Introduction

These notes were written to accompany my ten lectures on tight closure at the Institute for Studies in Theoretical Physics and Mathematics (IPM), School of Mathematics, in Tehran, Iran, in January 2002. The participants ranged from researchers with knowledge in tight closure to graduate students new to tight closure and research in general. My goal was to introduce the theory of tight closure and show its utility and beauty through both its early and its recent applications. I worked through some more technical aspects of tight closure, for example test elements, because the theory of test elements is central to many applications, and furthermore, because it uses and develops some beautiful commutative algebra. Throughout I tried to emphasize the beauty of the commutative algebra as developed through tight closure, as well as the need to understand various aspects of commutative algebra to work in the area of tight closure (such as the need to understand excellent rings, Cohen-Macaulay rings, asymptotic properties of (Frobenius) powers of ideals, local cohomology, homology theory, etc.).

All this of course points to the depth and the acuity of the two originators of the theory, namely of Melvin Hochster and Craig Huneke. They started the theory in the mid 1980s. They took several standing proofs in commutative algebra, such as the proofs of the homological conjectures-theorems, the proof of Hochster and Roberts [HR1] that the ring of invariants of a reductive group acting on a polynomial ring is Cohen-Macaulay, and Huneke's proof [Hu1] about integral closures of the powers of an ideal, and from these standing proofs Hochster and Huneke pulled the essential ingredients to define a new notion. Not only were they then able to reprove all the mentioned theorems more quickly, but they were able to easily prove greater generalizations, and many new theorems as well. Among the first new theorems were new versions of the Briançon-Skoda theorem, and new results on the Cohen-Macaulayness of direct summands of regular rings, the vanishings of some Tor maps, etc. The new notion was worthy of a name, and Hochster and Huneke called it tight closure.

Tight closure continues to be a good tool in commutative algebra as well as in algebraic geometry. Hochster and Huneke themselves have published several hundred pages on tight closure, all the while developing beautiful commutative algebra. The theory of tight closure has also grown due to the work of Ian Aberbach, Nobuo Hara, Mordechai Katzman, Gennady Lyubeznik, Anurag Singh, Karen Smith, Kei-Ichi Watanabe, and other people. My early research was also in tight closure, as a result of which I became interested in asymptotic properties of powers of ideals. My work on asymptotic properties uses the techniques of tight closure, but not necessarily involve the statements about tight closure.

The extensive tight closure bibliography at the end of these notes is meant to facilitate finding more details and further information. But most of the material for these notes is taken from [HH2], [HH4], [HH9], [HH14], [HH15], [Hu3] and [Sm1], and with major input also from [ElSm], [Ho1], [Hu2], [K1], [Ku1], [Ku2], [Sm13]. The sections of these notes roughly follow my lectures, although some sections took more than one lecture. In particular, the long section onn test elements (Section 6) took about 2 lectures.

I thank IPM for hosting these talks. I thank the many participants who came through snow and dense traffic, made comments, asked questions, explained to me their work, asked for more... My stay in Iran was made very pleasant and worthwhile by all of them. I especially thank Kamran Divaani-Aazar and Siamak Yassemi for organizing the workshop and for organizing my mathematical activities in Iran, as well as for introducing me to non-mathematical Iran. The ten days of my lectures on tight closure were interspersed with talks on commutative algebra by Rahim Zaare-Nahandi, Kamran Divaani-Aazar, Kazem Khashyarmanesh, Hassan Haghighi, Leila Khatami, Javad Asadollahi, and Tirdad Sharif, and I thank them all for their nice talks. I also thank Mohamad-Taghi Dibaei, Kamran Divaani-Aazar, Hassan Haghighi, Leila Khatami, Tirdad Sharif, Siamak Yassemi, and Hossein Zakeri for their participation in the workshop, and for their personal warmth. I am grateful to Craig Huneke and Melvin Hochster for teaching me about tight closure, and to Steve and Simon Swanson for putting up with my yet another absence. Finally, many thanks go also to Kamran Divaani-Aazar and Siamak Yassemi for the helpful feedback and proofreading of these notes.

1. Tight closure

Throughout these notes all rings are Noetherian commutative with $1 \neq 0$. In this first lecture I will introduce the basics, as well as indicate the early applications of tight closure.

Definition 1.1: *For any ring R, R° denotes the subset of R consisting of all the elements which are not contained in any minimal prime ideal of R. When R is an integral domain, then $R^\circ = R \setminus \{0\}$.*

During most of my talks, and in particular in the first seven sections of these notes, all the rings will have positive prime characteristic p. This means that $\mathbb{Z}/p\mathbb{Z}$ is a subring of such a ring.

Definition 1.2: *Let R be a ring and let I be an ideal of R. An element x of R is said to be in the* **tight closure**, *I^*, of I, if there exists an element $c \in R^\circ$ such that for all sufficiently large integers e, $cx^{p^e} \in (i^{p^e} | i \in I)$.*

First of all, we establish some notation: the ideal $(i^{p^e} | i \in I)$ is denoted by $I^{[p^e]}$ and is called the eth **Frobenius power** of I. Then name follows because of the **Frobenius functor** $x \mapsto x^p$, which is a morphism over rings of characteristic p. The eth iterate of the Frobenius functor takes x to x^{p^e}. In particular, if $I = (a_1, \ldots, a_r)$, then $I^{[p^e]} = (a_1^{p^e}, \ldots, a_r^{p^e})$.

To simplify notation, we will write q to stand for a power p^e of p. Then $I^{[p^e]} = I^{[q]}$.

For any ideals I and J, $I^{[q]} + J^{[q]} = (I+J)^{[q]}$, $I^{[q]}J^{[q]} = (IJ)^{[q]}$. Also, if n is any positive integer, $(I^n)^{[q]} = (I^{[q]})^n$.

Now back to the definition of tight closure: one is expected to solve infinitely many equations in infinitely many unknowns:

$$cx^q = b_{e1}a_1^q + \cdots + b_{er}a_r^q, \text{ for all } q >> 0,$$

and this is highly nontrivial in general.

Example: Let $R = \mathbb{Z}/p\mathbb{Z}[x,y,z]/(x^3+y^3+z^3)$, p a prime integer. Then $x^2 \in (y,z)^*$. Here is a proof. Note that $x \in R^\circ$, so that $x^6 \in R^\circ$. For all $q \geq p$, write $q = 3n+\delta$ for some non-negative integers n, δ, with $\delta \in \{0,1,2\}$. Then

$$x^6 x^{2q} \in x^{6-2\delta} x^{6n+2\delta} R = x^{6(n+1)} R \subseteq (y^3, z^3)^{2(n+1)} R = \sum_{i=0}^{2n+2} y^{3i} z^{3(2n+2-i)} R.$$

But every summand lies in $I^{[q]}$, for otherwise there exists an index i such that $3i \leq q-1$ and $3(2n+2-i) \leq q-1$, so that (by adding) $6n+6 \leq 2q-2 = 6n+2\delta-2$, but this is a contradiction.

(Verify that $x \notin (y,z)^*$ when p is different from 3.)

One cannot claim that the definition of tight closure is "pretty", and it is quite hard to work with. However, tight closure appears naturally in many contexts, and the techniques of tight closure were used well before tight closure was defined (and then Hochster and Huneke extracted the gist of several proofs):

- (Kunz [Ku1]) Let R be a ring of positive prime characteristic p. Then the Frobenius homomorphism F is flat over R if and only if R is a regular ring. Kunz did not use tight closure, but this fact about the Frobenius homomorphisms that he proved is used a lot in tight closure. Also, Kunz proved in [Ku2] that if R is module-finite over the subring $\{r^p | r \in R\}$, then R is an excellent ring.

- Peskine and Szpiro [PS] proved that applying the Frobenius functor to bounded acyclic complexes of finitely generated free modules preserves acyclicity. They applied this to prove several homological conjectures in characteristic p. Rudiments of tight closure show in their proofs.

- (Hochster-Roberts [HR1]) Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay.

