Powers of Ideals:
Primary Decompositions, Artin-Rees Lemma and Regularity

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Let R be a Noetherian ring and I an ideal in R. Then there exists an integer k such that for all $n \geq 1$ there exists a primary decomposition $I^n = q_1 \cap \cdots \cap q_s$ such that for all i, $\sqrt{q_i}^{nk} \subseteq q_i$.

Also, for each homogeneous ideal I in a polynomial ring over a field there exists an integer k such that the Castelnuovo-Mumford regularity of I^n is bounded above by kn.

The regularity part follows from the primary decompositions part, so the heart of this paper is the analysis of the primary decompositions. In [S], this was proved for the primary components of height at most one over the ideal.

This paper proves the existence of such a k but does not provide a formula for it. In the paper [SS], Karen E. Smith and myself find explicit k for ordinary and Frobenius powers of monomial ideals in polynomial rings over fields modulo a monomial ideal and also for Frobenius powers of a special ideal first studied by Katzman. Explicit k for the Castelnuovo-Mumford regularity for special ideals is given in the papers by Chandler [C] and Geramita, Gimigliano and Pitteloud [GGP].

Another method for proving the existence of k for primary decompositions of powers of an ideal in Noetherian rings which are locally formally equidimensional and analytically unramified is given in the paper by Heinzer and Swanson [HS].

The primary decomposition result is not valid for *all* primary decompositions. Here is an example: let I be the ideal (X^2, XY) in the polynomial ring $k[X, Y]$ in two variables X and Y over a field k. For each positive integer m, $I = (X) \cap (X^2, XY, Y^m)$ is an irredundant primary decomposition of I. However, for each integer k there exists an integer m, say $m = k + 1$, such that $(X, Y)^k \not\subseteq (X^2, XY, Y^m)$. Hence the result can only

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hold for *some* primary decompositions.

Section 1 contains the necessary background and some reductions towards the proof of the main theorem on the primary decompositions. The reductions make it necessary to prove a version of the "linear uniform Artin-Rees lemma" in the spirit of Huneke's paper [Hu]. This is done in Section 2. Section 3 contains the main results concerning the primary decompositions and the regularity of powers of an ideal. Section 4 contains several more versions of the uniform Artin-Rees lemma relating to powers of ideals and products of powers of various ideals and some related questions.

1. Background

We first make the observation that if $\phi : R \to S$ is a ring homomorphism and I an ideal in S, then any primary decomposition $I = q_1 \cap \cdots \cap q_s$ of I gives a possibly redundant primary decomposition $\phi^{-1}(I) = \phi^{-1}(q_1) \cap \cdots \cap \phi^{-1}(q_s)$ of the contraction of I to R. Namely, each $\phi^{-1}(q_i)$ is primary to $\phi^{-1}(\sqrt{q_i})$. In all our applications of this, ϕ will be the inclusion homomorphism: by the observation then any primary decomposition of an ideal I in S contracts to a primary decomposition of the contraction $IS \cap R$.

In the proof of the first theorem to come, we need the following well-known lemma:

Lemma 1.1: *If a prime ideal in a Noetherian ring R is associated to some ideal generated by a non zerodivisor, then it is associated to every ideal generated by a non zerodivisor that it contains.*

With this we prove a generalization of Ratliff's theorem which is needed in Sections 3 and 4:

Theorem 1.2: *(Ratliff [R]) Let $I_1, \ldots, I_d$ be ideals in a Noetherian ring R. Then the set $\cup_{n_1,\ldots,n_d} Ass\,(R/I_1^{n_1} \cdots I_d^{n_d})$ of all associated prime ideals of $R/I_1^{n_1} \cdots I_d^{n_d}$ as the n_i vary over non negative integers, is finite.*

Proof: First we reduce to the case when all the ideals I_i are principal and generated by non zerodivisors. This we may do by first passing to the extended Rees ring $S = R[I_i t, t^{-1}]$: if we can prove the result for ideals $I_1 S, \ldots, I_{i-1} S, t^{-1} S, I_{i+1} S, \ldots, I_d S$, then as primary decompositions of ideals in S contract to a primary decomposition in R and as $I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} t^{-n_i} I_{i+1}^{n_{i+1}} \cdots I_d^{n_d} S \cap R = I_1^{n_1} \cdots I_d^{n_d}$, we are done.

Thus we may assume that all the ideals I_i are principal and generated by non zerodivisors.

To finish the proof of the theorem it now suffices to prove that when all the $n_i \geq 1$, $\mathrm{Ass}\,(R/I_1^{n_1} \cdots I_d^{n_d})$ is contained in $\mathrm{Ass}\,(R/I_1 \cdots I_d)$. But this follows by the lemma. ∎

The main results of this paper are about "linear" properties of powers of an ideal. The following result of Katz and McAdam is the first step in this direction:

Theorem 1.3: *(Katz-McAdam [KM, (1.5)]) Let R be a Noetherian ring. For any ideal I there exists an integer l such that $I^n : J^{nl} = I^n : J^{nl+1}$ for all ideals J and all $n \geq 0$.*

Actually, Katz and McAdam prove the existence of such an integer l depending on I and J, but their argument can be easily extended to show that there is an l independent of J. A proof of this generalization appeared in [S], but I include it here for completeness:

Proof: Let $S = R[It, t^{-1}]$, where t is an indeterminate over R. It is clear that $I^n :_R J = (t^{-n}S :_S J) \cap R$ for all n and all ideals J of R. Thus it suffices to show that there exists an integer l such that $t^{-n}S :_S J^{nl} = t^{-n}S :_S J^{nl+1}$ for all n and all ideals J of S. So by replacing R by S we may assume without loss of generality that I is a principal ideal generated by a regular element a.

