \\ w(ay+bx) - a(yw+zv) &= x(bw+cv) - v(az+cx). \end{aligned}$$

This theorem is in contrast to the case of determinantal ideals for which the determinants themselves form the reduced Gröbner basis in any diagonal order. The determinantal result was proved independently by Caniglia, Guccione and Guccione [CGG] and by Sturmfels [S].

Remark. It is an easy consequence of this theorem that, if $m, n \geq 4$, then $P_2(M)$ is not integrally closed. Lemma 2.2 shows that $x_{41}x_{32}x_{23}^2$, $x_{41}x_{32}x_{14}^2$ are both contained in $P_2(M)$. Then the element $x_{41}x_{32}x_{23}x_{14}$ is in the integral closure of $P_2(M)$ (as its square equals the product of the first two monomials). However, the previous theorem says that $x_{41}x_{32}x_{23}x_{14}$ is not a multiple of any initial term of any element of the minimal Gröbner basis of $P_2(M)$, thus it is not in $P_2(M)$.

A slight modification of the proof of Theorem 3.1 also gives:

Theorem 3.2. *Assume that $m, n \geq 3$. Let I be the ideal of R generated by elements of $P_2(M)$ and all the products of three entries of M taken from three distinct rows and three distinct columns. The following collection of polynomials is a minimal reduced Gröbner basis for I with respect to any diagonal ordering:*

(1) *The subpermanents $x_{ij}x_{kl} + x_{kj}x_{il}, i < k, j < l$;*

(2) $x_{i_1j_1}x_{i_1j_2}x_{i_2j_3}, i_1 > i_2, j_1 < j_2 < j_3$;
(3) $x_{i_1j_1}x_{i_2j_2}x_{i_2j_3}, i_1 > i_2, j_1 < j_2 < j_3$;
(4) $x_{i_1j_1}x_{i_2j_1}x_{i_3j_2}, i_1 < i_2 < i_3, j_1 > j_2$;
(5) $x_{i_1j_1}x_{i_2j_2}x_{i_3j_2}, i_1 < i_2 < i_3, j_1 > j_2$;
(6) $x_{i_1j_1}x_{i_2j_2}x_{i_3j_3}, i_1 < i_2 < i_3, j_1 > j_2 > j_3$.

We will show in Theorem 5.4 that the ideal I is equal to the radical of $P_2(M)$ (when $m, n \geq 3$). With reasoning as in the remark above we get the following information about the module $I/P_2(M)$:

Theorem 3.3. *Let $m, n \geq 3$. Let I be the ideal of Theorem 3.2. Then*

(1) *The elements of type (6) as in Theorem 3.2 are a set of minimal generators of the module $\frac{I}{P_2(M)}$. Thus the number of minimal generators of $\frac{I}{P_2(M)}$ is $\binom{m}{3}\binom{n}{3}$.*
(2) *The module $\frac{I}{P_2(M)}$ has finite length as an F-vector space. In fact, $\frac{I}{P_2(M)}$ is minimally generated over F by the products of the anti-diagonal entries of each $(i \times i)$ submatrix of M as i varies from 3 to $\min\{m, n\}$. Thus*

$$\text{length}\left(\frac{I}{P_2(M)}\right) = \sum_{i\geq 3} (\text{number of } (i \times i) \text{ submatrices of } M) = \sum_{i\geq 3} \binom{m}{i}\binom{n}{i}.$$

IV. MINIMAL PRIMES.

We compute the primes of R minimal over $P_2(M)$. Our computation does not rely on any monomial ordering of the variables, thus in the proofs we may and do take transposes of M and we do permute the columns and rows of M as needed.

Theorem 4.1. *Let $m, n \geq 2$. Each of the prime ideals P of R minimal over $P_2(M)$ is one of the following:*

(1) *If $n \geq 3$, then P is generated by all the indeterminates in $m-1$ of the rows of M;*
(2) *if $m \geq 3$, then P is generated by all the indeterminates in $n-1$ of the columns of M;*
(3) *P is generated by the permanent of one (2×2)-submatrix of M and all the entries of M outside of this submatrix.*

Moreover, each of the primes in (1), (2) and (3) is minimal over $P_2(M)$.

