Linear Bounds on Growth of Associated Primes

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Let I be an ideal in a commutative Noetherian ring R. We study the growth of associated primary components of powers of I. Swanson [S] and Heinzer and Swanson [HS] have shown that there exists an integer l such that each power of I has a primary decomposition

$$I^n = q_1 \cap \cdots \cap q_s$$

where $(\sqrt{q_i})^{nl} \subset q_i$ for each of the primary components q_i. In other words, the primary components of I^n "grow linearly" in n. An interesting question of practical concern is *what is* l?

Swanson was partially motivated by the analogous question for "Frobenius powers", which has important applications to the theory of tight closure (the so-called "localization problem;" see [HH, Section 4]). Suppose now that R has prime characteristic $p > 0$ and q is a power of p. The Frobenius power $I^{[q]}$ of an ideal I is the ideal generated by all the qth powers of the elements (equivalently, the generators) of I.

We ask the same question: can we find an integer l such that for each q there is a primary decomposition of $I^{[q]}$

$$I^{[q]} = q_1 \cap \cdots \cap q_s$$

with each $(\sqrt{q_i})^{[q]l} \subset q_i$? This Frobenius analog of [S] may be quite difficult. One difficulty is that, unlike with the case of ordinary powers of I (proved in [R]), the Frobenius powers of I can have infinitely many distinct associated primes as q varies over all powers of p. An ideal with this property was first found by Katzman [K]. We analyze the primary decompositions of Frobenius powers of Katzman's ideal in Section 2 and prove that the question above has an affirmative answer for this ideal.

In Section 1 we solve the problem of linear growth for primary components of both ordinary and Frobenius powers of monomial ideals in polynomial rings modulo a monomial ideal by giving an explicit value l that works. More precisely, we analyze the ideals generated by monomials in a ring of the form $R = S/J$, where S is a polynomial ring over a field and J is an ideal generated by monomials in the given variables. Because primary decompositions of monomial ideals are easy to understand, we are able to give a constructive bound that works in both the ordinary and the Frobenius power case. We point out how this can be used to verify that tight closure commutes with localization in this special case.

In Section 3 we discuss the Castelnuovo-Mumford regularity of powers of monomial ideals. For a given monomial ideal I we explicitly find an integer B such that the regularity of the nth power of I is bounded above by Bn. We believe that in general B can be improved. However, in many cases the given B is a sharp upper bound.

1. Primary decompositions and localization of tight closure on monomial ideals

Throughout in this section S is a polynomial ring $k[x_1, \ldots, x_d]$, where k is a field and $x_1, \ldots, x_d$ are variables over k. Let I and J be ideals generated by monomials in these variables.

First some notation. An ideal is said to be *irreducible* if it cannot be written as an intersection of two strictly larger ideals. It is well-known and easy to show that every irreducible monomial ideal is of the form $(x_{i_1}^{a_1}, x_{i_2}^{a_2}, \ldots, x_{i_t}^{a_t})$ for some integers $t \leq d$, $i_1 < i_2 < \cdots < i_t$, and $a_i \geq 1$. The irreducible monomial ideals can be indexed by monomials: $(x_{i_1}^{a_1}, x_{i_2}^{a_2}, \ldots, x_{i_t}^{a_t})$ corresponds uniquely to the monomial $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_t}^{a_t}$, where each a_i is a positive integer. This is a one-to-one correspondence, so we will write an irreducible monomial ideal as J_m for a monomial m.

Consider an arbitrary monomial ideal I in the polynomial ring $k[x_1, x_2, \ldots, x_d]$. The ideal I can be decomposed into irreducible ideals

$$I = J_{m_1} \cap J_{m_2} \cap \cdots \cap J_{m_r}$$

for some monomials $m_1, \ldots, m_r$ (see for example Eagon-Hochster [EH], Sturmfels-Trung-Vogel [STV], or Heinzer-Ratliff-Shah [HRS]). Assume this decomposition is *minimal,* so that no J_{m_i} can be omitted, and no m_i can be replaced with a proper monomial factor.

