Quilting semi-regular tessellations

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This will be the first three sections of a chapter in the book "Crafting by Concepts", edited by sarah-marie belcastro and Carolyn Yackel and published by A K Peters.

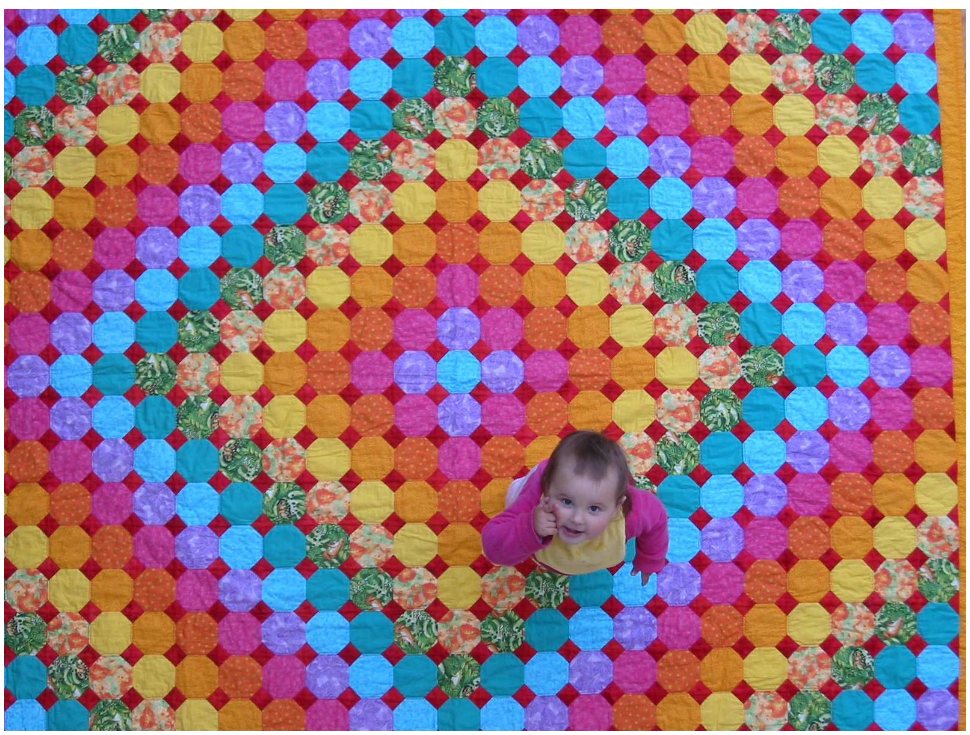

Figure 1: The project for this chapter: The cheerful trip-around-the-world pattern in 4.8.8 tessellation.

1 Overview

There are whole-cloth quilts and patchwork quilts; there are monochromatic quilts and multicolored quilts; there are functional bed quilts and wall quilts; there are treasured never-used quilts and well-used quilts; there are quilts with completely random and non-repeating patterns and there are quilts with repetitions; and of course there are many more ways of looking at quilts.

There are uncountably many patterns, but life is short, so we cannot make them all into quilts. One has to pick and choose a finite number (first one, then another...). Below is a systematic reduction of possible patterns to a doable finite number; in the process of this reduction we will learn some mathematical concepts and even prove that the reduction is indeed systematic. (If your resulting quilts are not seamed absolutely completely perfectly, at least you can have the satisfaction that you were systematic in choosing the quilt patterns!?)

We begin by restricting ourselves to the (smaller) infinite number of quilts that have regularly repeating patterns, and that are composed entirely of regular polygons: a *regular polygon* is a many-sided figure in which all interior angles are the same and all sides have the same length. Every regular polygon is convex, meaning that a needle whose ends are set inside the polygon also has its entire length inside the polygon (see Figure 2). Among the regularly

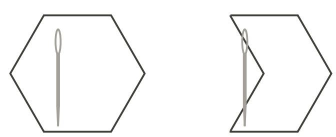

Figure 2: A convex regular hexagon (left) and a non-convex hexagon (right).

repeating quilt patterns whose parts are composed of regular polygons we will only examine those in which any two adjacent regular polygons meet exactly edge-to-edge. But even with all of these restrictions, there are still uncountably many different quilts that can be made, even if one does not account for the infinite variety of fabrics, colors, and fabric embellishments! We explain this in Section 2.

The mathematical goal of this chapter is to narrow down the quilt patterns to the so-called semi-regular tessellations, as there are only finitely many of those, and we may have some hope of making a quilt from each of these patterns before our time on earth runs out. This narrowing down will comprise the bulk of Section 2, where we discover that there are no more than twelve of the desired quilt patterns. In Section 2.5, we briefly discuss a slightly more general class of quilt patterns about which very little is known.

Teachers of all levels will find useable ideas in Section 3. Quilting concepts are used in elementary education more often than most other fiber arts concepts. In this chapter we present some of those, but some more advanced concepts arise as well, ranging from approximations of reals by rationals, to continued fractions, to undergraduate research in k-uniform tessellations.

The sewing goal of this chapter is to give shortcut sewing instructions for most of the semi-regular tesselation quilt patterns, and this will comprise the bulk of Section **??**. The advantage of the shortcut techniques is not just in saved time but also in increased accuracy. Namely, fabric is stretchable, and the more one handles it by sewing and ironing it, the more distortion can occur in its size, especially along the edges, where it matters most, and the resulting quilt may not lie flat. With the methods described in this chapter, the pieces are often not cut until a later stage, which means that there are fewer edges to distort, or the patterns are broken up into perhaps unusual but stabler shapes, also contributing to less distortion in the intermediate stages. A disadvantage of some of the presented techniques is that they use more fabric, some for the tucks and some for leftovers for another quilt top. (For years I have been making "left-over" quilts: they allow for greater freedom in placing pieces and colors and can be more pleasing than the more laboriously produced original quilts.)

Some of the described techniques are known among quilters, but most of them are probably not. They are presented here in the order of increasing difficulty. Readers are encouraged to experiment with these techniques, and to begin by trying the bed-sized squares-and-octagons quilt project in Sections **??**. Avid quilters should dive in head first (after reading the introductory sections about specialized techniques).