Now let $(a) = q_1 \cap \cdots \cap q_s$ be a primary decomposition of $I = (a)$. Let l be such that $(\sqrt{q_i})^l \subseteq q_i$ for all $i = 1, \ldots, s$.

We prove by induction on n that this l works for all ideals J. First assume that $n = 1$. If $J \subseteq \sqrt{q_i}$ then by the choice of l we have $J^l \subseteq q_i$. If, however, $J \not\subseteq \sqrt{q_i}$, then $q_i : J^l = q_i : J^{l+1} = q_i$ as q_i is primary. Thus $I : J^l = (q_1 : J^l) \cap \cdots \cap (q_s : J^l) = (q_1 : J^{l+1}) \cap \cdots \cap (q_s : J^{l+1}) = I : J^{l+1}$, so we are done.

Now let $n \geq 1$. It is enough to show that $I^n : J^{nl+1} \subseteq I^n : J^{nl}$. Let x be an element of $I^n : J^{nl+1}$. Then $x \in I^{n-1} : J^{nl+1}$ and by induction assumption x lies in $I^{n-1} : J^{(n-1)l}$. Let y be an element of $J^{(n-1)l}$ and z an element of J^{l+1}. Then xy lies in I^{n-1}, so $xy = ba^{n-1}$ for some b. As yz is in J^{ln+1}, we get that xyz lies in I^n, so $xyz = ca^n$ for some c. These two equations say that $ca^n = bza^{n-1}$, hence that $ca = bz$. Since z is an arbitrary element of J^{l+1}, this says that $b \in (a) : J^{l+1}$. So by induction assumption for $n = 1$ we get that $b \in (a) : J^l$. Now let w be an element of J^l. Then $xyw = ba^{n-1}w \in (a^n)$. This holds for arbitrary $y \in J^{(n-1)l}$ and arbitrary $w \in J^l$, so $x \in I^n : J^{nl}$. ∎

In particular, if R is a local ring with maximal ideal m and I is a prime ideal of dimension one, then $m^{kn}I^{(n)} \subseteq I^n$, where $I^{(n)}$ denotes the n'th symbolic power of I. This answers a question of Herzog's in [He].

With this, we are ready for some reductions towards the main theorem:

Main Theorem: *(see Theorem 3.4) For every ideal I in a Noetherian ring R there exists an integer k such that for all $n \geq 1$ there exists an irredundant primary decomposition $I^n = q_1 \cap \cdots \cap q_s$ such that $\sqrt{q_i}^{nk} \subseteq q_i$ for all i.*

We start outlining the proof here, but we complete it only in Section 3.

Let P be a prime ideal associated to some R/I^n. We want to prove that we can find an integer k, independent of n but possibly dependent on P, for which there exists a primary decomposition $I^n = q_1 \cap \cdots \cap q_s$ such that whenever $P = \sqrt{q_i}$, then $P^{nk} \subseteq q_i$. If we can find such a k for each P, then as there are only finitely many such P (see Theorem 1.2), we have proved the theorem.

Our strategy is to first localize at P: if we can prove that $P^{nk} R_P \subseteq q_i R_P$, then certainly $P^{nk} \subseteq q_i R_P \cap R = q_i$. Thus we may assume that R is a local ring and that P is the maximal ideal of R. We may also assume that the result is already known for Q-primary components, Q properly contained in P, as those components do not affect any possible P-primary components.

Let l be the Katz-McAdam's constant for I, i.e., l has the property that for all integers n and all ideals J,

$$I^n : J^{ln} = I^n : J^{ln+1}.$$

Thus in particular if P is the maximal ideal in the ring, $I^n : P^{ln} = I^n : P^{ln+1}$, and this is the intersection of all the primary components of I^n not primary to P. To simplify notation we denote this ideal as $I^{<n>}$. Note that $I^n = I^{<n>} \cap (I^n + P^m)$ for all large integers m and that this intersection decomposes into a primary decomposition of I^n. In order to prove the main theorem we have to bound the least possible m above by a linear function in n. One way of accomplishing this is to find some sort of "linear Artin-Rees" lemma: namely, we want to find an integer k such that

$$P^m \cap I^{<n>} \subseteq P^{m-kn} I^{<n>} \qquad (*)$$

for all $m \geq kn$ and all n. If we can do that, then by the choice of l for all $m \geq kn + ln$, $P^m \cap I^{<n>} \subseteq P^{m-kn-ln} I^n$. Hence $I^n = I^{<n>} \cap (I^n + P^{(k+l)n})$ gives a desired primary decomposition of I^n.

We prove the linear Artin-Rees statement (*) for special I in Section 3 (see Theorem 3.3) where we then also finish the proof of the main theorem for all I (see Theorem 3.4).

2. Artin-Rees Lemma (following [Hu])

The goal of this section is the following generalized Artin-Rees Lemma along the lines of Huneke's "Uniform bounds in Noetherian rings":

> **(See Theorem 2.7.)** *Let R be a complete Noetherian local ring with infinite residue field. Let $N \subseteq M$ be finitely generated R modules. Then there exists an integer k such that for all a satisfying a technical condition to be described later, for all proper ideals J in R and for all integers $m \geq k$, $aJ^m M \cap N \subseteq aJ^{m-k}N$.*
>
> The case $a = 1$ was proved with less restrictive hypotheses in [Hu, Theorem 4.12].

The case we need for the main theorem of this paper is when $a = 1$, $M = R$ and N equals $I^{<n>}$ for some n. The integer k in the statement of the goal certainly depends on n. By using the statement for a large set of elements a, we prove in Theorem 3.3 that for special I we can bound $k(n)$ above by a linear function in n. The connection between the statement and the promised linear bound for $k(n)$ may not be immediately obvious and we explain it in Section 3.

This section is somewhat technical and the reader may skip the rest of it.