(1.1) Lemma: *With irreducible decomposition of* I *as above, let* l *be the largest of the powers of a variable appearing as minimal generators of the* J_{m_i} *in this minimal decomposition. (Equivalently,* l *is the largest exponent of any variable appearing in the indexing monomials* $m_1, m_2, \ldots, m_r$*.) Let* k *be the largest exponent of a variable in a minimal generating set of* I*. Then* l *equals* k*.*

Proof: Note that I is generated by least common multiples of elements $\alpha_1, \ldots, \alpha_r$ as the α_i run through the monomial generators of J_{m_i}. Thus in the monomial generating set of I obtained in this way, all the variables have exponent at most l, so each minimal generator of I will have all exponents bounded above by l. This proves that $k \leq l$.

Suppose that $k < l$. By reindexing, if necessary, l may be assumed to be the exponent of x_1 in m_1. Let $m_1' = m_1/x_1^l$. Then certainly

$$I = J_{m_1} \cap J_{m_2} \cap \cdots \cap J_{m_r} \quad \supseteq \quad I' = J_{m_1'} \cap J_{m_2} \cap \cdots \cap J_{m_r}.$$

By assumption, for each minimal generator $m = x_1^{a_1} \cdots x_d^{a_d}$ of I, we have $a_1 < l$. Thus m is a multiple of the monomials in J_{m_1} other than x_1^l which means that m lies in $J_{m_1'}$. This says that each of the minimal generators for I is actually in I', whence $I = I'$, contradicting the minimality of our irreducible decomposition. ■

This lemma makes the proof of the linear growth of primary decompositions of both ordinary and Frobenius powers very easy: for let I and J be monomial ideals in S. Let l be the largest exponent appearing in a set of minimal monomial generators for $I + J$. Then the corresponding largest exponent for $I^q + J$ and $I^{[q]} + J$ is no more than ql. If we decompose $I^q + J$ (or $I^{[q]} + J$) efficiently into irreducibles, then each irreducible component J_m involves powers of variables of degree at most ql. Clearly $(\sqrt{J_m})^{qld} \subseteq J_m$. Thus:

(1.2) Theorem: *Let* I *and* J *be monomial ideals in* $S = k[x_1, \ldots, x_d]$*. Let* l *be the largest power of any variable occurring in a generating set of* I *or of* J*. Then for each* n *there exists a primary decomposition* $I^n + J = q_1 \cap \cdots \cap q_s$ *such that* $\sqrt{q_i}^{nld} \subseteq q_i$ *for all* i*. If the characteristic* p *of* k *is positive, then for any power* q *of* p *there exists a primary decomposition* $I^{[q]} + J = q_1 \cap \cdots \cap q_s$ *such that* $\sqrt{q_i}^{nld} \subseteq q_i$ *for all* i*. Moreover, for* n *and* q *sufficiently large we may take* l *to be the largest power of any variable occurring in the minimal generators for* I*.*

Proof: Note that a primary decomposition is obtained from an irreducible one by simply intersecting the irreducible components with the same radical. The result is now immediate from the lemma and the discussion above. ■

This proves the linear growth of primary decompositions of ordinary and Frobenius powers of a monomial ideal I in S/J, where J is a monomial ideal in the polynomial ring $S = k[x_1, \ldots, x_d]$ over a field k.

This theorem also has an immediate application to the localization problem in tight closure. Tight closure is a closure operation performed on ideals in a commutative Noetherian ring of prime characteristic. For an ideal $I \in R$, an element $z \in R$ is in the tight closure I^* of I if there exists an element c not in any minimal prime of R such that $cz^q \in I^{[q]}$ for all $q = p^n >> 0$. See [HH].

One of the most persistent open problems in the theory of tight closure has been the "localization problem:" given an ideal $I \subset R$ and a multiplicative system $U \subset R$, is $I^*U^{-1}R = (IU^{-1}R)^*$? The difficulty in understanding the growth of associated primes of monomial ideals is one of the key obstructions to settling this problem.In the next corollary, we indicate how our result on linear growth of associated primes can be applied to see that tight closure commutes with localization for monomial rings.

(1.3) Corollary: *Let I be an ideal generated by monomials in $R = S/J$ where J is a monomial ideal in $S = k[x_1, \ldots, x_d]$. Then for any $u \in S$,*

$$I^*R_u = (IR_u)^*;$$

that is, tight closure commutes with localization at multiplicatively closed sets of the form $\{1, u, u^2, \ldots\}$ for monomial ideals in rings of the type S/J.