2 Mathematics

The aim of this section is to review the basics of tessellations, and to systematically pare down the infinite list of tessellating choices to a finite list. Much more on the subject of tessellations can be found in Grünbaum and Shephard [5], especially in Chapter 2.

2.1 Tessellations by Regular Polygons

A *tiling* or a *tessellation* of the plane is a countable collection of closed sets such that each point of the plane is in one of the closed sets, but that no point is in the interior of any two closed sets. A point may, of course, lie on edges of two different closed sets.

For our tessellations, all closed sets will be convex regular polygons. We also require that the intersection of any two distinct sets, if not empty, is either a single point or an edge of each of the two shapes. Such tessellations are called *edge-to-edge*. (This is in contrast to tessellations in which the intersection of two closed sets might be a partial edge, such as when in a square grid we shift alternate rows over by half the edge-distance.)

How many different regular polygon configurations can occur at a vertex of an edge-to-edge tessellation? If an n_1-gon, n_2-gon, $\ldots$, n_r-gon, meet at this vertex, the interior angles of the polygons have to add up to the full circle because there are no overlaps in the polygons. This is expressed succinctly in Theorem 2, which first needs the following lemma:

Lemma 1. *A regular polygon with n sides has interior angles measuring $\pi\frac{n-2}{n}$ radians (or $180 - \frac{360}{n}$ degrees).*

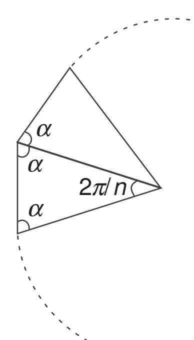

Figure 3: The interior angle of a polygon has measure twice α.

Proof. Connect the vertices of the polygon to the center of the polygon. This produces n congruent isoceles triangles, each of which has its two equal sides incident at the center of the polygon. Each central angle is $2\pi/n$ radians. The other angles in the triangles, labeled α in Figure 3, have measure exactly one half of the interior angle of the polygon. In other words, the sum of the two non-central angles in each triangle equals the interior angle of the polygon. Since the sum of all the angles in a triangle is π radians, we conclude that the interior angle of a regular polygon with n sides equals $\pi - \frac{2\pi}{n} = \pi\frac{n-2}{n}$ radians. $\square$

From Lemma 1, Theorem 2 follows directly: add the angles around a vertex and divide by π.

Theorem 2. *A regular n_1-gon, n_2-gon, ..., and an n_r-gon meet at a vertex without overlaps and without gaps if and only if*

$$\frac{n_1 - 2}{n_1} + \frac{n_2 - 2}{n_2} + \cdots + \frac{n_r - 2}{n_r} = 2.$$

We say that a point in a tessellation has *vertex configuration* $n_1, \ldots, n_r$ if regular n_1-, $\ldots, n_r$-gons meet with their vertices at the point without any overlaps and gaps (and the order in which the vertices meet is irrelevant). In the next subsection we determine all possible vertex configurations (see Theorem 3), and subsequently we examine when each vertex configuration can be extended to a tessellation of the plane.

2.2 Possible Vertex Configurations $\mathbf{n_1, \ldots, n_r}$

The goal is to find all integers $r \geq 3$ and $n_1, n_2, \ldots, n_r \geq 3$ that are solutions to the equation $\frac{n_1 - 2}{n_1} + \frac{n_2 - 2}{n_2} + \cdots + \frac{n_r - 2}{n_r} = 2$ from Theorem 2. It turns out that there are only finitely many solutions, and we list them below in Theorem 3. The list of course contains all the *regular* tessellations, namely tessellations using only one regular polygon: the vertex configurations are $3, 3, 3, 3, 3, 3$ for equilateral triangles, $4, 4, 4, 4$ for squares, and $6, 6, 6$ for hexagons. There are no regular tessellations using any other regular polygons.

Theorem 3. *The following is a complete list of integer solutions of the equation $\frac{n_1 - 2}{n_1} + \frac{n_2 - 2}{n_2} + \cdots + \frac{n_r - 2}{n_r} = 2$, with $n_1, \ldots, n_r \geq 3$ and $r \geq 3$, written in lexicographic (dictionary) order:*

(a) $3,3,3,3,3,3$
(b) $3,3,3,3,6$
(c) $3,3,3,4,4$
(d) $3,3,4,12$
(e) $3,3,6,6$
(f) $3,4,4,6$
(g) $3,7,42$
(h) $3,8,24$
(i) $3,9,18$
(j) $3,10,15$
(k) $3,12,12$
(l) $4,4,4,4$
(m) $4,5,20$
(n) $4,6,12$
(o) $4,8,8$
(p) $5,5,10$
(q) $6,6,6$

Proof. We proceed using a case-by-case analysis.

First suppose that all except possibly one n_i is 3. Namely, say $n_2 = n_3 = \cdots = n_r = 3$. We would like to bound r. Thus

$$\frac{n_1 - 2}{n_1} + \frac{n_2 - 2}{n_2} + \cdots + \frac{n_r - 2}{n_r} = 2$$

becomes

$$\frac{n_2 - 2}{n_2} + \cdots + \frac{n_r - 2}{n_r} = 2 - \frac{n_1 - 2}{n_1}$$

and then

$$\frac{3 - 2}{3} + \cdots + \frac{3 - 2}{3} = 2 - \frac{n_1 - 2}{n_1},$$

which simplifies to

$$\frac{1}{3} + \cdots + \frac{1}{3} = \frac{r - 1}{3} = 2 - \frac{n_1 - 2}{n_1}.$$

Equivalently,

$$\frac{r - 1}{3} = 2 - \frac{n_1 - 2}{n_1} = \frac{n_1 + 2}{n_1},$$

so that $(r-1)n_1 = 3(n_1+2)$. This means that either $r-1$ or n_1 is an integer multiple of 3. If $r - 1 = 3a$ for some integer a, then $3an_1 = 3(n_1 + 2)$ gives that $n_1(a - 1) = 2$, whence $n_1 \leq 2$, which is a contradiction. So necessarily $n_1 = 3a$ for some positive integer a. Then $3a(r - 1) = 3(3a + 2)$, so that $a(r - 1) = 3a + 2$ and $a(r - 4) = 2$, which implies that $1 \leq a \leq 2$. If $a = 1$, necessarily $r - 4 = 2$ so that $r = 6$ and $n_1 = 3$, and we get the configuration of 6 regular triangles meeting at the vertex; and if $a = 2$, we get $r = 5$ and $n_1 = 6$, so that there are $4 = 5 - 1$ triangles and one hexagon meeting at the vertex. This produces the solutions (a) and (b).