- (Huneke [HU1]) Let R be a Cohen-Macaulay local ring containing a field, $I = (x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ form a system of parameters. Then $\overline{I^n} \cap I^{n-1} = I^{n-1}\overline{I}$ for all $n \geq 1$.

- (Skoda-Briançon [SB], Lipman-Sathaye [LS], Lipman-Teissier [LT], Rees-Sally [RS]) Briançon-Skoda theorems ($\overline{I^n} \subseteq I$ for good n.) The proof of Lipman and Sathaye [LS] does not use the techniques of tight closure, but is a rich source of "test" elements, and helped tremendously in the development of the theory of tight closure.

- If S is an integral domain which is a module-finite overring over a ring R, then for any ideal I in R, $IS \cap R \subseteq I^*$.
- (Persistence of tight closure) If S is an R-algebra, then under some conditions on R, for any ideal I in R, $I^*S \subseteq (IS)^*$. This is not obvious if R° does not map to S°.
- (Colon capturing) Let $x_1, \ldots, x_n$ generate an ideal of height n (so $x_1, ldots, x_n$ are **parameters**). Then under some conditions on R (such as module-finite over a regular domain A containing the x_i), $(x_1, \ldots, x_{n-1})R :_R x_n \subseteq ((x_1, \ldots, x_{n-1})R)^*$.
- (Hochster [Ho3]) Big Cohen-Macaulay algebras exist.
- (With a few more definitions) Tight closure has had applications to rational singularities, multiplier ideals, symbolic powers of (prime) ideals, Kodaira vanishing theorem... (See [Sm5], [Sm7], [Sm9], [Sm11], [Ha4], [HuSm2], [HH14], [ELSm], [W5]...)

In these notes and lectures we will touch on some of these aspects of tight closure, but first we need to get the basics down.

Proposition 1.3: *Let R be a ring, x an element and I an ideal of R. Then $x \in I^*$ if and only if for all minimal primes P of R, the image of x modulo P is in the tight closure of IR/P.*

Proof: Let $P_1, \ldots, P_r$ be all the minimal primes of R (R is Noetherian). If $x \in I^*$ then there exists $c \in R$, c not in any P_i, such that for all $q >> 0$, $cx^q \in I^{[q]}$. Then for each P_i, the image of c in R/P_i is in $(R/P_i)^\circ$, and $cx^q + P_i \in I^{[q]} + P_i$. Thus the image of x modulo P_i is in the tight closure of IR/P_i.

Conversely, assume that for each $i = 1, \ldots, r$, the image of x modulo P_i is in the tight closure of IR/P_i. Let $c_i \in R \setminus P_i$ such that for all $q >> 0$, $c_i x^q \in I^{[q]} + P_i$. By Prime Avoidance there exists $d_i \in R$ which is not an element of P_i but is in every other minimal prime of R. Then $d_i c_i x^q \in I^{[q]} + \prod_j P_j \subseteq I^{[q]} + \sqrt{0}$. Set $c = \sum d_i c_i$. Then $c \in R^\circ$ and $cx^q \in I^{[q]} + \sqrt{0}$ for all $q >> 0$. As R is Noetherian, there exists q' such that $(\sqrt{0})^{q'} = 0$. Hence for all $q >> 0$,

$$c^{q'} x^{qq'} \in I^{[qq']},$$

which proves that $x \in I^*$. ■

The proposition above simplifies a lot of the theory and concrete computations, as it says that it (often) suffices to study tight closure in integral domains. Next we will verify that tight closure is indeed a closure operation.

Proposition 1.4: *Let I and J be ideals of R.*

(i) I^* *is an ideal and* $I \subseteq I^*$.

(ii) If $I \subseteq J$, *then* $I^* \subseteq J^*$.

(iii) $I^* = I^{**}$.

(iv) $(I \cap J)^* \subseteq I^* \cap J^*$ *(strict inclusion a possibility).*

(v) $(I + J)^* = (I^* + J^*)^*$.

(vi) $(I \cdot J)^* = (I^* \cdot J^*)^*$.

(vii) $(0)^* = \sqrt{0}$. *Also, for every ideal* I, I^* *is the natural preimage of the tight closure of* $IR/\sqrt{0}$ *in* $/\sqrt{0}$.

(viii) If I *is tightly closed, so is* $I : J$.

(ix) The intersection of tightly closed ideals is tightly closed.

(x) $I \subseteq I^* \subseteq \overline{I} \subseteq \overline{\sqrt{I}}$.

(xi) If R *is a regular ring, then every ideal is tightly closed.*

(xii) In an integrally closed domain, every principal ideal is tightly closed.

Proof: Clearly I^* is closed under multiplication by elements of R. If $x, y \in I^*$, then there exist elements $c, d \in R^\circ$ such that for all $q >> 0$, $cx^q, dy^q \in I^{[q]}$. Hence $cd(x + y)^q \in I^{[q]}$, $cd \in R^\circ$, and so $x + y \in I^*$. The rest of (i) and (ii) is trivial.

If $x \in (I^*)^*$, there exists an element $c \in R^\circ$ such that for all $q >> 0$, $cx^q \in (I^*)^q$. As R is a Noetherian ring, as established in the previous paragraph there exists an element $d \in R^\circ$ such that for all $q >> 0$, $d(I^*)^{[q]} \subseteq I^{[q]}$. Hence

$$cdx^q \in I^{[q]}$$

for all $q >> 0$, which proves (iii).

Then (iv) is obvious. Examples of strict inequalities can be given once we build up more examples from theory (rather than hard computation).

To prove (v) and (vi), let $\diamond$ stand for either $+$ or $\cdot$. By (ii) then $(I \diamond J)^* = (I^* \diamond J^*)^*$. If $x \in (I^* \diamond J^*)^*$, there exists an element $c \in R^\circ$ such that for all $q >> 0$, $cx^q \in (I^* \diamond J^*)^{[q]}$. As in the proof of (iii), there exists an element $d \in R^\circ$ such that for all $q >> 0$, $d(I^*)^{[q]} \subseteq I^{[q]}$ and $d(J^*)^{[q]} \subseteq J^{[q]}$. Then

$$dcx^q \in d(I^* \diamond J^*)^{[q]} = d\left((I^*)^{[q]} \diamond (J^*)^{[q]}\right) \subseteq I^{[q]} \diamond J^{[q]} = (I \diamond J)^{[q]},$$

whence $x \in (I \diamond J)^*$.

For every $x \in \sqrt{0}$, there exists an integer n such that $x^n = 0$. Hence for all $q >> 0$, $x^q \in (0)$, so that $x \in (0)^*$. The second part of (vii) is equally easy.

If I is tightly closed and J is an arbitrary ideal, let $x \in (I : J)^*$. Then there exists an element $c \in R^\circ$ such that for all $q >> 0$, $cx^q \in (I : J)^{[q]}$. Then $cx^q J^{[q]} \subseteq I^{[q]}$. As I is tightly closed, this implies that $xJ \subseteq I$, or that $x \in I : J$. Thus $I : J$ is tightly closed, proving (viii).

Let $I = \cap I_i$, where each I_i is tightly closed (i.e., $I_i^* = I_i$). Let $x \in I^*$. Then there exists an element $c \in R^\circ$ such that for all $q >> 0$, $cx^q \in I^{[q]}$. Then $cx^q \in I_i^{[q]}$ for all i and all $q >> 0$, so that $x \in I_i^* = I_i$ for all i, whence $x \in I$. This proves (ix).

For (x), to prove that $I^* \subseteq \overline{I}$, recall that $\overline{I}$ is uniquely determined by all the DVRs V which are R-algebras and contain R/P for some minimal prime ideal P of R:

$$\overline{I} = \cap_V \{r \in R \,|\, rV \subseteq IV\}.$$

If $cx^q \in I^{[q]}$, then $cx^q V \subseteq I^{[q]} V = I^q V$. By assumption $cV \neq 0V$. But then $cx^q V \subseteq I^q V$ for all $q >> 0$ implies that $xV \subseteq IV$, which proves (x).