Proof: The inclusion $I^*(R_u) \subset (IR_u)^*$ is obvious. Suppose that $\frac{z}{1} \in (IR_u)^*$. Then for each $q = p^n$, there exists an integer $N(n)$ such that $cz^q u^{N(n)} \in I^{[q]}$. *A priori,* we know only that c is not in any minimal prime of R not containing u, but by replacing c by $c + \delta$ where δ belongs precisely to those minimal primes not containing c and no others, we may assume that c is in no minimal prime of R (see the proof of Proposition 4.14 in [HH]). If cz^q is in each primary component of $I^{[q]}$, then $cz^q \in I^{[q]}$, and we are done. Otherwise, the element u is in every associated prime of $I^{[q]}$ such that cz^q is not in the corresponding primary component. By Theorem 1.2, u^{qN} is in each of these primary components, where N is a fixed integer independent of q. Therefore, $c(zu^N)^q \in I^{[q]}$, whence $zu^N \in I^*$ and $z \in I^*R_u$. ∎

Despite the interesting method of proof, which we hope may eventually be more broadly applicable, the corollary does not really prove anything new about localization. It is easy to see that tight closure commutes with localization at an arbitrary multiplicative system for any ideal I in a monomial ring R: in these rings, tight closure has a simple description as $I^* = \cap_{P \in minspecR}(I + P)$, and localization is immediate. This description of tight closure follows from the fact that tight closure can be computed modulo minimal primes and the fact that all ideals are tightly closed in a regular ring. More explicitly, $z \in I^*$ if and only if the image of z is in $(IR/P)^*$ for each minimal prime of R, and when R is a monomial ring, each R/P is a polynomial ring, so $(IR/P)^* = IR/P$. See also [T], [K2].

2. Primary decompositions of Katzman's ideal

Let k be a field of characteristic $p > 0$, t, x and y indeterminates over k, $A = k[t]$ and $R = A[x, y]$. Katzman's example is as follows. Set $I_q = (x^q, y^q, xy(x-y)(x-ty))R$. As q varies over powers of p (or all integers), the set of associated primes of the I_q is infinite. In particular, this means that the set of associated primes of all the Frobenius powers of $I = (x, y)$ in the ring $k[t, x, y]/(xy(x-y)(x-ty)$ is infinite.

We now show that although the set of associated primes is infinite, there nonetheless exists an integer l such that for each q, there is a primary decomposition of $I^{[q]}$:

$$I^{[q]} = q_1 \cap \cdots \cap q_s$$

with $(\sqrt{q}_i)^{lq} \subset q_i$ for all q_i. In fact, we show that $l = 2$.

Consider the elements $\tau_q = 1 + t + t^2 + \cdots + t^{q-2}$ and $G_q = x^2y^{q-1}$ of R. Katzman showed that $\tau_q G_q \in I_q$. Define $J_q = I_q + x^2y^{q-1}R$. We show below that J_q is a primary component of I_q.

(2.1) Lemma: *Let f be any nonzero element in A. Then $J_q : f = J_q$ for all q.*

Proof: We have to prove that $J_q : f \subseteq J_q$. Let $\alpha \in J_q : f$. Without loss of generality we may assume that α is homogeneous of degree d under the (x, y)-grading. Note that $(x, y)^{q+1} \subseteq J_q$. Thus without loss of generality we may assume that $d \le q$ and $f\alpha$ is a multiple of $(xy(x-y)(x-ty))$. But $f, xy(x-y)(x-ty)$ is a regular sequence, so $\alpha \in (xy(x-y)(x-ty))$. ∎

In particular, $J_q : \tau_q = J_q$, hence $I_q : \tau_q \subseteq J_q : \tau_q = J_q \subseteq I_q : \tau_q$. Moreover, $I_q : \tau_q^2 = (I_q : \tau_q) : \tau_q = J_q : \tau_q = J_q = I_q : \tau_q$, which implies that $I_q = (I_q : \tau_q) \cap (I_q + (\tau_q))$.

So in order to analyze the primary components of I_q it suffices to analyze the primary components of $I_q : \tau_q$ and $I_q + (\tau_q)$ separately.

First we analyze the primary components of $J_q = I_q : \tau_q$. As J_q is homogeneous under the (x, y)-grading, all the prime ideals associated to it are also homogeneous. We claim that J_q is (x, y)-primary. If it is not, then there exists an element in A which is a zero-divisor on R/J_q. But this is impossible by Lemma 2.1. Now observe that $(x,y)^{q+1}$ and hence $(x,y)^{2q}$ is contained in J_q. This means that the linear growth property holds for the primary component J_q of I_q; namely $(\sqrt{J_q})^{2q} \subset J_q$.