Proof. The proof proceeds by induction on $n+m$. If $m = n = 2$, then it is easy to see that $P_2(M)$ is prime, and of the form (3). So assume that $n+m \geq 5$. Without loss of generality we may assume that $n \geq m$ and $n \geq 3$. Let P be a prime ideal containing $P_2(M)$. We first show that P contains all the entries from some row or from some column of M.

Assume that there is no column all of whose entries are in P. If P contains all but one entry from each row of M, then, since $m \leq n$, there necessarily exists a (2×2)-submatrix

$$\begin{pmatrix} x_{ij} & x_{ik} \\ x_{lj} & x_{lk} \end{pmatrix}$$

of M such that the entries of either the diagonal or codiagonal do not lie in P, and the other two entries are elements of P, say x_{ij} and x_{lk} do not lie in P. Since $x_{ij}x_{lk} + x_{lj}x_{ik} \in P_2(M) \subset P$ and $x_{lj}x_{ik} \in P$, it follows that $x_{ij}x_{lk} \in P$, which is a contradiction, since P is prime.

So, necessarily, there is a row of M, which has two entries not in P, say $x_{11}, x_{12} \notin P$. By Lemma 2.1, $x_{11}x_{12}x_{ij} \in P_2(M) \subset P$ for all $i > 1, j > 2$. This implies that $x_{ij} \in P$ for all $i > 1, j > 2$. By assumption every column of M contains at least one entry not in P, so that $x_{1j} \notin P$ for all $j > 2$. Now consider x_{21} and x_{22}. Since $n \geq 3$, we know by Lemma 2.1 that $x_{21}x_{22}x_{13} \in P$. Since $x_{13} \notin P$, this implies that one of the other two factors, say x_{21}, is in P. Since $x_{11}x_{22} + x_{21}x_{12} \in P$, this implies that $x_{11}x_{22} \in P$. Since $x_{11} \notin P$, we have $x_{22} \in P$. Hence P contains all the entries from the second row of M. Similarly, one shows that it contains all the entries from rows $3, \ldots, m$. A similar argument applies if P does not contain all the entries from some row of M.

So we may assume that P contains an ideal J generated by all the entries from one row or one column of M. By transposing M, if necessary, we assume that P contains a row of M. Of course, P also contains the permanental ideal P_2 of the submatrix of M obtained by deleting that row. Let Q be a prime ideal contained in P which is minimal over P_2. We have the containments

$$P_2(M) \subset P_2 + J \subset Q + J \subset P,$$

and P is minimal over each of the smaller ideals. We know the structure of Q by induction hypothesis, whence we know the structure of $Q + J$. In particular, we deduce that $Q + J$ is a prime ideal. As P is minimal over it, $Q + J = P$. If Q is of types (1) or (3), respectively, so is P. Now suppose that Q is of type (2). Then P is the ideal generated by all the entries in $n - 1$ columns of M, plus an extra entry. But this extra entry makes P not minimal over $P_2(M)$, contradicting the assumption. Thus Q cannot be of type (2). This completes the proof.

Remark. Observe that the three types of primes in Theorem 4.1 satisfy no inclusion relations. Primes of type (1) have height $(m-1)n$, those of type (2) have height $m(n-1)$, and type (3) primes have height $mn - 4 + 1 = mn - 3$. These heights are different in general. Hence, $R/P_2(M)$ is not equidimensional, that is, $\dim(R/P)$ is different for different minimal primes P. This implies that neither $P_2(M)$ nor its radical is Cohen-Macaulay. In contrast, determinantal ideals are all radical [BV, Corollary 5.8] and Cohen-Macaulay [BV, Theorem 5.3 and Corollary 5.17].

Corollary 4.2. *If $(m, n) \neq (2, 2)$ and if $(m, n) \neq (3, 3)$, then $R/P_2(M)$ is not equidimensional, hence not Cohen-Macaulay.*

Corollary 4.3. *The number of minimal components of* $P_2(M)$ *is*

(1) $m+n+\binom{m}{2}\binom{n}{2}$ *if* $m,n \geq 3$,
(2) $m+\binom{m}{2}\binom{n}{2}$ *if* $m \geq 3, n=2$,
(3) $n+\binom{m}{2}\binom{n}{2}$ *if* $m=2, n \geq 3$,
(4) 1 *if* $m=n=2$.