It remains to determine all the solutions where at least two n_i are strictly bigger than 3.

So suppose that $n_1, n_2 \geq 4$. For now, let us also suppose that $n_1 \geq 13$ (and later we will handle the cases where all n_i are at most 12). Rewrite

$$\frac{n_1 - 2}{n_1} + \frac{n_2 - 2}{n_2} + \cdots + \frac{n_r - 2}{n_r} = 2$$

as

$$\frac{n_3 - 2}{n_3} + \cdots + \frac{n_r - 2}{n_r} = 2 - \frac{n_1 - 2}{n_1} - \frac{n_2 - 2}{n_2}. \qquad (*)$$

Note that $n \geq 3$ implies that $\frac{1}{n} \leq \frac{1}{3}$, so $-\frac{2}{n} \geq -\frac{2}{3}$, and $\frac{n-2}{n} \geq \frac{1}{3}$. The left-hand side of $(*)$ has $r-2$ terms, each of which is $\geq \frac{1}{3}$, so we see that

$$\frac{r-2}{3} \leq \frac{n_3 - 2}{n_3} + \cdots + \frac{n_r - 2}{n_r} = 2 - \frac{n_1 - 2}{n_1} - \frac{n_2 - 2}{n_2} = \frac{2}{n_1} + \frac{2}{n_2} \leq \frac{2}{4} + \frac{2}{13} = \frac{17}{26},$$

so that $r - 2 \leq \frac{51}{26}$, and since r is an integer, this means that $r - 2 \leq 1$, so that $r \leq 3$, and thus necessarily $r = 3$. This means that there exists an integer $n_3 \geq 3$ such that $\frac{n_3 - 2}{n_3} = \frac{2}{n_1} + \frac{2}{n_2}$ holds. Also, $\frac{n_3 - 2}{n_3} \leq \frac{17}{26}$, so that cross-multiplying and solving gives $n_3 \leq \frac{52}{9}$, whence $n_3 \leq 5$. Thus it suffices to find all integer solutions $n_1 \geq 13$ and $n_2 \geq 4$ such that with $n_3 \in \{3, 4, 5\}$, $\frac{n_3 - 2}{n_3} = \frac{2}{n_1} + \frac{2}{n_2}$. In case $n_3 = 3$, $\frac{1}{3} = \frac{2}{n_1} + \frac{2}{n_2}$, so that $\frac{1}{n_1} = \frac{1}{6} - \frac{1}{n_2} = \frac{n_2 - 6}{6n_2}$ and so $n_1 = \frac{6n_2}{n_2 - 6} = \frac{6(n_2 - 6)}{n_2 - 6} + \frac{36}{n_2 - 6} = 6 + \frac{36}{n_2 - 6}$, from which we can read off the only possible positive integer solutions (with $n_1 \geq 13$): (g) $n_1 = 42, n_2 = 7, n_3 = 3$; (h) $n_1 = 24, n_2 = 8, n_3 = 3$; (i) $n_1 = 18, n_2 = 9, n_3 = 3$; (j) $n_1 = 15, n_2 = 10, n_3 = 3$. In case $n_3 = 4$, we similarly get $n_1 = \frac{4n_2}{n_2 - 4} = 4 + \frac{16}{n_2 - 4}$, with one allowed solution (m) $n_1 = 20, n_2 = 5, n_3 = 4$. Finally, in case $n_3 = 5$, we get $n_1 = \frac{10n_2}{3n_2 - 10} \geq 13$, which yields $4 \leq n_2 \leq \frac{130}{29}$, so that $n_2 = 4$, but then $n_1 = \frac{10n_2}{3n_2 - 10}$ is not an integer, so the case $n_3 = 5$ is not possible.

So far we have handled all the cases where all but one of the n_i is 3, and all the cases where at least two n_j are strictly greater than 3 with at least one 13 or higher. It remains to examine all the possible integers $r \geq 3$ and $n_1, \ldots, n_r \geq 3$ with all $n_i \leq 12$ and satisfying $\frac{n_1-2}{n_1} + \frac{n_2-2}{n_2} + \cdots + \frac{n_r-2}{n_r} = 2$.

Say the n_i are sorted in the order from the largest to the smallest.

We have found the solution $n_1 = 3$: then $r = 6$ and all $n_i = 3$.

If $n_1 = 4$ (and all n_i are either 3 or 4), the equation $\displaystyle\sum_{i=1}^{r} \frac{n_i - 2}{n_i} = 2$ forces an even number of n_i to be 4 (so that the fractions add to an integer). Clearly at most four n_i can be 4, in which case we get the vertex configuration (l) $4, 4, 4, 4$, or else exactly two of the n_i are 4, and the rest have to be 3. This gives the vertex configuration (c) $3, 3, 3, 4, 4$.

If $n_1 = 5$ (and $n_1 \geq n_i$ for all i), similar denominator-clearing reasoning as above forces the number of 5-gons to be a multiple of 5, but this is impossible—the interior angles add up to more than 2π. Similarly we can eliminate the cases $n_1 = 7$, $n_1 = 9$, and $n_1 = 11$.