Now consider the primary components of $I_q + (\tau_q)$. Because the radical of this ideal includes x, y and τ_q, it is clear that this ideal is height three in $k[t,x,y]$, and thus cannot have any embedded primary components. Let $\tau_q = \prod_{i=1}^r \sigma_i$ be a factorization of $\tau_q \in k[t]$ into distinct irreducible polynomials. Each $\sqrt{I_q + (\sigma_i)} = (x, y, \sigma_i)$ is a maximal ideal containing $I_q + (\tau_q)$ and these are the only maximal ideals containing $I_q + (\tau_q)$. An (x, y, σ_i)-primary component of $I_q + (\tau_q)$ has to contain σ_i, so we get a primary decomposition

$$I_q + (\tau_q) = (I_q + (\sigma_1)) \cap (I_q + (\sigma_2)) \cap \cdots \cap (I_q + (\sigma_r)).$$

Now it is clear that $(x, y, \sigma_i)^{2q} \subset (x^q, y^q, \sigma_i) \subset I_q + (\sigma_i)$. Therefore the same linear bound $2q$ that worked for the primary component J_q also works for $I_q + (\tau_q)$.

In summary, Katzman's examples I_q decompose as

$$I_q = J_q \cap q_1 \cap \cdots \cap q_s$$

where $J_q = I_q : \tau_q = I_q + (x^2 y^{q-2})$ is primary with $(\sqrt{J_q})^{2q} \subset J_q$, and each $q_i = I_q + (\sigma_i)$ is primary with $(\sqrt{q_i})^{2q} \subset q_i$.

3. Castelnuovo-Mumford regularity of monomial ideals

This section is a preliminary attempt at understanding how for a given monomial ideal I, the Castelnuovo-Mumford regularity varies with powers of I. If I is Borel-fixed, then the Eliahou-Kervaire resolution gives that $\operatorname{reg}(I^n) \leq n \operatorname{reg}(I)$ (see [EK]). We do not know whether this is true for arbitrary monomial ideals.

Here we do the following: given a monomial ideal I in $k[x_1, \ldots, x_d]$, we want to find the integer B such that for all n, the Castelnuovo-Mumford regularity of I^n is bounded above by Bn. Such an integer B exists by Theorem 3.6 in [S] but the arguments in [S] do not show how to calculate B. We show in this section that for monomial ideals one can calculate such a B.

In the course of the proof we found it necessary to determine an upper bound on the Castelnuovo-Mumford regularity for a more general class of ideals. Namely we prove:

(3.1) Theorem: *Let R be $k[x_1, \ldots, x_d]$, a polynomial ring in d variables over a field k. Let $I_1, \ldots, I_m$ be monomial ideals in R. Let l be the largest exponent of a variable occurring in any of the generating sets for the I_j. Also assume $x_1, \ldots, x_r$ all lie in the radical of*

$\sum_j I_j$. *For a subset* S *of the variables and* $x_q \in S$, *define* $I_{S,x_q} = \sum_{x_i \in S \setminus \{x_q\}} x_i^l R$. *Let*

$$L = \max_{S \subseteq \{x_{r+1},\ldots,x_d\}} \max_{x_q \in S} \{(d-|S|-r)l + (m+|S|-2+d)\,reg\left(((\sum_{j=1}^m I_j) : x_q^l) + I_{S,x_q}\right)\}.$$

Then $reg\left(\sum_{j=1}^m I_j^n\right) \leq n \max\{dl, L\}$.

Note that dl and L are both computable and that the ideals $((\sum_{j=1}^m I_j) : x_q^l) + I_{S,x_q}$ involve at most $d-1$ variables.

Thus for a single monomial ideal I such that l is the largest exponent of a variable occurring in a monomial generating set and such that $x_1, \ldots, x_r$ all lie in the radical of I we get that

$$reg(I^n) \leq n \max\{dl, \max_{S \subseteq \{x_{r+1},\ldots,x_d\}} \max_{x_q \in S} \{(d-|S|-r)l + (|S|-1+d)\operatorname{reg}\left((I : x_q^l) + I_{S,x_q}\right)\}.$$

Before we prove Theorem 3.1 we need a few lemmas:

(3.2) Lemma: *Let* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be a short exact sequence of graded finitely generated* R*-modules such that all the maps are homogeneous of degree 0. Then*

(i) $reg\,A \leq \max\{reg\,B, reg\,C + 1\}$,

(ii) $reg\,B \leq \max\{reg\,A, reg\,C\}$.