The inclusion $cx^q \in I^{[q]}$ implies that $c \in I^{[q]} : x^q$. If R is a regular ring, by a result of Kunz $I^{[q]} : x^q$ equals $(I : x)^{[q]}$. Thus if $cx^q \in I^{[q]}$ for all $q >> 0$, then $c \in \cap_{q>>0} (I : x)^q$. After localizing at each maximal ideal, as c is not in any minimal prime ideal, this implies that locally $I : x$ is the whole ring, so that $x \in I$. Thus in a regular ring $I^* = I$.

Finally, for any element x in a ring, $(x) \subseteq (x)^* \subseteq \overline{(x)}$, and in an integrally closed domain, $\overline{(x)} = (x)$. ■

Thus in regular rings tight closure is (much) smaller than the integral closure. For example, $\overline{(x^2, y^2)}$ contains $(x, y)^2)$, which in turn strictly contains $(x^2, y^2) = (x^2, y^2)^*$. Thus tight closure gives a "tighter" fit over ideals, hence the name of the closure.

Properties (i), (ii), (iii), (vii), (viii), (x), (xi) above, as well as "colon capturing" and "persistence" (to be seen in later sections), are the most crucial properties of tight closure which give it its power. See the results and their proofs below for evidence of this. (Side remark: and try to find a mixed characteristic notion with these properties. Cf. [Ho7], [Mc2], [Mc3].)

Remark 1.5: *Let $J \subseteq I$ be ideals in R. Then I^* need NOT be the preimage of the tight closure of IR/J in R/J! For example, if $\mathfrak{m}$ is a maximal ideal in R and $J \subseteq I$ both $\mathfrak{m}$-primary ideals, the tight closure of IR/J is $\mathfrak{m}R/J$, but in general the tight closure of I is much smaller than $\mathfrak{m}$.*

And here is another natural way in which tight closure arises:

Theorem 1.6: *Let $R \subseteq S$ be a module-finite extension of integral domains. Then for any ideal I of R, $(IS)^* \cap R \subseteq I^*$. Furthermore, if T is an integral domain which is an integral extension of R, then for every ideal I of R, $IT \cap R \subseteq I^*$.*

Proof: Let $x \in (IS)^* \cap R$. Then there exists $c \in S^\circ$ such that for all $q >> 0$, $cx^q \in I^{[q]}S$. Let r be the rank of S as an R-module, i.e., $r = \dim_{Q(R)}(S \otimes_R Q(R))$. Then $0 \longrightarrow R^r \longrightarrow S$ is exact, and there exists a non-zero element d of S such that $dS \subseteq R^r$. Then for all $q >> 0$, $dcx^q \in I^{[q]}R^r \cap R = I^{[q]}$. As dc is non-zero, it has a non-zero multiple in R, so without loss of generality $dc \in R$. Hence $x \in I^*$.

The second part follows as there exists a module-finite extension S of R inside T such that $IT \cap R = IS \cap R$. ∎

2. Briançon-Skoda theorem, rings of invariants...

Briançon-Skoda theorem started in complex analysis: it is straightforward to prove that whenever f is a convergent power series in n variables $x_1, \ldots, x_n$ over $\mathbb{C}$, if $f(\underline{0}) = 0$, then

$$f \in \overline{\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)}.$$

Thus according to the theory of integral closures of ideals, there exists an integer k such that $f^k \in \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$. Analysts wanted to know if there is in general an upper bound on k, independent of f. A simple example of this result is when f is a homogeneous polynomial of degree d. Then it is easy to prove the following Euler's formula:

$$df = x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \cdots + x_n \frac{\partial f}{\partial x_n},$$

so that for f homogeneous, k can be taken to be 1. In 1974, Briançon and Skoda [SB] proved that k can always be taken to be n.

Here is an example showing that in general k cannot be taken to be smaller than n. Let $R = \mathbb{Q}\{x, y\}$, $f = x^3y^2 + y^5 + x^7$. The Jacobian ideal $J = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is generated by $3x^2y^2 + 7x^6$ and $2x^3y + 5y^4$. All these f and its partial derivatives are polynomials, so we can use finite methods for computation. The computer program Macaulay2 [GS] shows that $J : x^{10}$ contains $1715 * 9 * x * y - 49 * 54$, which is a unit in R, so that JR is the same as the ideal after coloning out with this unit. The ideal $J : x^{10}$ in the polynomial ring is computed by Macaulay2 to be $(x^3y + 5/2y^4, x^2y^3, y^6, x^6 + 3/7x^2y^2)$, an ideal primary to $(x, y)\mathbb{Q}[x, y]$, from which it can be computed that $f \notin J$ but $f^2 \in J$.

The proof of Briançon and Skoda was analytic, relying on some deep results of Skoda. Lipman and Teissier [LT] wrote that an absence of an algebraic proof of this algebraic statement was for algebraists "something of a scandal – even an insult – and certainly a challenge". In 1981, Lipman and Teissier in one paper [LT], and Lipman and Sathaye in another [LS], proved a generalized Briançon-Skoda theorem algebraically. Rees and Sally [RS] proved another generalization, again algebraically, but with a very different proof, in 1988, and at about the same time, Hochster and Huneke [HH4] gave the shortest proof via tight closure. These four proofs are all very different. All the algebraic versions of the Briançon-Skoda theorem imply the analytic result of Briançon and Skoda, but the other versions are all different. The tight closure version is limited to rings containing fields, whereas the other versions work over all regular rings.

Here is the tight closure version for rings in positive prime characteristic p:

Theorem 2.1: *(Briançon-Skoda theorem, due to Hochster-Huneke [HH4]) Let R be a Noetherian ring of positive prime characteristic p, and I an ideal generated by at most n elements. Then for all $m \geq 0$, $\overline{I^{m+n}} \subseteq (I^{m+1})^*$.*

In particular, if R is a regular ring, then for all $m \geq 0$, $\overline{I^{m+n}} \subseteq I^{m+1}$.

Proof: Modulo each minimal prime ideal P, I is generated by at most n elements, and if $x \in \overline{I^{m+n}}$, then x is still in the integral closure of the image of I^{m+1}. If we can prove that x is in the tight closure of the image of I^{m+n} modulo each minimal prime, we would be done. So we may assume that R is an integral domain.

If $n = 0$, as (0) is a prime ideal, the theorem follows trivially. So we may assume that $n > 0$.

Let $x \in \overline{I^{m+n}}$. By the theory of the integral closures of ideals, there exists an integer l such that for all $k \geq 1$,

$$(I^{m+n} + xR)^{k+l} = I^{(m+n)k}(I^{m+n} + xR)^l \subseteq I^{(m+n)k}.$$

If $I = (a_1, \ldots, a_n)$, then $I^{(m+n)k} \subseteq (a_1^k, \ldots, a_n^k)^{m+1}$. Proof of this fact: it suffices to prove that $a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n}$, with $\sum b_i = (m+n)k$, is contained in $(a_1^k, \ldots, a_n^k)^{m+1}$. Let $c_i = \lfloor \frac{b_i}{k} \rfloor$. Then $c_i + 1 > b_i/k$, so that $\sum(c_i + 1) > \sum b_i/k = m + n$, and $\sum c_i = (\sum(c_i + 1)) - n > m$. It follows that

$$a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n} \in (a_1^{c_1} a_2^{c_2} \cdots a_n^{c_n})^k R \subseteq (a_1^k, \ldots, a_n^k)^{m+1}.$$

Thus $(I^{m+n} + xR)^{k+l} \subseteq (a_1^k, \ldots, a_n^k)^{m+1}$. Set $c = x^l \in R^\circ$. Then when $k = q$,

$$cx^q \in (a_1^q, \ldots, a_n^q)^{m+1} = (I^{[q]})^{m+1} = (I^{m+1})^{[q]},$$

so that $x \in (I^{m+1})^*$. ∎

This raises some questions: in what rings is every ideal tightly closed? How difficult is it to find the element c for each tight closure containment? What is the proof in characteristic 0? We'll get to some partial answers soon.