If $n_1 = 6$ (and $n_1 \geq n_i$ for all i), the equation $\displaystyle\sum_{i=1}^{r} \frac{n_i - 2}{n_i} = 2$ forces, as above, all n_i to not be 5. One can also easily verify that there is no possible vertex configuration with two hexagons and one square at a vertex. We are here assuming that at least two n_i are 4 or greater. We have $n_1 = 6$, and we may, by possibly reindexing, assume that $n_2 \geq 4$. Since 5 is eliminated, we either have $n_2 = 4$ or $n_2 = 6$. If $n_2 = 6$, then $\displaystyle\sum_{i=1}^{r} \frac{n_i - 2}{n_i} = 2$ becomes $\displaystyle\sum_{i=3}^{r} \frac{n_i - 2}{n_i} = 2 - \frac{6-2}{6} - \frac{6-2}{6} = \frac{2}{3}$. From this we get the possibilities $r = 3$, $n_3 = 6$, giving the vertex configuration (q) $6, 6, 6$; and $r = 4$, $n_3 = n_4 = 3$, giving the vertex configuration (e) $3, 3, 6, 6$, and no other possibilities. The remaining cases with $n_1 = 6$ are with all other n_i at most 4 and $n_2 = 4$. Then $\displaystyle\sum_{i=1}^{r} \frac{n_i - 2}{n_i} = 2$ becomes $\displaystyle\sum_{i=3}^{r} \frac{n_i - 2}{n_i} = 2 - \frac{4-2}{4} - \frac{6-2}{6} = \frac{5}{6}$. For denominator clearing necessarily one n_i has to be 3 and another n_i must be 4, and this forces the vertex configuration (f) $3, 4, 4, 6$.

A similar check produces the vertex configurations (o) $4, 8, 8$ if $n_1 = 8$; (p) $5, 5, 10$ if $n_1 = 10$; and (d) $3, 3, 4, 12$, (k) $3, 12, 12$, (n) $4, 6, 12$ if $n_1 = 12$. $\square$

The list in the theorem gives all the solutions to a vertex configuration of an edge-to-edge tessellation using only regular polygons when we look at one vertex at a time.

2.3 Tessellating the Whole Plane

In this section we determine which vertex configurations from Theorem 3 extend to tessellations of the plane.

Consider strips made up of regular 3- and 4-gons as in Figure 4. The sides

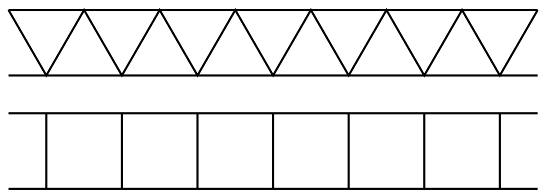

Figure 4: Strips of triangles and strips of squares.

of the triangles and of the squares have equal lengths, so these strips can be pasted together in any order by aligning the vertices to get edge-to-edge tessellations. If we align only the strips of triangles, we get the constant vertex configuration $3, 3, 3, 3, 3, 3$; if we align only the strips of squares, we get the constant vertex configuration $4, 4, 4, 4$; if we alternate the rows of triangles and of squares, we get the constant vertex configuration $3, 3, 3, 4, 4$; but there are also uncountably many other ways to arrange the strips (in correspondence with binary representations of real numbers in the unit interval).

Partially in the interest of being able to make a series of quilts in a finite amount of time, we now restrict our attention to tessellations with the same set of polygons at each vertex. (This prohibits arbitrarily mixing strips of squares and triangles.)

But even with this restriction, there are uncountably many quilts to make from regular polygons in an edge-to-edge manner. Namely, consider the strip of equilateral triangles and hexagons as in Figure 5. Such strips can

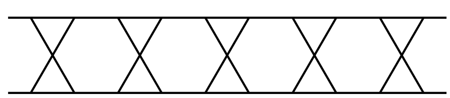

Figure 5: A strip of hexagons, with corner triangles

be stacked in two different ways that have the same vertex configuration. See Figure 6. Thus even with this restriction we still get uncountably many distinct tessellations. How can we eliminate the uncountability of options?

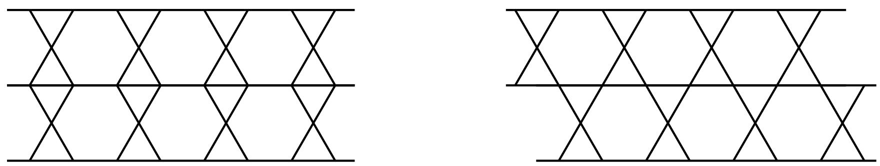

Figure 6: Two stackings of a strip of hexagons

We make one final restriction on the type of tessellations: we require that not only do the same polygons appear at each vertex, but that these polygons appear in the same order when enumerated either clockwise or counterclockwise. We call such tessellations *semi-regular*.

We establish new notation to record the polygons appearing in their order: a vertex has the *cyclic vertex configuration* $n_1.n_2.\ldots.n_r$ if either clockwise or counterclockwise around the vertex, the regular polygons appear in the order n_1-gon through n_r-gon. For example, in the tessellation on the left in Figure 6, the cyclic vertex configuration of a vertex on the central horizontal line is 3.3.6.6, 3.6.6.3, 6.6.3.3, or 6.3.3.6, and for a vertex between two horizontal lines it is 3.6.3.6 or 6.3.6.3. It is standard to record the configurations with the r-tuple that is the smallest in the lexicographic ordering. For the two vertices above, one would thus record 3.3.6.6 and 3.6.3.6, respectively. Thus, a tessellation is semi-regular if and only if all vertices have the same cyclic vertex configuration.

2.4 Determining All Semi-Regular Tessellations

We just defined a tessellation to be semi-regular if the closed sets in the tessellation are all regular polygons, the polygons meet edge-to-edge, and at all vertices the cyclic vertex configurations are the same. Not all of the 17 vertex configurations on the list in Theorem 3 can be made into semi-regular tessellations. In this section we review the list and determine which extend to one or more semi-regular tessellations.

Clearly (a) $3, 3, 3, 3, 3, 3$ makes the regular tessellation 3.3.3.3.3.3 by equilateral triangles (and there is no choice for how the triangles are joined). See the first tessellation in Figure 15.