Lemma 2.1: *(Compare with [Hu, Proposition 2.2].) Let $N \subseteq K \subseteq M$ be R modules and let a be an element of R. Assume that there exist integers h and k such that for all ideals J in R, $aJ^m M \cap K \subseteq aJ^{m-h}K$ for all $m \geq h$ and $aJ^m K \cap N \subseteq aJ^{m-k}N$ for all $m \geq k$. Then $aJ^m M \cap N \subseteq aJ^{m-h-k}N$ for all $m \geq h+k$.*

The proof is straightforward. An important consequence is

Proposition 2.2: *(Compare with [Hu, Proposition 2.2].) Let R be a Noetherian ring, a an element of R and $N \subseteq M$ two finitely generated R modules. Assume that $0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_t = M/N$ is a filtration of M/N such that $K_i/K_{i-1} \cong R/P_i$, where P_i is an ideal of R. Let k_i be an integer satisfying*

$$aJ^m \cap P_i \subseteq aJ^{m-k_i}P_i$$

for all ideals J and all $m \geq k_i$. Then for all J and all $m \geq k_1 + \cdots + k_t$,

$$aJ^m M \cap N \subseteq aJ^{m-k_1-\cdots-k_t}N.$$

Proof: It suffices to prove that if $M/N \cong R/P$ for some ideal P in R, then $aJ^m \cap P \subseteq aJ^{m-k}P$ implies $aJ^m M \cap N \subseteq aJ^{m-k}N$.

By assumption $M = N + xR$ for some $x \in M$ such that $N :_R x = P$. Then $aJ^m M \cap N = aJ^m N + aJ^m x \cap N$. Let $y \in aJ^m x \cap N$. Write $y = arx$ for some $r \in J^m$. Then ar is an element of $aJ^m \cap (N :_R x) = aJ^m \cap P \subseteq aJ^{m-k}P$. Thus $y = arx$ lies in

$aJ^{m-k}Px \subseteq aJ^{m-k}N$. ∎

The following is a technical lemma which will allow us to conclude the main result for all ideals primary to the maximal ideals once we know the result for any minimal reduction:

Lemma 2.3: *(Compare with [Hu, Lemma 3.1].) Let J, L, and P be ideals in a ring R and let a be an element of R. Suppose that*

(i) $aL^m \cap P \subseteq aJ^{m-k}P$ for all $m \geq k$,

(ii) $L^m \cap P \subseteq J^{m-k}P$ for all $m \geq k$, and

(iii) $J^m \subseteq L^{m-h} + P$ for all $m \geq h+1$.

Then $aJ^m \cap P \subseteq aJ^{m-k-h-1}P$ for all $m \geq k+h+1$.

Proof: We first prove

$$J^m \subseteq L^{m-h} + J^{m-k-h-1}P \text{ for all } m \geq h+k+1. \qquad (\#)$$

If $m = h+k+1$, this is just (iii). Now assume (#) for some $m \geq h+k+1$. Multiply through by J and use (iii):

$$J^{m+1} \subseteq (JL^{m-h} + J^{m-k-h}P) \cap (L^{m+1-h} + P).$$

Let $x \in J^{m+1}$. Write $x = u + v = y + z$ for some $u \in JL^{m-h}$, $v \in J^{m-k-h}P$, $y \in L^{m+1-h}$ and $z \in P$. Then

$$\begin{aligned} u - y = z - v &\in (JL^{m-h} + L^{m+1-h}) \cap P \\ &\subseteq L^{m-h} \cap P \\ &\subseteq J^{m-h-k}P \text{ by (ii).} \end{aligned}$$

Thus $x = (u-y) + y + v \in J^{m-h-k}P + L^{m+1-h}$, which finishes the induction.

Hence $aJ^m \cap P \subseteq a(L^{m-h} + J^{m-k-h-1}P) \cap P \subseteq aL^{m-h} \cap P + aJ^{m-k-h-1}P \subseteq aJ^{m-h-k}P + aJ^{m-k-h-1}P$. ∎

In the proof of the main result of this section we proceed by induction on the dimension of M/N: important ingredients in the induction step are elements which may be thought of as "Cohen-Macaulayfiers". We denote their set as $CM(R)$:

Definition 2.4: (See [Hu, 2.11].) Let

$$\mathbb{F} : 0 \to F_t \xrightarrow{f_t} F_{t-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0$$

be a complex of finitely generated free R modules. We say that $\mathbb{F}$ satisfies *the standard condition on rank* if $\operatorname{rank}(f_t) = \operatorname{rank} F_t$ and for $1 \leq i < t$, $\operatorname{rank}(f_i) + \operatorname{rank}(f_{i+1}) = \operatorname{rank} F_i$.

Let $I(f_i)$ denote the ideal generated by the rank-size minors of f_i. We say that $\mathbb{F}$ satisfies *the standard condition on height* if $\operatorname{ht}(I(f_i)) \geq i$ for all i.

With this, we define $CM(R)$ to be the set of all elements $x \in R$ such that for all complexes $\mathbb{F}$ satisfying the standard rank and height conditions, $xH_i(\mathbb{F}) = 0$ for all $i \geq 1$.

Lemma 2.5: *(Compare with [Hu, Lemma 3.3].) Let R be a Noetherian ring and P a prime ideal in R. Let a and c be elements of R not contained in P such that the image of c in R/P lies in $CM(R/P)$. Let L be any ideal in R generated by elements $a_1, \ldots, a_d$ such that $ht(a_1, \ldots, a_d)R/P = d$. Then if there exists an integer k such that $aL^m \cap (P + cR) \subseteq aL^{m-k}(P + cR)$ for all $m \geq k$, then also*

$$aL^m \cap P \subseteq aL^{m-k}P$$

for all $m \geq k$.