Note that whenever I has an l-generated reduction, the Briançon-Skoda theorem says that

$$\overline{I^{m+l}} \subseteq (I^{m+1})^*.$$

In particular, in a polynomial ring $k[x,y]$, k a field of arbitrary characteristic, every ideal has a two-generated reduction after passing to the faithfully flat extension $\overline{k}[x,y]$ and localizing, so that for any elements $f,g,h \in k[x,y]$, $f^2g^2h^2 \in (f^3,g^3,h^3)$. (Here, $I = (f^3,g^3,h^3)$, $m=0$, $l=2$.) Hochster posed a challenge to find an elementary proof for this.

Corollary 2.2: *For any element $x \in R$, $\overline{(x)} = (x)^*$.*

Proof: By the Briançon-Skoda theorem, with $n = m = 1$, $\overline{(x)} \subseteq (x)^*$, and the other inclusion always holds. ∎

In particular, in every integrally closed domain, every principal ideal is tightly closed.

Further generalizations of the Briançon-Skoda theorem can be found in [AHu1], [AHu2], [AHuT], [Sw1], [Sw2].

Definition 2.3: *If every ideal in a ring R is tightly closed, then R is said to be* **weakly F-regular**. *If every localization of R is weakly F-regular, then R is said to be* **F-regular**.

It is suspected that tight closure commutes with localization (I will talk about this problem in greater detail). If that is so, then weakly F-regular is the same as F-regular. Unfortunately, for now we have to make a distinction between these two notions (and a few others, such as strong F-regularity).

It is known that weak F-regularity equals F-regularity for uncountable affine algebras (Murthy), for Gorenstein and $\mathbb{Q}$-Gorenstein rings ([HH4], [AKM]), and for rings of dimension at most 3 which are images of Gorenstein rings ([Wi]). We will prove later that a weakly F-regular Gorenstein ring is F-regular (see Theorem 3.3).

Proposition 2.4: *A weakly F-regular ring is reduced and normal.*

Proof: As $\sqrt{0} = (0)^* = (0)$, R is reduced. Let $r, s \in R$ with $s \in R^\circ$, and assume that $\frac{r}{s}$ is integral over R. Then $r \in \overline{sR}$, so that by Corollary 2.2,

$r \in (s)^* = (s)$, so that $\frac{r}{s} \in R$. ■

Proposition 2.5: *Let $(R, \mathfrak{m})$ be a Gorenstein local ring. Then R is weakly F-regular if and only if every parameter ideal in R is tightly closed. Furthermore, R is weakly F-regular if and only if one parameter ideal in R is tightly closed.*

Proof: Assume that some parameter ideal $(x_1, \ldots, x_d)$ in R is tightly closed. As the x_i form a regular sequence, then also $(x_1^{t_1}, \ldots, x_d^{t_d})$ is tightly closed for all $t_i \geq 1$ (proof by induction). Let I be an $\mathfrak{m}$-primary ideal. Choose t such that $J = (x_1^t, \ldots, x_d^t) \subseteq I$. By the Gorenstein property, $J : (J : I) = I$. But then if J is tightly closed, so is $I = J : (J : I)$. This proves that every $\mathfrak{m}$-primary ideal is tightly closed. But since every ideal is the intersection of $\mathfrak{m}$-primary ideals, thus the intersection of tightly closed ideals, it follows that every ideal is tightly closed. ■

Theorem 2.6: *A direct summand of a (weakly) F-regular domain is (weakly) F-regular.*

More generally, if A is a direct summand of R, $A^\circ \subseteq R^\circ$, and R is (weakly) F-regular, then A is also (weakly) F-regular.

Proof: Note that it is enough to prove the weak F-regularity part. By the direct summand assumption there exists an A-module homomorphism $\varphi : R \to A$ such that φ composed with the inclusion i is the identity function on A.

Let I be an ideal in A, and $x \in I^*$. Then there exists $c \in A^\circ = A \setminus \{0\}$ such that for all $q >> 0$, $cx^q \in I^{[q]}$. As $c \in R^\circ$, this says that $x \in (IR)^* = IR$. But then $x = \varphi \circ i(x) \in \varphi(IR) = I\varphi(R) = I$. ■

The next goal is to prove that direct summands of good rings are Cohen-Macaulay, but for that we will need the following useful theorem which also shows how tight closure arises naturally in many contexts:

Theorem 2.7: *(Colon capturing) Let R be a Noetherian equidimensional local ring of positive prime characteristic which is a homomorphic image of a local Cohen-Macaulay ring S. Then for any part of a system of parameters $x_1, \ldots, x_n$,*

$$(x_1, \ldots, x_{n-1}) :_R x_n \subseteq (x_1, \ldots, x_{n-1})^*.$$

Proof: Write $R = S/Q$, where Q is an ideal of S of height m. Then there exist preimages x_i' in S of the x_i and elements $y_1, \ldots, y_m \in Q$ such that

$x_1', \ldots, x_n', y_1, \ldots, y_m$ generate an ideal in S of height $m+n$ (to prove this use Prime Avoidance) and such that there exists c in S not in any minimal prime ideal over Q and an integer k with $cQ^k \subseteq (y_1, \ldots, y_m)$ (because of equidimensionality). Let $r \in S$ such that the image of r in R lies in $(x_1, \ldots, x_{n-1}) :_R x_n$. Thus we can write $rx_n = \sum_{i=1}^{n-1} r_i x_i' + y$ for some $r_i \in S$ and $y \in Q$. Hence for all $q \geq 1$, $r^q x_n^q = \sum_{i=1}^{n-1} r_i^q {x_i'}^q + y^q$. For any q such that $q \geq k$, $cy^q = \sum_j s_j y_j$ for some $s_j \in S$. Thus for all $q >> 0$,

$$cr^q x_n^q = \sum_{i=1}^{n-1} cr_i^q {x_i'}^q + \sum_j s_j y_j.$$

As $x_1', \ldots, x_n', y_1, \ldots, y_m$ is a regular sequence in S, this says that for all $q >> 0$, $cr^q \in ({x_1'}^q, \ldots, {x_{n-1}'}^q) + (y_1, \ldots, y_m)$ (in S). Let d be the image of c in R. Then by the choice of c, $d \in R^\circ$, so that the image of r in R lies in $(x_1, \ldots, x_{n-1})^*$, as was to be proved. ∎

Theorem 2.8: *A direct summand of a regular ring is Cohen-Macaulay.*

Proof: Let $A \subseteq R$ be rings, A a direct summand of R as an A-module, and R a regular ring. There exists an A-module homomorphism $\varphi : R \to A$ such that φ composed with the inclusion is the identity function on A. Everything remains unchanged under localization at multiplicatively closed subsets of A, so without loss of generality A is a local ring with maximal ideal $\mathfrak{m}$. As R is regular, it is reduced, so A is reduced. Then

$$A \subseteq R = R/P_1 \times R/P_2 \times \cdots \times R/P_k,$$

where $P_1, \ldots, P_k$ are all the minimal prime ideals of R. Each R/P_i is a regular domain. We claim that A is a direct summand of some direct summand S of R for which $A^\circ \subseteq S^\circ$. Suppose not. Then, by possibly renumbering the P_i, there exists $a \in A^\circ \cap P_1$. Set $S = R/P_2 \times \cdots \times R/P_k$, a proper regular summand of R, and ψ the restriction of φ to S. Let j be the composition of the inclusion i of A in R with the natural surjection π of R onto S. If for some non-zero $b \in A$, $j(b) = 0$, then $i(b) \in \cap_{j>1} P_j$, so that $ab = \varphi \circ i(ab) = \varphi(ai(b)) = \varphi(0) = 0$, contradicting the assumptions that b is non-zero, $a \in A^\circ$, and A is reduced. Thus necessarily j is an injection. Furthermore, for any $b \in A$,

$$a\psi \circ j(b) = \psi \circ j(ab) = \psi \circ \pi \circ i(ab) = \varphi \circ i(ab) = ab,$$

so that as a is a non-zerodivisor on A, $\psi \circ j(b) = b$.