Proof: See for example Corollary 20.19 in Eisenbud [E]. ∎

(3.3) Lemma: *Let* I *be a homogeneous ideal in* R, x_q *a variable and* l *a positive integer. Then*

(i) $reg\,(I \cap x_q^l R) = l + reg\,(I : x_q^l)$.

(ii) $reg\,(I : x_q^l) \leq \max\{0, reg\,(I) - l, reg\,(I + x_q^l R) + 1 - l\}$.

(iii) $reg\,(I) \leq \max\{reg\,(I : x_q^l) + l, reg\,(I + x_q^l R)\}$.

Proof: (i) follows from the elementary facts $I \cap x_q^l R = x_q^l (I : x_q^l)$ and $\operatorname{reg}(x_q^l J) = l + \operatorname{reg} J$ for any ideal J.

For (ii) and (iii), we use Lemma 3.2, (i) and the short exact sequence

$$0 \longrightarrow I \cap x_q^l R \longrightarrow I \oplus x_q^l R \longrightarrow I + x_q^l R \longrightarrow 0,$$

to get that

$$\begin{aligned}\operatorname{reg}(I : x_q^l) &= \operatorname{reg}(I \cap x_q^l R) - l \\ &\le \max\{\operatorname{reg}(I \oplus x_q^l R) - l, \operatorname{reg}(I + x_q^l R) + 1 - l\} \\ &= \max\{\operatorname{reg}(I) - l, \operatorname{reg}(x_q^l) - l, \operatorname{reg}(I + x_q^l R) + 1 - l\},\end{aligned}$$

which proves (ii). Similarly, (iii) follows. ■

Now we prove the main theorem of this section, namely Theorem 3.1:

Proof: We proceed by double induction on d and $d-r$. If $d-r=0$, then $r=d$ so $\sum_{j=1}^m I_j$ is primary to $(x_1,\ldots,x_d)$. In that case, by assumption on l, $(x_1^l,\ldots,x_d^l) \subseteq \sum_{j=1}^m I_j$. Hence $(x_1,\ldots,x_d)^{dln} \subseteq (x_1^{nl},\ldots,x_d^{ln})$. As x_j^l lies in some I_p, it follows that x_j^{ln} lies in I_p^n, hence $(x_1,\ldots,x_d)^{dln} \subseteq \sum_{j=1}^m I_j^n$. By Bayer-Stillman [BS, Lemma 1.7] it follows now that $\operatorname{reg}\left(\sum_{j=1}^m I_j^n\right) \le dln$, which proves the theorem in the case $d=r$. Note that this also proves the case $d=1$.

Now assume that $d>1$, $r<d$. By Lemma 3.3,

$$\operatorname{reg}\big(\sum_{j=1}^m I_j^n\big) \le \max\left\{\operatorname{reg}\left(\big(\sum_{j=1}^m I_j^n\big) : x_{r+1}^{ln}\right) + ln, \operatorname{reg}\left(\sum_{j=1}^m I_j^n + x_{r+1}^{ln}R\right)\right\}.$$

Now we use some facts about monomial ideals (which definitely fail for arbitrary ideals). First of all, $\left(\sum_{j=1}^m I_j^n\right) : x_{r+1}^{ln}R = \sum_{j=1}^m (I_j^n : x_{r+1}^{ln})$. Moreover, by the choice of l, $I_j^n : x_{r+1}^{ln} = (I_j : x_{r+1}^l)^n$ can be identified with the ideal I_j^n after we set x_{r+1} to 1.

Thus to find the regularity of $\sum_j (I_j : x_{r+1}^l)^n$, we may work in the polynomial ring $k[x_1,\ldots,x_r,x_{r+2},\ldots,x_d]$ with one fewer variable. Let $I=\sum_{j=1}^m I_j$ and set

$$L_1 = \max_{S\subseteq\{x_{r+2},\ldots,x_d\}} \max_{x_q\in S}\{(d-1-|S|-r)l + (m+|S|-3+d)\operatorname{reg}\left(I : x_{r+1}^l x_q^l) + I_{S,x_q}\right)\},$$

$$L_2 = \max_{S\subseteq\{x_{r+2},\ldots,x_d\}} \max_{x_q\in S}\{(d-|S|-r-1)l + (m+|S|-1+d)\operatorname{reg}\left((I : x_q^l) + x_{r+1}^l R + I_{S,x_q}\right)\}.$$