Note that this lemma does the inductive step: assuming the main result of this section (Theorem 2.7) for all (N, M) such that $\dim(M/N) < \dim(R/P)$, we can prove it also for the pair (P, R), but only for the special ideals L. However, Lemma 2.3, together with some more technicalities, takes care of the rest of the ideals.

Proof of Lemma 2.5: First note that $aL^m \cap P = aL^m \cap (P+cR) \cap P \subseteq aL^{m-k}(P+cR) \cap P = aL^{m-k}P + aL^{m-k}cR \cap P$. Thus it suffices to prove that $aL^{m-k}cR \cap P$ lies in $aL^{m-k}P$. First note that $aL^{m-k}cR \cap P$ equals $c(aL^{m-k} \cap P)$. Thus by the choice of c it suffices to prove that $\frac{aL^{m-k} \cap P}{aL^{m-k}P}$ lies in some $H_i(\mathbb{F})$ for some $\mathbb{F}$ satisfying standard rank and height conditions and for some $i \geq 1$.

First let $\mathbb{G}$ be a minimal homogeneous free resolution of $(X_1, \ldots, X_d)^{m-k}$ in the polynomial ring $T = \mathbb{Z}[X_1, \ldots, X_d]$. Let the maps in the complex be called g_i. It is well-known that $\sqrt{I(g_i)} = (X_1, \ldots, X_d)$ for all i (cf. the proof of Lemma 3.3 in [Hu]). Let $\mathbb{G}' = \mathbb{G} \otimes_T R$, where the map $T \to R$ takes X_i to a_i. Let $\mathbb{F}'$ be the same as $\mathbb{G}'$ except that the last map is composed with multiplication by a. Namely, $\mathbb{F}'$ equals

$$\mathbb{F}' : 0 \to G_t \xrightarrow{g_t \otimes 1} G_{t-1} \to \cdots \to G_1 = R^{\binom{m-k+d-1}{m-k-1}} \xrightarrow{af_1} G_0 = R \to 0,$$

where the entries of (the matrix) f_1 are all the elements of a generating set of L^{m-k}. Note that $H_0(\mathbb{F}') = R/aL^{m-k}$. We compare $\mathbb{F}'$ with a free resolution $\mathbb{H}$ of R/aL^{m-k}:

$$\begin{array}{ccccccccc} \mathbb{F}' : & \cdots & G_2 & \to & R^{\binom{m-k+d-1}{m-k-1}} & \xrightarrow{af_1} & R & \to & 0 \\ & & \downarrow & & \| & & \| & & \\ \mathbb{H} : & \cdots & R_2 & \to & R^{\binom{m-k+d-1}{m-k-1}} & \xrightarrow{af_1} & R & \to & 0 \end{array}$$

where the leftmost vertical arrow exists because G_2 is free and the second complex is exact. Now tensor both complexes with $\otimes_R R/P$ and set $\mathbb{F}$ to be $\mathbb{F}' \otimes_R R/P$. We get a surjective map from $H_1(\mathbb{F}) = H_1(\mathbb{F}'\otimes_R R/P)$ to $H_1(\mathbb{H}\otimes_R R/P) = \operatorname{Tor}_1^R(R/aL^{m-k}, R/P) = \frac{aL^{m-k}\cap P}{aL^{m-k}P}$. As by assumption $\mathbb{F}$ satisfies the standard conditions on height and rank, then $cH_1(\mathbb{F}) = 0$, so the lemma is proved. ∎

Note that if P is the only maximal ideal of the ring, the conlusion of the lemma is trivially true with $k = 1$ for all a in the ring. However, the conclusion of this lemma is false if a is an element of P and P is not the maximal ideal. For this reason we need to put restrictions on a in the main theorem (Theorem 2.7) of this section:

Definition 2.6: Let R be a complete Noetherian local ring and let $N \subseteq M$ be finitely generated R modules. We define the condition (C) inductively on dimension of M/N: if $\dim(M/N) = 0$, we say that all the elements of R satisfy the condition (C) with respect to the pair (N, M). Now suppose that M/N has positive dimension. We say that an element a satisfies the condition (C) with respect to (N, M) if there exists a prime filtration $0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_t = M/N$ of M/N (i.e., for each i, $K_i/K_{i-1} \cong R/P_i$ for some prime ideal P_i in R), such that a satisfies the condition (C) with respect to each (P_i, R). It remains to define the condition (C) with respect to (P, R) where P is a nonmaximal prime ideal in R. By [Hu, Proposition 4.5 (i)] and the Cohen Structure Theorem, $CM(R/P)$ is nonzero. We say that a satisfies the condition (C) with respect to (P, R) if a is not contained in P and if there exists an element c of R whose image in R/P is nonzero in $CM(R/P)$ such that a satisfies the condition (C) with respect to $(P + cR, R)$.

Note that an element satisfies the condition (C) with respect to (N, M) if and only if it avoids a finite set of nonmaximal primes. Thus 1 satisfies (C) for all (N, M), and an element a satisfies (C) with respect to (N, M) if and only if all of its powers satisfy (C) with respect to (N, M). Note that by the Prime Avoidance Theorem there are always elements satisfying (C) with respect to some (N, M) even in the maximal ideal.

With this we are ready to prove the main result of this section:

Theorem 2.7: *(Compare with [Hu, Theorem 4.12].) Let R be a complete Noetherian local ring with infinite residue field. Let $N \subseteq M$ be finitely generated R modules. Then there exists an integer k such that for all proper ideals J in R, for any element a in R satisfying condition (C) with respect to (N, M), and for all $m \geq k$, $aJ^mM \cap N \subseteq aJ^{m-k}N$.*

Proof: First note that it suffices to prove the theorem for ideals primary to the maximal ideal $\mathbf{m}$ only, because then for an arbitrary J we have: $aJ^mM \cap N \subseteq \bigcap_l a(J + \mathbf{m}^l)^m M \cap N \subseteq \bigcap_l a(J + \mathbf{m}^l)^{m-k}N \subseteq \bigcap_l (aJ^{m-k} + \mathbf{m}^l)N = aJ^{m-k}N$.