Thus by induction on k we may assume that R is a regular integral domain and $A^\circ \subseteq R^\circ$. Then by Theorems 4.5 and 2.6, A is an F-regular integral domain. So every principal ideal is tightly closed, so that by Corollary 2.2, every principal ideal is integrally closed.

Let $x_1, \ldots, x_n$ be a part of a system of parameters in A. Assume that $x \in (x_1, \ldots, x_n)^*$. As $A^\circ \subseteq R^\circ$,

$$x \in ((x_1, \ldots, x_n)R)^* = (x_1, \ldots, x_n)R,$$

so that $x = \varphi \circ i(x) \in \varphi((x_1, \ldots, x_n)R) = (x_1, \ldots, x_n)$. This proves that every parameter ideal is tightly closed in A. In particular, by Theorem 2.7, $(x_1, \ldots, x_{n-1}) :_A x_n = (x_1, \ldots, x_{n-1})$, which proves that A is Cohen-Macaulay. ∎

In particular, a ring of invariants of a linearly reductive linear algebraic group over a field K acting K-rationally on a K algebra R is a direct summand of R. Thus if R is a polynomial ring, the ring of invariants is Cohen-Macaulay.

Definition 2.9: *For every power q of p, define R^q to be the subring of R consisting of the qth powers of all the elements of R. If R is a reduced ring, define $R^{1/q}$ to be the set of all elements in an algebraic closure of the total field of fractions of R whose qth power lies in R.*

There are obvious Frobenius morphisms $R \to R^q$ and $R^{1/q} \to R$. The relation of R to R^q is the same as the relation of $R^{1/q}$ to R, except that the latter relation was defined only for reduced rings.

Kunz proved in [Ku2] that if R is module-finite over some R^q, then R is excellent.

There is another notion that is often the same as (weak) F-regularity:

Definition 2.10: *A ring is said to be* **F-finite** *if R is module-finite over R^p. An F-finite reduced ring R of characteristic p is said to be* **strongly F-regular** *if for every $c \in R^\circ$ there exists an integer e such that the R-linear map $R \to R^{1/q}$ mapping 1 to $c^{1/q}$ splits as a map of R-modules.*

Proposition 2.11: *If R is strongly F-regular, so is every localization of R. If $R^{1/p}$ is module-finite over R, then the strongly F-regular locus of R is open. If for some $c \in R^\circ$, the R-linear map $R \to R^{1/q}$ mapping 1 to $c^{1/q}$ splits as a map of R-modules, then the R-linear map $R \to R^{1/q'}$ mapping 1* *to $c^{1/q'}$ splits as a map of R-modules for all $q' \geq q$. If for every prime ideal P, R_P is strongly F-regular, then R is strongly F-regular.*

Proof: The last part follows as $\operatorname{Spec} R$ is quasi-compact. Now assume that for some $c \in R^\circ$, the R-linear map $R \to R^{1/q}$ mapping 1 to $c^{1/q}$ splits as a map of R-modules. Let $f : R^{1/q} \to R$ split the inclusion 1 to $c^{1/q}$. Then for any $q' = p^{e'}$, the map

$$R^{1/qq'} \xrightarrow{f^{1/q'}} R^{1/q'} \xrightarrow{F^{e'}} R$$

splits the map $R \to R^{1/qq'}$ mapping 1 to $c^{1/qq'}$.

The rest of the proof is straightforward commutative algebra. ■

Proposition 2.12: *Let R be a reduced ring. If R is strongly F-regular, it is F-regular.*

Proof: After localizing, the hypotheses remain unchanged. So it suffices to prove that every ideal in R is tightly closed. Let I be an ideal in R, $x \in I^*$. Then there exists $c \in R^\circ$ such that for all $q >> 0$, $cx^q \in I^{[q]}$. Thus $c^{1/q}x \in IR^{1/q}$. By the strongly F-regular assumption, for all $q >> 0$, there exists an R-module map $\psi_q : R^{1/q} \to R$ mapping $c^{1/q}$ to 1. Hence $x \in IR$. ■

3. The localization problem

As mentioned already, it is not known whether tight closure commutes with localization. There are no counterexamples, and there are many special cases in which commutation has been proved. It is known that tight closure commutes with localization in F-regular rings, on principal ideals (since tight closure is the same as the integral closure), on ideals generated by regular sequences (Hochster-Huneke [HH9, Theorem 4.5]), on ideals generated by a system of parameters in an excellent semi-local domain (Smith via plus closure [Sm1]), for ideals of finite phantom projective dimension in an excellent semi-local domain (Aberbach [A2], based on Smith's result), in affine rings which are quotients of a polynomial ring over a field by a monomial or a binomial ideal (Smith [Sm13]), in all Artinian and all 1-dimensional rings (exercise), in weakly F-regular ($\mathbb{Q}$)-Gorenstein rings (Aberbach-MacCrimmon [AM]), on Katzman's example below (Smith-Swanson [SmSw]), etc. Here I want to present some of what is known.

First of all, if R is a ring, S a multiplicatively closed subset, I an ideal of R, and $x \in I^*$, then $\frac{x}{1} \in (S^{-1}I)^*$. This follows as $c \in R^\circ$ and $cx^q \in I^{[q]}$ for all $q >> 0$ imply that $\frac{c}{1} \in S^{-1}R^\circ = (S^{-1}R)^\circ$ and $\frac{cx^q}{1} \in S^{-1}I^{[q]} = (S^{-1}I)^{[q]}$ for all $q >> 0$.

But the other inclusion, namely $(S^{-1}I)^* \subseteq S^{-1}(I^*)$, does not follow so easily. See further discussion below of the associated problems. First I want to present some positive results.

To see if tight closure commutes with localization on an ideal I in a ring R, it suffices to check that tight closure commutes with localization at prime ideals. Namely, if S is a multiplicatively closed subset and $x \in (S^{-1}I)^* \cap R$ with $x \notin S^{-1}(I^*)$, we have seen that there exists $c \in R^\circ$ such that for all $q >> 0$, $cx^q \in S^{-1}I^{[q]}$. For each such q there exists $s_q \in S$ such that $cs_qx^q \in I^{[q]}$. Then it is straightforward to prove that an ideal P maximal with respect to the property of containing $I^* : x$ and being disjoint from all the s_q is a prime ideal. By replacing S with $R \setminus P$ yields $x \in (S^{-1}I)^* \cap R \setminus S^{-1}(I^*)$.

Thus if tight closure does not commute with localization on an ideal I, it does not commute with localization at a prime ideal on I.

Special case is localization at maximal ideals, for very special ideals:

Proposition 3.1: *Let R be a Noetherian ring of positive prime characteristic p and let I be an ideal primary to a maximal ideal M. Then $(IR_M)^* \cap R = I^*$ and thus $I^* R_M = (IR_M)^*$.*

Proof: Let $x \in R$ such that $\frac{x}{1} \in (IR_M)^*$. Then there exists $c \in (R_M)^\circ \cap R$ such that for all $q >> 0$, $\frac{cx^q}{1} \in I_M^{[q]}$. Let $d \in R$ be an element which is in precisely those minimal prime ideals of R which do not contain c. Then $\frac{d}{1}$ is nilpotent in R_M, and by replacing it by its own power we may assume that it is actually zero. Then $c' = c + d$ is an element of R° and thus

$$c' x^q \in I_M^{[q]} \cap R = I^{[q]}.$$

The last equality uses the fact that I is primary to the maximal ideal M. But this says that $x \in I^*$. ∎

Thus tight closure commutes with localization on ideals primary to a maximal ideal: if the multiplicatively closed subset contains elements of the maximal ideal, then both the localization of the tight closure and the tight closure of the localization blow up to the whole ring; the case when the multiplicatively closed subset contains no elements of the maximal ideal is handled by the proposition above.

Another corollary of the proposition gives more indications of how tight closure behaves under localization:

Corollary 3.2: *Let R be a Noetherian ring of positive prime characteristic p. The following statements are equivalent:*

(i) R is weakly F-regular.