Next on the list in Theorem 3 is $3, 3, 3, 3, 6$, so the hexagon has to be surrounded by triangles, as in Figure 7. Each of the triangles on the outside has to share a vertex with another hexagon. Thus there are two distinct

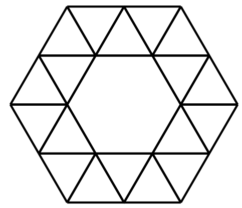

Figure 7: A hexagon surrounded by triangles

ways of continuing the construction with a hexagon at the rightmost vertex, see Figure 8. Once the placement of the second hexagon is chosen, the rest

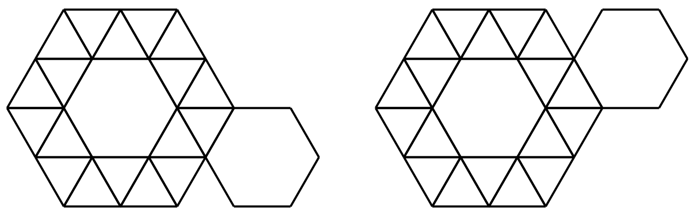

Figure 8: Two possible continuations of the hexagon-triangle combination

of the tessellation is uniquely determined: in Figure 9 there is an extension of the first continuation; the extension of the other 3.3.3.3.6 continuation

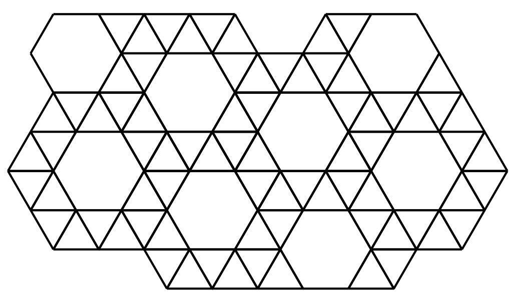

Figure 9: A 3.3.3.3.6 semi-regular tessellation

produces a reflection of this one; the two tessellations are not identical, but are isomorphic up to reflection.

The next vertex configuration on the list in Theorem 3 is $3, 3, 3, 4, 4$. This translates to possible cyclic configurations 3.3.3.4.4 and 3.3.4.3.4, and each gives a uniquely determined semi-regular tessellation, see the bottom two leftmost tessellations in Figure 15. As discussed earlier, the 3.3.3.4.4 tessellation exists and is uniquely determined by alternating square and triangle strips. We need to verify that the 3.3.4.3.4 tessellation can be constructed in only one way, and that it is given by the tessellation on the bottom left in Figure 15 (in particular, that the partially depicted tessellation can be extended to the whole plane). We start from scratch: because the vertex configuration is 3.3.4.3.4, somewhere in the tessellation there are two equilateral triangles sharing an edge, as seen in Figure 10. Since no vertex has

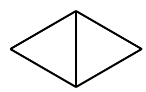

Figure 10: Building 3.3.4.3.4—start with two adjacent triangles

three triangles incident, these two triangles must be surrounded by squares, as in Figure 11. No squares are adjacent, so we have no choice also in sur-

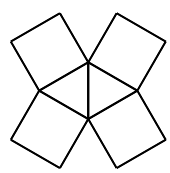

Figure 11: Building 3.3.4.3.4—squares surround the two starting triangles

rounding these last squares by triangles. Continue this process. The black lines in Figure 12 are obtained with this process. How can we be sure that by continuing this process, the entire plane will be tiled with the semi-regular tessellation 3.3.4.3.4? Observe that the regions marked in red are copies of the fundamental domain, and that a 90°-degree rotated copy of a red square precisely fills in the marked blue square. Thus if this part is forced and possible, so is the rest of the plane tessellation.

We proceed to the vertex configuration $3, 3, 4, 12$. Abstractly, the cyclic vertex configurations could be 3.3.4.12 or 3.4.3.12. We first attempt to produce a 3.3.4.12. At each vertex there would be two adjacent triangles, and up to reflection, a square and a dodecagon must be adjacent at one of the vertices, as on the left in Figure 13. Because triangles must be adjacent at each dodecagon vertex, this forces another triangle as on the right in Figure 13

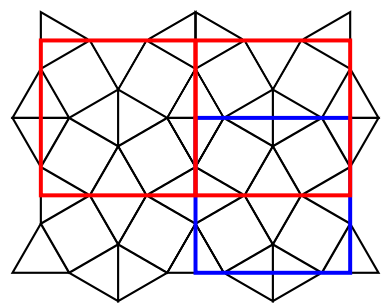

Figure 12: 3.3.4.3.4—two red squares and one blue square are marked above; the center any red-square edge could have been the start of the construction as in Figure 10. Rotate a red square by 90°, translate, and obtain the blue square.

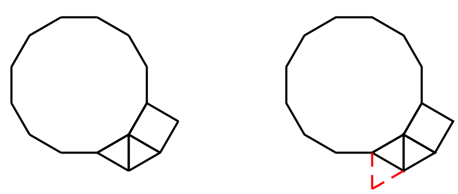

Figure 13: Attempting to build 3.3.4.12

in red color. This produces a 3.3.3... vertex, which is not allowed. Thus, 3.3.4.12 does not extend to a semi-regular tessellation. Also, 3.4.3.12 does not produce a semi-regular tessellation, as the squares share edges only with triangles, so a triangle-dodecagon-dodecagon vertex is forced as in Figure 14.

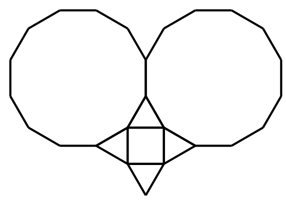

Figure 14: Attempting to build 3.4.3.12

Next on the list is the vertex configuration $3, 3, 6, 6$. One can show as above that 3.3.6.6 is not possible, as an attempt at construction forces some vertices to be 3.6.3.6. But the semi-regular tessellation 3.6.3.6 is possible, as shown on the right in Figure 6, or as in the second column and second row in Figure 15.

The next on the list is $3, 4, 4, 6$. There is no semi-regular tessellation 3.4.4.6, which can be seen using an argument similar to that used to prohibit 3.4.3.12. However, there is a unique semi-regular tessellation 3.4.6.4; see the first entry in the second column of Figure 15.