Consider a prime filtration of M/N as in the definition of the condition (C). Note that by the choice of P_i, a satisfies the condition (C) with respect to all these (P_i, R). Also, for each of these $P_i \neq \mathbf{m}$ there exists some c_i whose image is nonzero in $CM(R/P_i)$, such that a also satisfies the condition (C) with respect to $(P_i + c_i R, R)$.

Now we proceed as in [Hu, Theorem 3.4] using induction on $\dim(M/N)$. By using Proposition 2.2 and the given prime filtration of M/N, it suffices to prove that for each P_i there exists an integer k such that for all $\mathbf{m}$-primary ideals J and for all $m \geq k$, $aJ^m \cap P_i \subseteq aJ^{m-k} P_i$.

If $\dim(R/P_i) = 0$, then P_i is the maximal ideal $\mathbf{m}$ of R, and the theorem follows easily with $k = 1$ for all a.

Now assume that $d = \dim(R/P_i) > 0$. We rename P_i as P and c_i as c. By induction on dimension there exists an integer k_c such that for all a satisfying the condition (C) with respect to (N, M), for all $\mathbf{m}$-primary ideals J and for all $m \geq k_c$, $aJ^m \cap (P + cR) \subseteq aJ^{m-k_c}(P + cR)$.

Also, as in the proof of [Hu, Theorem 4.12], there exists an integer s and an element $t \in m \setminus P$ such that for all ideals J in R/P, t multiplies the integral closure of $J^n R/P$ into $J^{n-s} R/P$. (In the notation of [Hu], $t \in T_s(R/P)$. In fact, in this case $T_s(R/P) = R/P$ by the Uniform Briançon-Skoda Theorem [Hu, Theorem 4.13].) We may assume either by induction with $a = 1$ or by using Huneke's [Hu, Theorem 4.12] that there exists an integer k_t such that for all ideals J and for all $m \geq k_t$, $J^m \cap (P + tR) \subseteq J^{m-k_t}(P + tR)$.

We claim now that $aJ^m \cap P \subseteq aJ^{m-k_t-k_c-s-1} P$ for all $m \geq k_t + k_c + s + 1$ and for all $\mathbf{m}$-primary ideals J.

As R/P has an infinite residue field, we may choose elements $a_1, \ldots, a_d$ in J whose images in R/P generate a reduction of the image of J in R/P (see [NR, Section 5]). As we may assume that J is $\mathbf{m}$-primary, $\operatorname{ht}(a_1, \ldots, a_d)R/P = d$. Let $L = (a_1, \ldots, a_d)R$. Then by Lemma 2.5

$$aL^m \cap P \subseteq aL^{m-k_c} P \subseteq aJ^{m-k_c} P \qquad (i) \text{ and } (ii)$$

for all $a \in R$ satisfying condition (C) with respect to (N, M) and for all $m \geq k_c$.

Now note that the integral closures of JR/P and LR/P are the same, so by the choice of t, $t(JR/P)^m \subseteq (LR/P)^{m-s}$. Thus

$$\begin{aligned} tJ^m &\subseteq (L^{m-s} + P) \cap (P + tR) \\ &= L^{m-s} \cap (P + tR) + P \\ &\subseteq L^{m-s-k_t}(P + tR) + P \\ &= L^{m-s-k_t} tR + P \end{aligned}$$

Modulo P then $tJ^mR/P \subseteq L^{m-s-k_t}tR/P$, so as t is not in P, $J^mR/P \subseteq L^{m-s-k_t}R/P$. Thus

$$J^m \subseteq L^{m-s-k_t} + P \qquad (iii)$$

for all $m \geq s + k_t$.

Now we apply Lemma 2.3 to the displayed equations (i), (ii) and (iii) to obtain that $aJ^m \cap P \subseteq aJ^{m-k_t-k_c-s-1}P$. ∎

Note that the assumptions in this theorem are overly restrictive. However, they help keep the notation simple and the statement is enough for the main theorem of the paper which is to be proved in the next section.

3. Primary decompositions and regularity of powers of ideals

We first apply the main theorem of the previous section (Theorem 2.7) to the case when $M = R$ and N varies over powers of a special ideal:

Theorem 3.1: *Let R be a complete Noetherian local ring with infinite residue field and x an element of R which is not a zerodivisor. Then there exists an integer k such that for any proper ideal J of R, for all integers n, all $m \geq kn$ and any element a in R which satisfies the condition (C) with respect to the pair $((x), R)$,*

$$aJ^m \cap (x)^n \subseteq aJ^{m-kn}(x)^n.$$

Proof: Let $0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_t = R/(x)$ be a prime filtration of $R/(x)$, i.e., each K_i/K_{i-1} is isomorphic to R/P_i for some prime ideal P_i in R. The P_i may not be distinct, but to simplify notation we think of them as distinct.