(ii) For every maximal ideal M, every M-primary ideal is tightly closed.

(iii) For every maximal ideal M, R_M is weakly F-regular.

Proof: If R is weakly F-regular, every ideal in R is tightly closed. In particular, for every maximal ideal M and every M-primary ideal I, $I = I^*$. This proves that (i) implies (ii).

Next we assume (ii). By the previous proposition, for every maximal ideal M and every M-primary ideal I, $IR_M \cap R = I = I^* = (IR_M)^* \cap R$, hence $IR_M = (IR_M)^*$. Thus in R_M, every zero-dimensional ideal is tightly closed. But in a Noetherian local ring, every ideal is the intersection of zero-dimensional ideals, and since each one of the intersectands is tightly closed, so is every ideal. Thus R_M is weakly F-regular. Thus (iii) holds.

By the previous proposition (iii) implies (ii). Finally, (ii) implies (i) as every ideal in a Noetherian ring is the intersection of ideals primary to maximal ideals. ∎

Warning: this does not imply that localization at the maximal ideals makes tight closure commute with localization!

However, with some extra assumptions (existence of a test element, among others), localization at the maximal ideals makes tight closure commute with localization. (Discussion of test elements is postponed until Section 6.)

In weakly F-regular Gorenstein rings tight closure does commute with localization, due to the following result (and the fact that strong F-regularity implies F-regularity):

Theorem 3.3: *Let R be a Gorenstein ring. If R is weakly F-regular, it is F-regular.*

Proof: Let S be a multiplicatively closed subset of R. We need to prove that $S^{-1}R$ is weakly F-regular. As every ideal is the intersection of ideals primary to a maximal ideal, by Proposition 1.4 it suffices to prove that every ideal primary to a maximal ideal in $S^{-1}R$ is tightly closed. By Corollary 3.2 it then suffices to prove that every localization of $S^{-1}R$ at a maximal ideal is weakly F-regular.

In other words, the theorem is proved if we can show that for any prime ideal P of R, R_P is weakly F-regular. Let M be maximal ideal containing P. By Corollary 3.2, R_M is weakly F-regular. Without loss of generality we may assume that $R = R_M$.

By Proposition 2.5 it suffices to prove that some parameter ideal in R_P is tightly closed. Let h be the height of P and d the height of M. We can find a system of parameters $x_1, \ldots, x_d$ of R such that $x_1, \ldots, x_h$ is a system of parameters in R_P. We will prove that $(x_1, \ldots, x_h)R_P$ is tightly closed.

If $x \in ((x_1, \ldots, x_h)R_P)^* \cap R$, then there exists $c \in R^\circ$ such that for all $q >> 0$, $cx^q \in (x_1^q, \ldots, x_h^q)R_P \cap R$. Let $P_1, \ldots, P_k$ be the prime ideals in R different from P and minimal over $(x_1, \ldots, x_h)$. Let j be an integer such that for all $i = 1, \ldots, k$, P_i^j is contained in the P_i-primary component of $(x_1, \ldots, x_h)$. Then for all q, $(P_i^j)^{[q]}$ is contained in the P_i-primary component of $(x_1^q, \ldots, x_h^q)$. As R is Gorenstein, all the zerodivisors modulo $(x_1^q, \ldots, x_h^q)$ lie in P, $P_1, \ldots, P_k$, so that

$$cx^q((P_1 \cdots P_k)^j)^{[q]} \subseteq (x_1^q, \ldots, x_h^q).$$

Thus $x(P_1 \cdots P_k)^j$ is contained in $(x_1, \ldots, x_h)^*$, so by weak F-regularity $x(P_1 \cdots P_k)^j \subseteq (x_1, \ldots, x_h)$, so that $x \in (x_1, \ldots, x_h)R_P$. ∎

But even more holds:

Theorem 3.4: *Let R be a Gorenstein ring. If R is weakly F-regular and F-finite, it is strongly F-regular.*

Proof: By Corollary 3.2 the hypotheses localize at maximal ideals, and it suffices to prove that R is strongly F-regular after localizing at maximal ideals. Thus without loss of generality we may assume that R is local, Gorenstein, and weakly F-regular. Weak F-regularity implies that R is reduced.

Let $c \in R^\circ$. Let $x_1, \ldots, x_d$ be a system of parameters in R. Let u be the socle element for this system of parameters. Then for infinitely many q, $uc^{1/q} \notin (x_1, \ldots, x_d)R^{1/q}$ for otherwise for all $q >> 0$, $cu^q \in (x_1, \ldots, x_d)^{[q]}$, so that $u \in (x_1, \ldots, x_d)^* = (x_1, \ldots, x_d)$, contradiction.

Let q be one of these infinitely many q for which $uc^{1/q} \in (x_1, \ldots, x_d)R^{1]q}$. Let $n \in \mathbb{N}$. Let $i : R \to R^{1/q}$ be the inclusion $r \mapsto rc^{1/q}$. Tensor i with $R/(x_1^n, \ldots, x_d^n)$ to get

$$i_n : \frac{R}{(x_1^n, \ldots, x_d^n)R} \to \frac{R^{1/q}}{(x_1^n, \ldots, x_d^n)R^{1/q}}, \quad \overline{r} \mapsto \overline{rc^{1/q}}.$$

This is an inclusion for some $q >> 0$. For otherwise the socle element $(x_1 \cdots x_d)^{n-1}u$ of $\frac{R}{(x_1^n, \ldots, x_d^n)R}$ maps to 0. Hence

$$(x_1 \cdots x_d)^{n-1}uc^{1/q} \in (x_1^n, \ldots, x_d^n)R^{1/q},$$

whence $uc^{1/q} \in (x_1, \ldots, x_d)R^{1/q}$, contradiction. Thus as $\frac{R}{(x_1^n, \ldots, x_d^n)R}$ is self-injective, there exists $f_n : \frac{R^{1/q}}{(x_1^n, \ldots, x_d^n)R^{1/q}} \to \frac{R}{(x_1^n, \ldots, x_d^n)R}$ such that $f_n \circ i_n$ is the identity map. We get a composition

$$\begin{array}{ccccccccc} R & \xrightarrow{i} & R^{1/q} & \xrightarrow{surj} & \frac{R^{1/q}}{(x_1^n, \ldots, x_d^n)R^{1/q}} & \xrightarrow{f_n} & \frac{R}{(x_1^n, \ldots, x_d^n)R} \\ r & \mapsto & rc^{1/q} & \mapsto & rc^{1/q} + (x_1^n, \ldots, x_d^n)R^{1/q} & \mapsto & r + (x_1^n, \ldots, x_d^n), \end{array}$$

and all these maps are compatible to give

$$\begin{array}{ccccc} R & \xrightarrow{i} & R^{1/q} & \longrightarrow & \hat{R} \\ r & \mapsto & rc^{1/q} & \mapsto & (r, r, r, \ldots), \end{array}$$

so that the image is actually in $R \subseteq \hat{R}$. Now the theorem follows by [Ho1]. ■

By Corollary 2.2, tight closure commutes with localization on all principal ideals. More generally:

Theorem 3.5: *(Hochster and Huneke [HH9, Theorem 4.5]) Tight closure commutes with localization on all ideals generated by a regular sequence.*

Proof: Let $I = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ form a regular sequence. Let S be a multiplicatively closed set.

First of all, as the powers of I have only finitely many associated primes, there exists $s \in S$ contained in all the (finitely many) associated primes of all those I^m which intersect S. It is known that there exists an integer k such that for all $m \geq 1$, $I^m : s^\infty = I^m : s^{km}$. (Possibly $s = 1$.) By replacing s with s^k without loss of generality $k = 1$.