Thus by induction on d,

$$\operatorname{reg}\left(\big(\sum_{j=1}^m I_j^n\big) : x_{r+1}^{ln}\right) = \operatorname{reg}\sum_{j=1}^m (I_j : x_{r+1}^l)^n \le n\max\{(d-1)l, L_1\}.$$

and by induction on $d-r$, $\operatorname{reg}(\sum_{j=1}^m I_j^n + x_{r+1}^{ln}R) \le n\max\{dl, L_2\}$. Hence

$$\operatorname{reg}\big(\sum_{j=1}^m I_j^n\big) \le n\max\{dl, L_1+l, L_2\}.$$

Note that L_2 equals

$$\max_{x_{r+1}\in S\subseteq\{x_{r+1},x_{r+2},\ldots,x_d\}} \max_{x_q\in S\setminus\{x_{r+1}\}}\{(d-|S|-r)l + (m+|S|-2+d)\operatorname{reg}\left((I : x_q^l) + I_{S,x_q}\right)\}.$$

and this is bounded above by L.

Now we want to show that $L_1 + l \leq \max\{dl, L\}$. As x_{r+1} is not an element of S, $(I : x_{r+1}^l x_q^l) + I_{S,x_q} = ((I : x_q^l) + I_{S,x_q}) : x_{r+1}^l$, so that by Lemma 3.3, the regularity of $I : x_{r+1}^l x_q^l + I_{S,x_q}$ is bounded above by

$$\max\{0, \operatorname{reg}((I : x_q^l) + I_{S,x_q}) - l, \operatorname{reg}(((I : x_q^l) + I_{S,x_q}) + x_{r+1}^l) + 1 - l\}.$$

Thus $L_1 + l$ is bounded above by the maximum of $(d - |S| - r)l$,

$$(d - |S| - r)l + (m + |S| - 3 + d)(\operatorname{reg}((I : x_q^l) + I_{S,x_q}) - l)$$

and

$$(d - |S| - r)l + (m + |S| - 3 + d)(\operatorname{reg}(((I : x_q^l) + I_{S,x_q}) + x_{r+1}^l) + 1 - l),$$

as S varies over subsets of $\{x_{r+2}, \ldots, x_d\}$ and x_q over elements of S. Each of these expressions is clearly bounded above by $\max\{dl, L\}$, which finishes the proof of the theorem.
■

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Bibliography

[BS] D. Bayer and M. Stillman, A criterion for detecting m-regularity, *Invent. Math.*, **87** (1987), 1-11.

[EH] J. Eagon and M. Hochster, R-sequences and indeterminates, *Quarterly J. Math.*, **25** (1974), 61-71.

[E] D. Eisenbud, Commutative Algebra with a View toward Algebraic Geometry, Springer-Verlag, 1994.

[EK] S. Eliahou and M. Kervaire, Minimal resolutions of some monomial ideals, *J. Algebra*, **129** (1990), 1-25.

[HS] W. Heinzer and I. Swanson, Ideals contracted from 1-dimensional overrings with an application to the primary decomposition of ideals, preprint.

[HRS] W. Heinzer, L. J. Ratliff, Jr. and K. Shah, Parametric decomposition of monomial ideals (I), preprint.

[HH] C. Huneke, and M. Hochster, Tight closure, invariant theory, and the Briançon-Skoda theorem, *Jour. of Am. Math. Soc.* **3** (1990), 31-116.

[K] M. Katzman, Finiteness of $\cup_e \operatorname{Ass} F^e(M)$ and its connections to tight closure, *Illinois J. Math.*, **40** (1996), no.2, 330-337.

[K2] M. Katzman, An application of Gröbner bases to a tight closure question, preprint.

[R] L. J. Ratliff, Jr., On prime divisors of I^n, n large, *Michigan Math. J.*, **23** (1976), 337-352.

[STV] B. Sturmfels, N. V. Trung and W. Vogel, Bounds on degrees of projective schemes, *Math. Annalen*, **302** (1995), 417-432.

[S] I. Swanson, Powers of Ideals: Primary decompositions, Artin-Rees lemma and regularity, *Math. Annalen*, to appear.

[T] W. Traves, Differential operators on monomial rings, preprint.