The reader can verify that there are no semi-regular tessellations arising from configurations (g) $3, 7, 42$, (h) $3, 8, 24$, (i) $3, 9, 18$, and (j) $3, 10, 15$. Configurations (k) $3, 12, 12$ and (l) $4, 4, 4, 4$ give the unique semi-regular tessellations depicted in the second column of Figure 15. The reader can straightforwardly eliminate (m) $4, 5, 20$ from the list of semi-regular tessellations, but (n) $4, 6, 12$ and (o) $4, 8, 8$ give the unique semi-regular tessellations as the first two tessellations in the last column in Figure 15. Similarly, the reader can verify that (p) $5, 5, 10$ produces no semi-regular tessellation, but that (q) $6, 6, 6$ gives the honeycomb semi-regular tessellation, depicted in the last column of Figure 15.

We have thus (mostly) proved:

Theorem 4. *There are only* 12 *semi-regular tessellations, up to translations and rotations; and, there are only* 11 *semi-regular tessellations up to translations, rotations and reflections (*3.3.3.3.6 *has a distinct mirror image):*

(i) 3.3.3.3.3.3	*(v)* 3.4.6.4	*(ix)* 4.6.12
(ii) 3.3.3.3.6	*(vi)* 3.6.3.6	*(x)* 4.8.8
(iii) 3.3.3.4.4	*(vii)* 3.12.12	*(xi)* 6.6.6
(iv) 3.3.4.3.4	*(viii)* 4.4.4.4	

All this is illustrated in Figure 15. Note that in the semi-regular tessellation 4.6.12, if counting clockwise, some vertices have a square (4-gon) followed by a hexagon followed by a dodecagon, and some have this order if counting counterclockwise. For all other semi-regular tessellations, the orientation of counting is irrelevant.

2.5 k-uniform tessellations

For any two vertices in a given semi-regular tessellation, it is possible to send one to the other by translating, rotating and reflecting that maps the remainder of the tessellation onto itself. More generally we have the following definition:

Definition 5. A tessellation is called *k-uniform* if the vertices of the tessellation can be divided into k non-empty disjoint sets $V_1, \ldots, V_k$ such that for any two vertices v and w there exists a rigid motion of the plane carrying the tessellation to itself, and v to w, if and only if v and w are in the same V_i. (A more technical way of saying this is that the symmetry group of the tessellation has exactly k transitivity classes of vertices.)

With this terminology, we observe that all semi-regular tessellations are 1-uniform. Section 2.4 showed that there are exactly eleven 1-uniform edge-to-edge tessellations of the plane formed by regular polygons.

Grünbaum and Shephard [5] state that there are exactly twenty 2-uniform edge-to-edge tessellations of the plane formed by regular polygons. This was proved by Krötenheerdt [6] in 1969. Some readers may wish to carry out a proof of this case. Chavey [1] proved in the 1980s that there are exactly sixty-one 3-uniform tessellations. More recently, Galebach [2] wrote a program to compute the number of k-uniform edge-to-edge regular-polygon tessellations; the program computes the number of 4-uniform tessellations to be 151, the number for 5-uniform to be 332, and the number for 6-uniform to be 673 (after a month of computation). But there yet appears to be no *proof* of these results, and no further numbers are known as of this writing. The

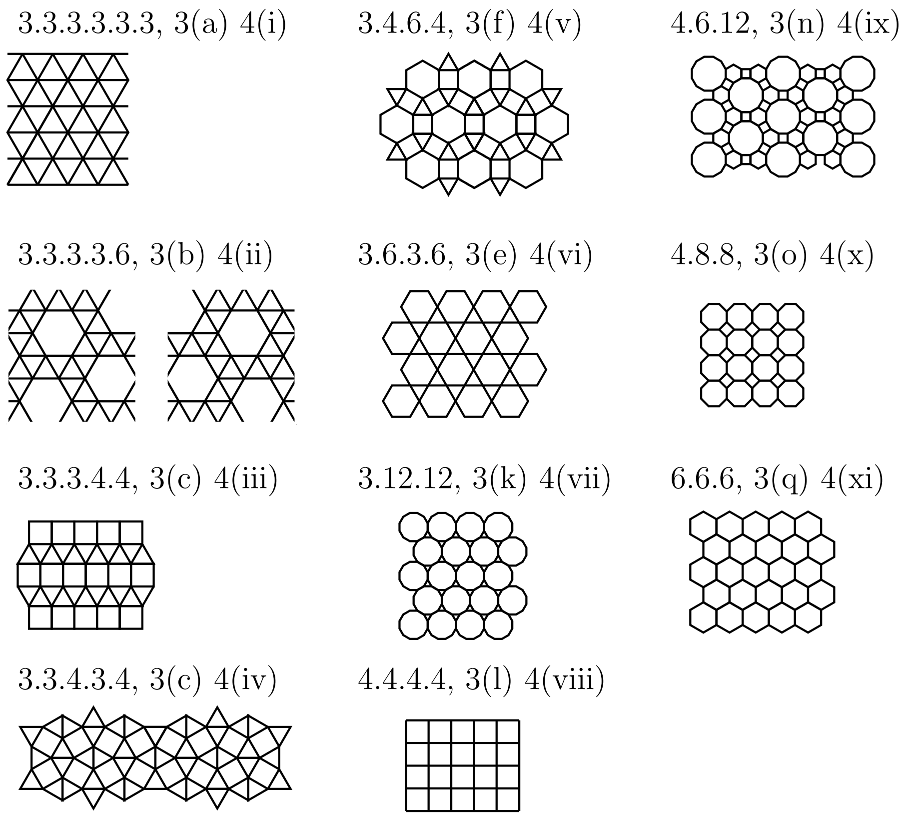

Figure 15: All the possible semi-regular tessellations: each tessellation is identified with its cyclic vertex configuration, its letter on the list in Theorem 3, and its roman numeral on the list in Theorem 4.

good news for a quilter is that for each k, the number of such k-uniform tessellations is finite—this is because there are only finitely many possible symmetry groups—so that only finitely many quilts have to be made for each k. The not so good news is that the number of quilts to be made is still very large for one's lifetime.