Claim: $\cup_{w \in S}(I^{[q]} : w) = I^{[q]} : s^{(n+1)q}$. Certainly the right-hand side is contained in the left-hand side. To prove the other inclusion, let $x \in I^{[q]} : w$ for some $w \in S$. Then by the choice of s, $s^q x \in I^q = I^{[q]} + I^q$. Assume that $s^h x \in I^{[q]} + I^h$ for some $h \geq q$. Write $s^h x = \sum_i a_i x_i^q + \sum_\nu a_\nu \underline{x}^\nu$ for some multi-indices $\nu = (\nu_1, \ldots, \nu_n)$ with $0 \leq \nu_i < q$, $\sum \nu_i = h$, and $a_i, a_\nu \in R$. Then

$$ws^h x = \sum_i w a_i x_i^q + \sum_\nu w a_\nu \underline{x}^\nu = \sum_i b_i x_i^q$$

for some $b_i \in R$. Since the x_i form a regular sequence, this says that $wa_\nu \in I$. Hence $sa_\nu \in I$, whence $s^{h+1}x \in I^{[q]} + I^{h+1}$. In particular, this holds for $h = q(1+n)$. But then $I^{[q]} + I^{h+1} \subseteq I^{[q]}$, which proves the claim.

Now let $x \in (S^{-1}I)^*$. Then for some $c \in R^\circ$ and each $q >> 0$, there exists $w_q \in S$, such that $w_q c x^q \in I^{[q]}$. By the claim above, $s^{q(n+1)} c x^q \in I^{[q]}$, whence $s^{n+1} x \in I^*$, and so $x \in S^{-1}I^*$. ■

Thus tight closure commutes with localization in many cases. What is then the difficulty of tight closure commuting with localization? Let I be an ideal in R, S a multiplicatively closed subset in R, and $x \in (S^{-1}I)^* \cap R$. Then there exists $c \in R^\circ$ (not just in $(S^{-1}R)^\circ \cap R$ – we have seen this trick in the proof of Proposition 3.1) such that for all $q >> 0$, $\frac{cx^q}{1} \in I^{[q]}S^{-1}R$. Then for each $q >> 0$, there exists $s_q \in S$ satisfying $s_q c x^q \in I^{[q]}$.

If s_q is not a zero-divisor modulo $I^{[q]}$, then $cx^q \in I^{[q]}$, and if this happens for all $q >> 0$, then $x \in I^*$, whence $\frac{x}{1} \in S^{-1}(I^*)$.

This, of course, is a very special case. In general s_q is a zero-divisor modulo $I^{[q]}$. For example, if $S = \{1, s, s^2, s^3, \ldots\}$, then $s_q = s^{n_q}$ for some integer n_q. If there exists an integer C such that $n_q \le Cq$, then $c(s^C x)^q \in s_q c x^q R \subseteq I^{[q]}$, whence $s^C x \in I^*$ and $x \in S^{-1}(I^*)$.

This brings to the following question:

Question 3.6: *Let I be an ideal in a Noetherian ring of positive prime characteristic p. Does there exist an integer C such that for all $q \ge 1$, there exists a primary decomposition of $I^{[q]}$:*

$$I^{[q]} = Q_{q1} \cap Q_{q2} \cap \cdots \cap Q_{qk_q}$$

such that for all $i = 1, \ldots, k_q$, $\sqrt{Q_{qi}}^{C[q]} \subseteq Q_{qi}$.

Proposition 3.7: *Let I be an ideal for which the answer to this question is yes. Let S be any multiplicatively closed subset in R which intersects non-trivially only finitely many of the associated primes of all the $I^{[q]}$. Then $S^{-1}(I^*) = (S^{-1}I)^*$.*

Proof: We have shown that there exists $c \in R^\circ$ such that for each $q >> 0$, there exists $s_q \in S$ satisfying $s_q c x^q \in I^{[q]}$. Let $s \in S$ be an element of all the possible finitely many associated prime ideals of all the $I^{[q]}$. By the assumption s^{Cq} lies in every primary component of $I^{[q]}$ that it can, and is a non-zerodivisor modulo the other components. But s_q is a non-zerodivisor modulo these same components, so that $s^{Cq} c x^q \in I^{[q]}$. Thus $s^C x \in I^*$, so that $x \in S^{-1}(I^*)$. ∎

Not much is known about the answer to the question raised above, and it seems to be a very hard question. The corresponding answer is true for ordinary (rather than Frobenius) powers of ideals and for monomial ideals in a polynomial ring modulo a monomial ideal (Swanson, Heinzer, Sharp, Smith...a [Sw3], [HSw], [SmSw], [Sh]). Ordinary powers of ideals are easier to work with as they determine finitely generated, Noetherian, Rees algebras, but there is no corresponding “Frobenius Rees algebra”. Furthermore, Katzman found an example of an ideal I for which the set of associated primes of all the $I^{[q]}$ is infinite:

Example: (Katzman [K1]) Let k be a field of positive prime characteristic p, t, x, y indeterminates over k, and $R = k[t, x, y]/(xy(x+y)(x+ty))$. Then $\cup_q \operatorname{Ass}(R/(x^q, y^q))$ is an infinite set. Here is a quick sketch of the proof:

for each $q \geq 1$, define $\tau_q = 1 + t + t^2 + \cdots + t^{q-2}$, and $G_q = x^2y^{q-1}$. Then $\tau_q G_q \in (x^q, y^q)R$, but G_q is not in $(x^q, y^q)R$. Thus τ_q is a zero-divisor modulo a Frobenius power of $(x, y)R$, and the different τ_q determine infinitely many associated prime ideals. Thus $\cup_q \operatorname{Ass}(R/(x^q, y^q))$ is an infinite set.

Here is another "example" of an ideal illustrating that the set of associated primes of quasi-Frobenius powers can be infinite (due to Hochster): let $R = \mathbb{Z}[x, y]$ and I the ideal $(x, y)R$. We take its generators to be x, y, and (redundantly) $x + y$. Then for any prime integer p, p divides $(x+y)^p - x^p - y^p$, so that p is a zero-divisor modulo $(x^p, y^p, (x+y)^p)$. Thus $\cup_p \operatorname{Ass}(R/(x^p, y^p, (x+y)^p))$ and $\cup_n \operatorname{Ass}(R/(x^n, y^n, (x+y)^n))$ are infinite sets, where p varies over all prime and n over all positive integers.

Apart from the obvious modifications of these two examples, there are no other known examples of an ideal for which the set of all associated primes of all the Frobenius powers if an infinite set. If you find another example, does Question 3.6 have an affirmative answer? (Katzman's example does, due to Smith-Swanson [SmSw]).

More on Katzman's example: note that modulo each minimal prime, the ring is an integrally closed domain and the image of each $I_q = (x^q, y^q)$ is a principal ideal. Thus modulo each minimal prime ideal, each I_q is integrally and tightly closed. Thus for all q, $\overline{I_q} = I_q^*$, whence we know that $\cup_q \operatorname{Ass}(R/(x^q, y^q)^*)$ is a finite set. Thus perhaps the following is a better, or a more reasonable, question:

Question 3.8: *Let I be an ideal in a Noetherian ring of positive prime characteristic p.*

(i) Does there exist an integer C such that for all $q \geq 1$, there exists a primary decomposition of $(I^{[q]})^*$:

$$(I^{[q]})^* = Q_{q1} \cap Q_{q2} \cap \cdots \cap Q_{qk_q}$$

such that for all $i = 1, \ldots, k_q$, $\sqrt{Q_{qi}}^{C[q]} \subseteq Q_{qi}$.

(ii) Is the set $\cup_q \operatorname{Ass}(R/(I^{[q]})^*)$ a finite set?

One may be able to prove that tight closure commutes with localization with a different approach. For example, the following is due to Smith:

Theorem 3.9: *(Smith [Sm13]) Let R be a ring with the property that for every minimal prime ideal P of R, R/P has a module-finite extension domain in which tight closure commutes with localization. Then tight closure commutes with localization in R.*

In particular, if tight closure commutes with localization in each R/P, as P varies over the minimal prime ideals of R, then tight closure commutes with localization in R.