Note that for every integer n, $R/(x)^n$ has a prime filtration in which each P_i occurs n times and no other prime ideals occur in the filtration. This follows by induction, using the fact that $(x^n)/(x^{n+1}) \cong R/(x)$ and using the short exact sequences

$$0 \longrightarrow (x)^n/(x)^{n+1} \longrightarrow R/(x)^{n+1} \longrightarrow R/(x)^n \longrightarrow 0.$$

By Theorem 2.7 there exists an integer k_i such that for all proper ideals J in R and all integers $m \geq k_i$,

$$aJ^m \cap P_i \subseteq J^{m-k_i}P_i.$$

Let $k = \sum_i k_i$. Then by Proposition 2.2, for all proper ideals J, all given elements a in R, all n and all $m \geq kn$, $aJ^m \cap (x)^n \subseteq aJ^{m-kn}(x)^n$. ∎

This proof also shows:

Proposition 3.2: *Let x be a non zerodivisor in a complete Noetherian ring R. Then an element a of R which satisfies the condition (C) with respect to the pair $((x), R)$ also satisfies the condition (C) with respect to all $((x^n), R)$.* ∎

A corollary is the promised uniform linear Artin-Rees lemma for symbolic powers:

Theorem 3.3: *Let I be a principal ideal generated by a non zerodivisor in a Noetherian local ring (R, P). Let $I^{<n>}$ denote the intersection of all the primary components of I^n which are not primary to P. Then there exists an integer k such that for all n and all $m \geq kn$,*

$$P^m \cap I^{<n>} \subseteq P^{m-kn} I^n.$$

Proof: Let X be an indeterminate over R and S the completion of $R[X]_{PR[X]}$ in the $PR[X]$-adic topology. As S is faithfully flat over R, $I^{<n>}S$ is the intersection of all primary components of $I^n S$ which are not primary to PS. Suppose that there exists an integer k such that $P^m S \cap I^{<n>} S \subseteq P^{m-kn} I^n S$. Then $P^m \cap I^{<n>} \subseteq P^m S \cap I^{<n>} S \cap R \subseteq P^{m-kn} I^n S \cap R =\subseteq P^{m-kn} I^n$, so we are done. Thus we may replace R by S and assume that R is a complete local ring with infinite residue field.

Let a be an element of R such that a satisfies the condition (C) with respect to (I, R), a lies in P but not in any other prime ideal associated to any power of I, and a is a non zerodivisor. As all of these conditions on a just mean that a has to avoid finitely many prime ideals properly contained in P, then by the Prime Avoidance Theorem a exists. Thus by Theorem 3.1 there exists an integer k such that

$$a^l P^m \cap I^n \subseteq a^l P^{m-kn} I^n$$

for all n, l, and all $m \geq kn$. As a is not contained in contained in any associated prime ideal of $I^{<n>}$, then $I^{<n>} = I^n : a^l$ for all sufficiently large l. Thus as a is not a zerodivisor, the displayed equation says that

$$P^m \cap I^{<n>} = P^m \cap (I^n : a^l)(a^l P^m \cap I^n) : a^l \subseteq P^{m-kn} I^n. \quad ∎$$

Now we can prove the main theorem:

Theorem 3.4: *For every ideal I in a Noetherian ring R there exists an integer k such that for all $n \geq 1$ there exists an irredundant primary decomposition $I^n = q_1 \cap \cdots \cap q_s$ such that $\sqrt{q_i}^{nk} \subseteq q_i$ for all i.*

Proof: Suppose we can prove the theorem for the ideal (t^{-1}) in $R[It, t^{-1}]$. Then as $t^{-n} R[It, t^{-1}] \cap R = I^n$ and as every primary decomposition of $t^{-n} R[It, t^{-1}]$ contracts to a primary decomposition of I^n, we have also proved the theorem for the ideal I in R. Thus without loss of generality we may assume that I is a principal ideal generated by a non zerodivisor x.

As in the reductions at the end of Section 1, we may localize and assume that R is a Noetherian local ring with maximal ideal P and it then suffices to find an integer k and primary decompositions $I^n = q_1 \cap \cdots \cap q_s$ such that whenever $\sqrt{q_i} = P$, then $P^{nk} \subseteq q_i$.

Set $I^{<n>}$ to be the intersection of those primary components of I^n which are not primary to P. Then by Theorem 3.3 there exists an integer k such that for all n and for all $m \geq kn$, $P^m \cap I^{<n>} \subseteq P^{m-kn} I^n$. This means that I^n equals $I^{<n>} \cap (I^n + P^{kn})$, which gives a desired primary decomposition of I^n. ∎

For general ideals, it is very difficult to find the integer k which satisfies the theorem. For one thing, computing the embedded components of primary decompositions of arbitrary ideals is very difficult (see [EHV]). However, for monomial ideals in polynomial rings, computing primary decompositions is quite fast, see for example [STV]. It turns out that one can obtain k for monomial ideals even without any primary decomposition algorithms: in [SS] Smith and I prove that if I and J are monomial ideals in the polynomial ring $K[x_1, \ldots, x_d]$ over a field K and if l is the maximum of the degrees of elements in a minimal monomial generating set, then for every integer n there exists a primary decomposition $I^n + J = q_1 \cap \cdots \cap q_s$ such that $\sqrt{q_i}^{2dln} \subseteq q_i$ for all i. Thus in this case, $k = 2dl$, which is easily computable. For a more precise statement and for the analogous result for Frobenius powers of I modulo J, refer to [SS].

A consequence of the existence of the integer k in Theorem 3.4 is that we can bound the Castelnuovo-Mumford regularity of powers of ideals linearly in the powers. Namely, for every homogeneous ideal I in a polynomial ring $R = K[x_1, \ldots, x_d]$ over a field K there exists an integer k such that the regularity of I^n is bounded above by kn (see Theorem 3.6 below). When $\dim(R/I) \leq 1$, Chandler [C] and Geramita, Gimigliano and Pitteloud [GGV] found explicit formulas for upper bounds of the regularity of I^n. In higher dimensions, the obstruction to finding explicit bounds is precisely the lack of understanding in which degrees I^n and the saturation of I^n start agreeing (in our notation the saturation of I^n is $I^{<n>} = \cup_m I^n : (x_1, \ldots, x_d)^m$).

We say that I is m-saturated if I and its saturation agree in degrees m and higher.