The explicit structure of the minimal primes over $P_2(M)$ gives yet another property of permanental ideals which contrasts with determinantal ideals. Glassbrenner and Smith proved in [GS] that determinantal varieties have systems of parameters that are highly symmetric and sparse. We now show that this is not the case for (2×2)-permanental varieties.

By a parameter we mean an element $a = \sum_{ij} a_{ij}x_{ij}$ with elements a_{ij} in R such that a is not in any minimal prime of $P_2(M)$. A parameter is called *sparse* if very few of the a_{ij} are non-zero. By Theorem 4.1, for each (2×2)-submatrix of M, one of the variables appearing in that submatrix must have a non-zero coefficient in a, for otherwise a lies in a minimal prime of type (3). Similarly, if $n \geq 3$, each row of M contains an entry with a non-zero coefficient in a, and if $m \geq 3$, each column of M contains an entry with a non-zero coefficient in a. Thus no parameter on the (2×2)-permanental variety is sparse.

V. PRIMARY DECOMPOSITION.

In this section we calculate a primary decomposition of $P_2(M)$ and its radical. Theorem 4.1 determines all the minimal primes over $P_2(M)$, which are of three types. We let $\mathcal{P}_i, i=1,2,3$, be the set of all minimal primes of type (i) as in Theorem 4.1. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$.

Proposition 5.1. *The primary components of* $P_2(M)$ *corresponding to the minimal primes over* $P_2(M)$ *are exactly the minimal primes in* $\mathcal{P}$ *themselves.*

Proof. The result is clear if $m=n=2$.

Let $P \in \mathcal{P}$. Let Q_P be the P-primary component of $P_2(M)$. Then $P_2(M) \subset Q_P \subset P$, and Q_P is characterized by the property that Q_P contains a power of P and, if $rs \in Q_P$ and $r \notin P$, then $s \in Q_P$. We will show that Q_P contains all the generators of P, hence is equal to P.

First let P be a minimal prime over $P_2(M)$ that is generated by all the entries of M except for those in one row, say the first one. Then, by assumption, $n \geq 3$. Let $x_{ij}, i > 1$, be an entry of M. Let $p \neq q \neq j \neq p$ be column labels (which exist, since $n \geq 3$). Then, by Lemma 2.1, the element $x_{ij}x_{1p}x_{1q}$ is in $P_2(M) \subset Q_P$. Furthermore, $x_{1p}x_{1q} \notin P$, hence $x_{ij} \in Q_P$. This proves that $P = Q_P$.

The case where P is generated by all entries of M except for those in one column is proved similarly.

Finally, suppose that P is generated by one (2×2)-subpermanent and all entries outside of this (2×2)-block, as in Part (3) of Theorem 4.1. As above, by using Lemma 2.1 and the defining property of Q_P repeatedly, all entries of M outside of the (2×2)-block are in Q_P. Since any (2×2)-subpermanent is in $P_2(M) \subset Q_P$, this completes the proof.

In order to compute the radical and a primary decomposition of $P_2(M)$ we need to compute the intersection of all the minimal primes. Define $I_i = \bigcap_{P \in \mathcal{P}_i} P$. We first calculate I_1, I_2 and I_3, which are the unmixed parts of $P_2(M)$ of various dimensions.

Lemma 5.2. *Let $m, n \geq 2$. Recall that I_1 is only defined if $n \geq 3$, and I_2 is only defined if $m \geq 3$. Then*

1. $I_1 = \langle x_{ij}x_{lk} | i \neq l, 1 \leq i, l \leq m, 1 \leq j, k \leq n \rangle$;
2. $I_2 = \langle x_{ij}x_{lk} | 1 \leq i, l \leq m, 1 \leq j, k \leq n, j \neq k \rangle$;
3. $I_3 = P_2(M) + \langle x_{ip}x_{jq}x_{kr} | 1 \leq i < j < k \leq m \rangle + \langle x_{ip}x_{jq}x_{kr} | 1 \leq p < q < r \leq n \rangle$,

where one or both of the first two ideals may be zero if m or n is 2.