3 Teaching ideas

Semi-regular tessellations can be studied in a classroom at several levels. Building tessellations both combinatorially and geometrically is highly illuminating. The explorations in Sections 3.1 and 3.2 can be done exhaustively by elementary-school students or in a deeper way by more advanced students; some of the aspects of these investigations will even be instructive to graduate students. The questions in Section 3.3 ask the reader to determine the finished size of the n-gons on a quilt top given the cut size of fabric n-gons, and, conversely, the size n-gons one needs to cut in order to wind up with a specific size n-gon on the quilt top. The results are used in in the planning of the quilts in Section **??**. The Pythagorean theorem and trigonometry are needed to successfully complete these investigations. Section 3.4 uses continued fractions to find rational approximations to irrational fabric proportions. It requires fairly good algebra skills but no concepts encountered after high school. The research questions in Section 3.5 require excellent geometric and organizational skills.

3.1 Exploring the Tessellations Geometrically

The goal of the quilting constructions given in Section **??** is to give ways to create semi-regular tessellations efficiently and to minimize the propagation of error. After all, when tessellating the plane, accuracy of the produced regular polygons (most with irrational heights) is paramount, as is aligning edges and vertices precisely. The same issues arise when attempting to construct the tessellations on paper or on a computer. Here are some exercises designed to explore building tessellations. For students using paper, magnets or feltboard, stacks of regular n-gons are needed, at least one for each of the various values of n. Suggested computer software includes Geogebra, Geometer's Sketchpad, or even Adobe Illustrator; each has its own challenges.

For each semi-regular tessellation a student tries to create, the following questions can be asked:

1. In the semi-regular tessellations, the only regular n-gons used are 3, 4, 6, 8, and 12-gons. Construct each of these shapes with a straightedge and compass.

2. One way of drawing a semi-regular tessellation consisting only of squares is to draw the squares one at a time and then align the edges, but there is obviously a faster (and more precise!) way of drawing this semi-regular tessellation—what is it?

3. Which semi-regular tessellations can be drawn more efficiently than one shape at a time (apart from the 4.4.4.4 tessellation mentioned above)? Discuss possibilities for shortcuts. Are there some parallel lines in the design? Can some lines or edges be extended?

4. If you are using a computer program for drawing, what is the best way to replicate a diagram? Should you make one line segment at a time? Should you make all parallel line segments first? Is there some other set of line segments to start with, such as all the lines coming out of a vertex? How should these lines be made so that they are accurate (i.e., of the right length and in the right locations relative to each other)? Should you make regular polygons and then copy and paste those together? (How would one do that?)

5. Suppose you were going to make a fancy floor (or a quilt) tiled with your tessellation. Try coloring your tessellations with colors (or fabrics) so that no two shapes sharing a whole edge have the same color. Which of the semi-regular tessellations can be colored in this way using exactly two colors? Which ones cannot be colored with two colors, but three colors suffice? Do any of them need four or even more colors? (There is a big theorem in mathematics saying that all tessellations by connected pieces, semi-regular or not, can be colored using no more than four colors. See [7].)

3.2 Exploring the Non-Tessellations

The following two problems encourage students to fill in the details of eliminating vertex configurations that do not extend to tessellations of the plane.

- In Theorem 3, seventeen potential semi-regular vertex configurations are enumerated. According to Section 2, seven cyclic vertex configurations cannot be realized as semi-regular tessellations. However, in Theorem 4, we found eleven different semi-regular tessellations. Yet $17 - 7 < 11$. Explain what is going on here.

- The same drawing methods that were used for building the semi-regular tessellations can also be used to help work through the proofs that some cyclic vertex congurations from Theorem 3 do not yield semi-regular tessellations. Show that the following vertex configurations do not yield semi-regular tessellations: (g) 3, 7, 42, (h) 3, 8, 24, (i) 3, 9, 18, (j) 3, 10, 15, (m) 4, 5, 20, and (p) 5, 5, 10. Note that two different cyclic vertex configurations may arise from (f) 3, 4, 4, 6, but (only) 3, 4, 4, 6 is impossible. Show that.

3.3 The Size of the Shapes

Quilters, and tessellaters, have to make a lot of calculations while doing their craft: what final size is needed, how many basic units, how much of each fabric and color, etc. Much of that arithmetic is quite elementary, but it can quickly get into harder mathematics. Prerequisites for this section are the Pythagorean theorem and trigonometry and a willingness to get one's hands dirty. Depending on the audience, some teachers may want to break down the questions for their students.

Question 1: If regular n-gons are to be cut from a strip of fabric (or paper) of width h inches, what is the largest possible side length of the n-gon? For example, if $n = 3$, then h denotes the height of the triangle, and the answer is $\frac{2}{\sqrt{3}}h$. If $n = 4$, then certainly the answer is simply h. Answer the question for $n = 6, 8, 12$.

One can also reverse the question:

Question 2: If we want the side length of a regular n-gon to be d inches, what is the smallest width of a strip of fabric (of paper) from which we can cut the n-gon?

Quilters have to be more careful than that, however! It is not enough to cut a piece of fabric to the finished size; extra fabric is needed because seam must be sewn on the interior of the fabric in order to hold. For quilting, it is traditional to add a quarter-inch seam allowance to all edges. For example, if the finished square is supposed to be 3 inches by 3 inches, one needs to cut the square of 3.5 inches by 3.5 inches of fabric, a quarter of an inch to be eaten away by the seaming along each of the four sides.

The previous two questions can be rephrased for quilting purposes as follows:

Question 1q: If regular n-gons are to be cut from a strip of fabric of width h inches, and if $\frac{1}{4}$ inch is reserved along each edge of the n-gon for seam allowance, what is the largest possible finished side length of the n-gon? Do this for $n = 3, 4, 6, 8, 12$. (The answers might not be unique; see Figure **??**.)

Question 2q: If we want the finished side length of a regular n-gon to be d inches, what is the smallest width of a strip of fabric from which we can cut a polygon so that after subtracting $\frac{1}{4}$ inch along each side for seam allowance from this polygon we get the desired n-gon? Do this for $n = 3, 4, 6, 8, 12$.

For specific numerical values of h and d in the questions above, the "answers" may be provided by geometric construction instead of numerically. Namely, one may want to draw the finished and cut size on paper, then transfer the drawn lengths as needed.

3.4 Approximating Fabric Proportions via Continued Fractions

Quilting provides an excellent real-life application of continued fractions. The presentation below can be used within a number theory course, or in a recreational mathematics setting as early as middle school. In making quilts, one often has a predetermined finished size, depending on the size of the bed or on the size of the recipient, say. One also often starts with a predetermined idea what the building blocks of the quilt would look like, and then the mathematical input is to calculate how many of those building blocks are needed to fill the desired finished size.