Definition 3.5: Let I be a homogeneous ideal in the polynomial ring $R = K[x_1, \ldots, x_d]$ over a field K. Let

$$0 \longrightarrow F_d \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

be a minimal graded free resolution of I. Thus each $F_i = \oplus R(-b_{ij})$ for some integers b_{ij}. We say that I is m-regular if $b_{ij} - i \leq m$ for all i and all j. The (Castelnuovo-Mumford) regularity $\operatorname{reg}(I)$ of I is then defined to be the least integer m for which I is m-regular.

Theorem 3.6: *Let I be a homogeneous ideal in the polynomial ring $R = K[x_1, \ldots, x_d]$ of d variables over a field K. Then there exists an integer k such that $reg(I^n) \leq kn$ for all n.*

Proof: We proceed by induction on $\dim(R/I)$. If $\dim(R/I) = 0$ (or 1 also) this is true by [C] (or Theorem 1.1 in [GGP]). Thus we may assume that $\dim(R/I) > 0$.

Note that the hypotheses or the conclusion of the theorem are unaffected if we replace K by an infinite field extension. Thus we may assume that K is infinite to start with. Hence there exists a homogeneous element h of degree 1 in R which avoids all the finitely many prime ideals different from $(x_1, \ldots, x_d)$ which are associated to the powers of I. By induction on the dimension then there exists an integer k_1 such that $\operatorname{reg}(I^n + hR) \leq k_1 n$ for all n. Also, by Theorem 3.4 there exists an integer k_2 such that I^n and the saturation $I^{<n>}$ of I^n agree in degrees $k_2 n$ and larger. Now let $k = \max\{k_1, k_2\}$. Then by assumptions I^n is kn saturated and $I^n + h$ is kn-regular. Now we use a lemma of Bayer and Stillman ([BS, (1.8) Lemma]) which says that if h is a form of degree one in R which is not a zerodivisor on R modulo the saturation of J such that J is m-saturated and $J + hR$ is m-regular, then J is m-regular. By applying that we get that I^n is kn-regular. ∎

4. More uniform Artin-Rees lemmas for powers of ideals

Similar techniques as in the proof of Theorem 3.4 show:

Theorem 4.1: *Let I be an ideal in a Noetherian ring R. Then there exists an integer k such that for all ideals J in R, for all integers n and for all $m \geq kn$,*

$$J^m \cap I^n \subseteq J^{m-kn} I^n.$$

Proof: As in the proof of Theorem 3.4 we may assume that I is principal and generated by a non zerodivisor. To prove that $J^m \cap I^n \subseteq J^{m-kn} I^n$, it suffices to prove the inclusion after localization at every prime ideal associated to $J^{m-kn} I^n$. By Theorem 1.2 there are only finitely many such prime ideals, thus if we find a k for each one of them, we are done. Hence we may localize and assume that R is a Noetherian local ring with, say, maximal ideal P. As in the proofs in Section 3 we may pass to the completion of $R[X]_{PR[X]}$ in the $PR[X]$-adic topology, whence we may apply Theorem 3.1 to complete the proof. ∎

Moreover, a similar but messier proof shows that given ideals $I_1, \ldots, I_l$ in a Noetherian ring R, there exist integers k_i, $i = 1, \ldots, l$, such that for all ideals J, all integers $n_i \geq 1$ and all $m \geq k_1 n_1 + \cdots + k_l n_l$,

$$J^m \cap I_1^{n_1} \cdots I_l^{n_l} \subseteq J^{m-k_1 n_1 - \cdots - k_l n_l} I_1^{n_1} \cdots I_l^{n_l}.$$

The only new observation we need for this is that if $x_1, \ldots, x_l$ are non zerodivisors in R, then $R/x_1^{n_1} \cdots x_l^{n_l} R$ has a prime filtration in which each of the prime ideals occurring in a prime filtration of $R/x_i R$ occurs exactly n_i times.

And this can be easily generalized to the following: Let $I_1, \ldots, I_l$ be ideals in a Noetherian ring R. Then there exist integers k_i, $i = 1, \ldots, l$, such that

$$\begin{aligned} I_1^{n_1} \cap \cdots \cap I_l^{n_l} &\subseteq I_1^{n_1} \cap \cdots \cap I_{l-2}^{n_{l-2}} \cap I_{l-1}^{n_{l-1}-k_l n_l} I_l^{n_l} \\ &\subseteq I_1^{n_1} \cap \cdots \cap I_{l-3}^{n_{l-3}} \cap I_{l-2}^{n_{l-2}-k_{l-1}(n_{l-1}-k_l n_l)-k_l n_l} I_{l-1}^{n_{l-1}-k_l n_l} I_l^{n_l} \quad \text{etc.} \end{aligned}$$

This brings us to the multi-ideal version of the uniform Artin-Rees lemma:

Theorem 4.2: *(Compare with [Hu, Theorem 4.12].) Let R be a Noetherian ring satisfying at least one of the following conditions:*

(i) R is essentially of finite type over a Noetherian local ring.

(ii) R is a ring of positive prime characteristic p and is module-finite over R^p (R^p is the subring generated by pth powers of elements of R).

(iii) R is essentially of finite type over $\mathbb{Z}$.

Then given any finitely generated R modules $N \subseteq M$ there exists an integer k such that for all integers l, for all ideals $J_1, \ldots, J_l$ in R and all integers $n_i \geq k$, $i = 1, \ldots, l$,

$$J_1^{n_1} \cdots J_l^{n_l} M \cap N \subseteq J_1^{n_1-k} \cdots J_l^{n_l-k} N.$$

Moreover, if R is a complete local Noetherian ring with infinite residue field and a satisfies the condition (C) with respect to (N, M), then also (for a possibly different k)

$$aJ_1^{n_1} \cdots J_l^{n_l} M \cap N \subseteq aJ_1^{n_1-k} \cdots J_l^{n_l-k} N.$$

In Section 2 we proved the special case of this when $l = 1$ and the ring is complete and local.