Proof. (1) Each $P \in \mathcal{P}_1$ contains any product of entries from two different rows, which shows that the right-hand side is contained in I_1. To show the other inclusion, observe that each $P \in \mathcal{P}_1$ is a monomial ideal, and the intersection of monomial ideals is again monomial. Let $u \in I_1$ be a monomial generator. Then u is a product of entries of M. Without loss of generality assume that $u = x_{11}v$ for some monomial v. It is now sufficient to show that v is contained in the ideal generated by the entries of the last $m-1$ rows of M. But this is clear, since $x_{11}v$ is contained in all primes in $\mathcal{P}_1$, in particular in the ideal that is generated by all entries except those in the first row.

The proof of (2) is similar to that of (1).

Our original proof of part (3) involved multiple lengthy and uninstructive induction arguments. Instead we present Niermann's clever shortcut. Then the proof of (3) follows by an easy application of the following lemma from [N]:

Lemma 5.3. *(Niermann* [N, p. 103]*) Let R be an arbitrary ring and $I_1, \ldots, I_l$, $J_1, \ldots, J_l$ ideals in R such that $I_i \subseteq J_j$ if $i \neq j$. Then*

$$\bigcap_{i=1}^{l} (I_i + J_i) = I_1 + \cdots + I_l + \bigcap_{i=1}^{l} J_i.$$

The proof, given in [N], is a straightforward induction on l. This lemma finishes the proof of Lemma 5.2.

We are now ready to compute the radical and primary decomposition of $P_2(M)$.

Theorem 5.4. *If m or n is equal to 2, then rad$(P_2(M)) = P_2(M)$. If $m, n \geq 3$, then*

$$\begin{aligned} rad(P_2(M)) &= I_1 \cap I_2 \cap I_3 \\ &= P_2(M) + \langle x_{ip}x_{jq}x_{kr} | i \neq j \neq k \neq i, p \neq q \neq r \neq p \rangle. \end{aligned}$$

So $P_2(M)$ is a radical ideal if and only if $m \leq 2$ or $n \leq 2$.

Proof. The result is of course trivial if $m = n = 2$.

Assume that $m = 2$ and $n \geq 3$. Then $P_2(M)$ has minimal primes of types (1) and (3) only (cf. Theorem 4.1). Thus

$$\begin{aligned}\operatorname{rad}(P_2(M)) &= I_1 \cap I_3 \\ &= \langle x_{ij}x_{lk} | i \neq l, 1 \leq i, l \leq m, 1 \leq j, k \leq n\rangle \\ &\qquad \bigcap (P_2(M) + \langle x_{ip}x_{jq}x_{kr} | 1 \leq i < j < k \leq 2 \text{ or } 1 \leq p < q < r \leq n\rangle) \\ &= P_2(M) + \langle x_{ij}x_{lk} | i \neq l\rangle \cap \langle x_{ip}x_{jq}x_{kr} | 1 \leq p < q < r \leq n\rangle,\end{aligned}$$

and the last intersection is in $P_2(M)$ by Lemma 2.1 as $m = 2$.

Similarly, if $n = 2$, $P_2(M)$ is a radical ideal.

Now assume that $m, n \geq 3$. Then

$$\begin{aligned}\operatorname{rad}(P_2(M)) &= I_1 \cap I_2 \cap I_3 \\ &= \langle x_{ij}x_{lk} | j \neq k, i \neq l\rangle \cap I_3 \\ &= \langle x_{ij}x_{lk} | j \neq k, i \neq l\rangle \\ &\qquad \cap (P_2(M) + \langle x_{ip}x_{jq}x_{kr} | i, j, k \text{ distinct or } p, q, r \text{ distinct}\rangle) \\ &= P_2(M) + \langle x_{ip}x_{jq}x_{kr} | i, j, k \text{ distinct and } p, q, r \text{ distinct}\rangle,\end{aligned}$$

the latter equality by Lemma 2.1. Now, the monomials of the form $x_{ip}x_{jq}x_{kr}$ with distinct i, j, k and distinct p, q, r are not in $P_2(M)$, as can be easily verified via our Gröbner basis in Theorem 3.1. This finishes the proof of the theorem.

Thus, for $m, n \geq 3$, $P_2(M)$ is an example of an ideal whose radical requires generators of degree higher than those of the ideal itself. Thus permanental ideals might seem a possible candidate for a negative answer to a question of Ravi [R], whether for a homogeneous ideal its Castelnuovo-Mumford regularity is at least as big as the regularity of its radical. This is not the case, however.