Specifically, suppose we want to tessellate a quilt with equilateral triangles, each of side length d inches. How should we choose the number of triangles per row, and number of rows in the quilt, so that the finished size is as desired (or close enough)? For example, how should p, the number of triangle side lengths in a row, and q, the number of rows in the quilt, be chosen, so that the finished quilt is square?

The height of the quilt will be $pd\sqrt{3}/2$ inches and the width will be qd inches. Because $\sqrt{3}$ is irrational, it is *not* possible to find positive integers p and q such that the finished quilt would be square. However, with fabric stretchability, one can get close: If (p,q) is $(7,6)$ (see the example on the left in Figure **??**), then $7 \cdot \frac{\sqrt{3}}{2} \cong 6.06218$, which is very close to 6, and a quilt with $p = 7$ and $q = 6$ *looks* square. Similarly, $15 \cdot \frac{\sqrt{3}}{2} \cong 12.9904$ so $(15, 13)$ is a good approximation, and $97 \cdot \frac{\sqrt{3}}{2} \cong 84.0045$, so $(97, 84)$ is even closer to square. But how were these pairs (p,q) arrived at? Trial and error are trumped by the systematic method introduced next.

Definition 6. A *continued fraction* is an expression such as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}},$$

where a_0 is an integer, and all other a_i are non-negative integers. If $a_n = 0$ for some $n > 0$, then all subsequent a_i are also 0.

Let's compute an example, say for $x = \frac{2}{\sqrt{3}} \cong 1.15470053837925$. (The motivation for this x is that for a square quilt we want the ratio $\frac{p\sqrt{3}}{2} : q$ to be (close to) 1, i.e., we want $p : q$ to be (close to) $\frac{2}{\sqrt{3}}$.) The largest integer smaller than or equal to x is 1. Thus we set $a_0 = 1$. So we will want to write $x = 1 + \frac{1}{something}$, and solving gives $something = 1/(x-1) \cong 6.46410161513775$. We then have to write $something$ as $a_1 + \frac{1}{something\ else}$. The largest integer less than or equal to $something$ is 6, so we set $a_1 = 6$, and by solving we find $something\ else = \dfrac{1}{\frac{1}{x-1} - 6} \cong 2.15470053837925$, whence we set $a_2 = 2$. Note that the part to the right of the decimal point has appeared before!

Exercise: The two quantities above *look* similar. Prove that indeed they are the same. Namely, prove that

$$\frac{2}{\sqrt{3}} - 1 = \frac{1}{\frac{1}{\frac{2}{\sqrt{3}} - 1} - 6} - 2.$$

This justifies that all further a_{2n+1} are 6, and all a_{2n} are 2. One way to notate this continued fraction is as $2/\sqrt{3} = [1; 6, 2, 6, 2, 6, 2...]$. Just as we take decimal truncations of decimal expansions of real numbers, similarly we can take truncations of continued fractions:

1. The truncation $[1; 6] = 1 + \frac{1}{6}$ of $2/\sqrt{3} = [1; 6, 2, 6, 2, 6, 2...]$ gives $7/6$ (hence the pair $(7, 6)$).

2. The truncation $[1;6,2] = 1 + \frac{1}{6+\frac{1}{2}}$ of $[1;6,2,6,2,6,2...]$ gives 15/13.

3. $[1;6,2,6] = 1 + \frac{1}{6+\frac{1}{2+\frac{1}{6}}} = \frac{97}{84}$.

4. $[1;6,2,6,2] = \frac{209}{181}$, etc.

In general, the continued fraction of any real number x is obtained as follows: set a_0 to be the largest integer less than or equal to x; a_0 is the *floor* of x, denoted $\lfloor x \rfloor$. Notice that $x - a_0 \geq 0$. Let $a_1 = \lfloor 1/(x-a_0) \rfloor$. To continue, let $b_1 = 1/(x - a_0)$ and $b_n = 1/(b_{n-1} - a_{n-1})$. Then $a_n = \lfloor 1/(b_{n-1} - a_{n-1}) \rfloor$. If at any point a denominator is zero, stop—x is a rational number.

If $x = [a_0; a_1, a_2, a_3, \ldots]$ is a continued fraction expansion of a real number x, let a truncated continued fraction $[a_0; a_1, a_2, a_3, \ldots, a_n]$ be written as $\frac{p}{q}$ with integers p and q having no common factor. In a number theory class one may want to prove the following fact, but some may want to take it on faith: for any rational $\frac{r}{s}$ and $0 < s \leq q$,

$$|x - [a_0; a_1, a_2, a_3, \ldots, a_n]| \leq \left| x - \frac{r}{s} \right|.$$

In other words, the truncated continued fractions are the best rational approximations if the denominators of the rational approximations do not exceed the denominator of the truncated continued fraction approximation.

Further questions:

1. What if we wanted the finished quilt to be rectangular, with ratio 7 : 5 of height to width? What would be a good number q of rows with each row p triangle side lengths in width? Verify your answers numerically. (Hint: Do a continued fraction expansion of $\frac{7\cdot 2}{5\sqrt{3}}$.)

2. What if we modify the layout of the quilt and have q rows of equilateral triangles, each $p/2$ triangle side lengths wide?

3. How can continued fractions be used for tessellations with hexagons, or the other semi-regular tessellations? (You will need the calculation results from Subsection 3.3.)

4. Compute a few terms of the continued fraction of π, and compute its first few truncated continued fractions approximations. (Of course, you cannot get all the terms of the continued fractions, and since you have only a limited access to the real value of π, you cannot compute very many parts of the continued fraction.)

3.5 k-uniform tessellation projects

Just as we derived from scratch all the possible 1-uniform tessellations, one could determine from scratch all the possible 2-uniform tessellations, and higher as well. A big part of this discovery is keeping track of the discoveries, especially when the number of possible tessellations gets beyond twenty. Other possible searches might be for those k-regular tessellations that use only squares and triangles, or some other small combination of shapes.

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