The proof proceeds as in Section 2 or as in [Hu], with some modifications:

1) As in Proposition 2.2, we may assume that $M = R$ and N is a prime ideal P in R.

2) As in the proof of Theorem 2.7 we may assume that each J_i is primary to some maximal ideal in R.

3) We define

$$T^l(R) = \{c \in R : \exists t \text{ such that } c\overline{J_1^{n_1} \cdots J_l^{n_l}} \subseteq J_1^{n_1-t} \cdots J_l^{n_l-t} \text{ for all ideals } J_i, \text{ all } n_i\}$$

(Here, the overline above an ideal stands for the integral closure.) Then the natural generalizations of [Hu, Proposition 4.7] and [Hu, Theorem 4.10] (by using a multi-ideal version of the Briançon-Skoda Theorem for Proposition 4.7 and a multi-Rees ring for Theorem 4.10) show that for any R satisfying (i), (ii) or (iii), $T^l(R/P)$ is nonzero for all prime ideals P in R and, moreover, that $\cap_l T^l(R/P)$ is nonzero.

4) As in the proof of [Hu, Theorem 4.12], $CM(R/P)$ is nonzero for every prime ideal P in R.

5) (Compare with Lemma 2.3.) Let $L_1, \ldots, L_l$, $J_1, \ldots, J_l$, and P be ideals in R. For any l-tuple $(n_1, \ldots, n_l)$, $\underline{J}^{\underline{n}}$ stands for the ideal $J_1^{n_1} \cdots J_l^{n_l}$, and for any integer n, $\underline{n}$ stands for the l-tuple $(n, \ldots, n)$. Let h and k be integers. Assume that

   (i) $a\underline{L}^{\underline{m}} \cap P \subseteq a\underline{L}^{\underline{m}-\underline{k}} P$ for all $m_i \geq k$,

   (ii) $\underline{L}^{\underline{m}} \cap P \subseteq \underline{L}^{\underline{m}-\underline{k}} P$ for all $m_i \geq k$, and

   (iii) $\underline{J}^{\underline{m}} \subseteq \underline{L}^{\underline{m}-\underline{h}} + P$ for all $m_i \geq h+1$.

   Then $a\underline{J}^{\underline{m}} \cap P \subseteq a\underline{J}^{\underline{m}-\underline{k}-\underline{h}-\underline{1}} P$ for all $m_i \geq k+h+1$.

6) A multi-ideal version of Lemma 2.5 holds: Let elements a, c not lie in a prime ideal P and assume that the image of c in R/P lies in $CM(R/P)$. Let $L_1, \ldots, L_l$ be any d-generated ideals in R such that for each i, $i = 1, \ldots, l$, $\operatorname{ht}(L_i R/P) = d$. Suppose that for a given integer l there exists an integer k such that for all integers $m_1, \ldots, m_l \geq k$, $a\underline{L}^{\underline{m}} \cap (P + cR) \subseteq a\underline{L}^{\underline{m}-\underline{k}}(P + cR)$. Then for all $m_1, \ldots, m_l \geq k$,

$$a\underline{L}^{\underline{m}} \cap P \subseteq a\underline{L}^{\underline{m}-\underline{k}} P.$$

Now the proof proceeds as in Theorem 2.7, except that if the ring is not local, we also need to use some methods of the proof of [Hu, Theorem 3.4]. ∎

The corollary of this is that for every ideal I in a Noetherian ring R there exists an integer k such that for all ideals $J_1, \ldots, J_l$ and all $m_1, \ldots, m_l \geq k$,

$$\underline{J}^{\underline{m}} \cap I^n \subseteq \underline{J}^{\underline{m}-\underline{kn}} I^n.$$

A natural follow-up, which was actually my main motivation for studying these asymptotic properties of primary decompositions, is:

Open Question: Given elements $x_1, \ldots, x_l$ in a Noetherian ring R of positive prime characteristic p, does there exist an integer k such that for all integers n there exists a primary decomposition $(x_1^{p^n}, \ldots, x_l^{p^n}) = q_1 \cap \cdots \cap q_s$ such that $\sqrt{q_i}^{k\mu(\sqrt{q_i})p^n} \subseteq q_i$? Here, $\mu(\ )$ stands for the minimal number of generators.

In analogy with the proof of Theorem 2.7, one may want to start with something like

Theorem 1.2 for ideals $(x_1^{p^n}, \ldots, x_l^{p^n})$ as n varies. However, Katzman gave a counterexample in [K]: if R is $K[X,Y,t]/(XY(X+Y)(X+tY))$, where K is a field of characteristic p and X, Y, t are indeterminates, then the set $\cup_n \mathrm{Ass}\,(R/(X^{p^n}, Y^{p^n}))$ is infinite. Nevertheless, Smith and I verified that the stated open question and the analog of the Katz-McAdam's Theorem 1.3 do hold in this example (see [SS]). We also proved that the stated open question has a positive answer when the x_i are monomials and R is a polynomial ring modulo a monomial ideal (see [SS] for precise statements).

In general, even the analog of the Katz-McAdam theorem is an open question for ideals of the form $(x_1^{p^n}, \ldots, x_l^{p^n})$ as n varies. The following special case is known: if the x_i form a regular sequence, a similar argument as in Theorem 1.3, together with an argument from [HH, Theorem 4.5], gives:

Proposition 4.3: *Let R be a Noetherian ring and $x_1, \ldots, x_s$ a regular sequence in R. Let l be as in Katz-McAdam's theorem for $I = (x_1, \ldots, x_s)$. Then for any $n_1, \ldots, n_s \geq 1$ and any ideal J,*

$$(x_1^{n_1}, \ldots, x_s^{n_s}) : J^{lN} = (x_1^{n_1}, \ldots, x_s^{n_s}) : J^{lN+1},$$

where $N = (\sum n_i) - s + 1$. ∎

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