Corollary 5.5. *Let I be the radical of $P_2(M)$. Then*

(1) *$reg(P_2(M)) \geq reg(I)$;*
(2) *$reg(P_2(M)) \geq 1 + \sum_{i\geq 3} \binom{m}{i}\binom{n}{i}$.*

Proof. Consider the short exact sequence of graded R-modules

$$0 \longrightarrow I/P_2(M) \longrightarrow R/P_2(M) \longrightarrow R/I \longrightarrow 0.$$

Since $I/P_2(M)$ has finite length by Theorem 3.3, it follows from [E, Corollary 20.19.d] that

$$\operatorname{reg}(R/P_2(M)) = \max\{\operatorname{reg}(I/P_2(M), R/I\} \geq \operatorname{reg}(R/I).$$

As for any ideal $J \subset R$, $\operatorname{reg}(J) = \operatorname{reg}(R/J) + 1$, the first assertion of the corollary follows.

The displayed formula also shows that $\operatorname{reg}(R/P_2(M)) \geq \operatorname{reg}(I/P_2(M))$, and as $I/P_2(M)$ has finite length, $\operatorname{reg}(I/P_2(M)) = \operatorname{length}(I/P_2(M))$. Thus

$$\operatorname{reg}(P_2(M)) = 1 + \operatorname{reg}(R/P_2(M)) \geq 1 + \operatorname{length}(I/P_2(M)) = 1 + \sum_{i \geq 3} \binom{m}{i}\binom{n}{i}.$$

Here, the first inequality follows from the fact that $I/P_2(M)$ has finite length, and the second equality follows from Theorem 3.3.

Corollary 5.6. *$P_2(M)$ has embedded components if and only if $m, n \geq 3$.*

It follows that for all M with $m, n \geq 3$, $P_2(M)$ has embedded components. We now show that, in fact, $P_2(M)$ has exactly one embedded component.

Let Q be the ideal

$$Q = P_2(M) + \langle x_{ij}^2 | 1 \leq i \leq m, 1 \leq j \leq n \rangle.$$

Then for $m, n \geq 3$, we obtain from Theorem 5.4 that

$$\begin{aligned} Q \cap I_1 \cap I_2 \cap I_3 &= \left(P_2(M) + \langle x_{ij}^2 \rangle\right) \cap \Big(P_2(M) + \sum \langle x_{ij} x_{kl} x_{pq} \rangle\Big) \\ &= P_2(M) + \langle x_{ij}^2 \rangle \cap \Big(P_2(M) + \sum \langle x_{ij} x_{kl} x_{pq} \rangle\Big) \end{aligned}$$

The sum in this identity extends over all subscripts $i \neq k \neq p \neq i$ and $j \neq l \neq q \neq j$. To simplify the last intersection and prove that it lies in $P_2(M)$, we use Gröbner bases again, in any diagonal term order. It suffices to show that for any element f in the intersection, after reducing f with respect to the Gröbner basis of $P_2(M)$ as in Theorem 3.1, we get $f = 0$. If this is false, the leading monomial x of f is non-zero, and hence necessarily a multiple of one of the monomials of type (6) in the Gröbner basis for $I_1 \cap I_2 \cap I_3 = \sqrt{P_2(M)}$, as was computed in Theorem 3.2. Moreover, x has to be divisible by the square of a variable x_{ij}. But least common multiples of these two types of monomials are all in the ideal generated by monomials of types (2)-(6) in the Gröbner basis of $P_2(M)$ (as in Theorem 3.1), contradicting the initial statement that f is reduced with respect to this basis.

Since Q is primary to the homogeneous maximal ideal of all variables and each of I_1, I_2, I_3 is the intersection of distinct minimal components of $P_2(M)$, we have calculated a primary decomposition of $P_2(M)$.

Theorem 5.7. *Let $m, n \geq 3$. The intersection*

$$P_2(M) = Q \cap I_1 \cap I_2 \cap I_3,$$

after rewriting each I_i as the intersection of the minimal primes of type (i), is an irredundant primary decomposition of $P_2(M)$.

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