Proof: Let S be a multiplicatively closed subset of R, I an ideal in R, and $x \in R$ such that $\frac{x}{1} \in (S^{-1}IR)^*$. Then there exists $c \in R \cap (S^{-1}R)^\circ$ such that for all $q >> 0$, $\frac{cx^q}{1} \in S^{-1}I^{[q]}$. By the same tricks established earlier, we may actually assume that $c \in R^\circ$.

Let P be a minimal prime ideal of R, and let A be a module-finite extension domain of R/P in which tight closure commutes with localization. Then for all $q >> 0$,

$$\frac{cx^q}{1} + S^{-1}P \in S^{-1}I^{[q]} + S^{-1}P \subseteq S^{-1}I^{[q]}A.$$

Thus $x + P \in (S^{-1}IA)^*$. As tight closure commutes with localization in A, then $x + P \in S^{-1}(IA)^*$. Thus for some $s \in S$, $sx + P \in (IA)^* \cap R/P$. By Theorem 1.6, this intersection is contained in the tight closure of IR/P. Thus as R has only finitely many minimal prime ideals, for some $s \in S$, sx lies in the tight closure of the image of I modulo each prime ideal, so that sx lies in the tight closure of I. Thus $x \in S^{-1}I^*$. ∎

A similar proof shows that if for every minimal prime ideal P of R, R/P has an integral weakly F-regular extension domain, then tight closure commutes with localization in R.

Corollary 3.10: *(Smith [Sm13]) Let k be a field (arbitrary characteristic!), $x_1, \ldots, x_n$ variables over k, J an ideal in $k[x_1, \ldots, x_n]$ generated by monomials and binomials, and $R = k[x_1, \ldots, x_n]/J$. Then tight closure commutes with localization in R.*

Note that Katzman's example falls into this category.

Proof: Let $\overline{k}$ be a finite field extension of k such that in $\overline{k}[x_1, \ldots, x_n]$, the minimal prime ideals over the extension of J are all binomial ideals (Eisenbud-Sturmfels [ES]). Suppose that tight closure commutes with localization in $\overline{R} = \overline{k}[x_1, \ldots, x_n]/J$. Let I be an ideal of R, S a multiplicatively closed subset of R, and $x \in (S^{-1}IR)^*$. Then $x \in (S^{-1}I\overline{R})^*$ (observe that $R^\circ \subseteq \overline{R}^\circ$ by faithful flatness). Thus by the assumption, $x \in S^{-1}((I\overline{R})^*)$. Hence for some $s \in S$, $sx \in (I\overline{R})^*$. Then by Theorem 1.6, $sx \in I^*$, whence $x \in S^{-1}I^*$. Thus tight closure commutes with localization in R if it does so in $\overline{R}$.

By the previous theorem and the assumption on $\overline{R}$, without loss of generality J is a binomial prime ideal. Such a ring is isomorphic to a subring of a polynomial ring $k[y_1, \dots, y_m]$ generated by monomials in the y_i. In other words, such a ring is a monomial algebra. Its normalization is module-finite over it and also a monomial algebra. So by the previous theorem it suffices to prove that tight closure commutes with localization on normal monomial algebras. But a normal monomial algebra is a direct summand of some polynomial overring. Then by Theorem 2.6, all these rings are F-regular, hence tight closure commutes with localization on them. ■

Yet another approach to the problem of commutation of tight closure with localization in quotients of polynomial rings is using the Gröbner basis approach. Katzman explored this approach:

Question 3.11: *(Katzman [K4]) Let $R = k[x_1, \dots, x_n]$, and I and J ideals in R. Does there exist an integer C such that for all $q \geq 1$, every element of a minimal reduced Gröbner basis of $J + I^{[q]}$ under the reverse lexicographic order has x_n degree at most Cq?*

Katzman proved that an affirmative answer would imply that tight closure commutes with some localizations on R/J. Katzman proved that the question above has an affirmative answer when J is generated by monomials, or when J is generated by binomials and I by monomials. Very special cases have been proved also by Hermiller and Swanson.

4. Tight closure for modules

As a general rule, in commutative algebra a lot of better light can be shed onto ideals if one studies more general modules. Thus we look also at tight closure for modules, not just ideals. Throughout this section, rings are Noetherian of prime characteristic p. We first need more facts about the iterated Frobenius functor F^e, $e \in \mathbb{N}$.

Definition 4.1: *Let R be a Noetherian ring of positive prime characteristic p. Let S denote R viewed as an R-algebra via the eth power of the Frobenius endomorphism $r \mapsto r^p$. Then the functor F^e is simply $S \otimes_R$, a covariant right-exact functor from R-modules to S-modules. Sometimes it will be necessary to make the ring explicit, and then we will write F_R^e instead of F^e.*

Recall that always $q = p^e$.

Of course, every S-module is also an R-module, so another application of the iterated Frobenius functor is possible. Historically, Peskine and Szpiro were the first to explore these functors in greater depth.

Examples:

(i) Let $\varphi : R^n \to R^m$ be an R-module homomorphism given by the matrix $\varphi = (r_{ij})$. Then $F^e(\varphi) : F^e(R^n) \to F^e(R^m)$ can be viewed as the map from S^n to S^m, or even from R^n to R^m, given by the matrix (r_{ij}^q).

(ii) Thus by the right exactness of F^e, for any ideal I in R, $F^e(R/I) = R/I^{[q]}$.

(iii) Note that F^e transforms free modules into free modules, projective modules into projective modules, flat modules into flat modules.

(iv) Recall the Buchsbaum-Eisenbud exactness criterion: a bounded complex of finitely generated free modules can be written as

$$\mathbf{G} : 0 \to G_n \xrightarrow{\varphi_n} G_{n-1} \xrightarrow{\varphi_{n-1}} G_{n-2} \cdots G_1 \xrightarrow{\varphi_1} G_0 \to 0.$$

> The Buchsbaum-Eisenbud criterion says that $\mathbf{G}$ is exact if and only if for all i, $\operatorname{rank}(G_i) = \operatorname{rank}(\varphi_i) + \operatorname{rank}(\varphi_{i+1})$, and the depth of the ideal of minors of φ_i of size $rank(\varphi_i)$ is at least i.
>
> Then it is clear that the Frobenius functor maps finite acyclic complexes of finitely generated free modules into finite acyclic complexes of finitely generated free modules. Thus if I is an ideal of finite projective dimension, so is the ideal $I^{[q]}$. Furthermore, $\operatorname{Ass}(R/I) = \operatorname{Ass}(R/I^{[q]})$.

For any R-module M, $F^e(M) = S \otimes_R M$ has the following R-module structure: $r(s \otimes m) = (rs) \otimes m$ but $(s \otimes m)r = (sr^q) \otimes m$.

There is a canonical map $M \to F^e(M)$ taking $m \mapsto 1 \otimes m$. We denote this image also as m^q (without necessarily attaching meaning to the exponential notation!). If M is a submodule of a free R-module, then for $m = (r_1, \ldots, r_k)$, $m^q = (r_1^q, \ldots, r_k^q)$, and the r_i^q have the usual meaning. With this, for all $m, n \in M$ and $r \in R$, $(m+n)^q = m^q + n^q$, $(rm)^q = r^q m^q$.

Definition 4.2: *When $N \subset M$ are R-modules, then $N_M^{[q]}$ denotes*

$$Kernel(F^e(M) \to F^e(M/N)) = Image(F^e(N) \to F^e(M)).$$

When the context makes M understood, we will write $N^{[q]}$ instead of $N_M^{[q]}$.

With this notation, the previously used $I^{[q]}$ is really $I_R^{[q]}$.

Definition 4.3: *Let $N \subseteq M$ be modules over a ring in positive prime characteristic p. An element $x \in M$ is in the* **tight closure** *N_M^* of N if there exists $c \in R^\circ$ such that for all $q >> 0$, $cx^q \in N_M^{[q]}$. When the ambient M is understood, we will often write N^* instead of N_M^*.*

If $N = N^*$, then N is **tightly closed***.*

It is clear that with set-up as in the definition, $x \in N_M^*$ if and only if $x + N \in 0_{M/N}^*$. In particular, one may always translate a tight closure problem to the problem when N is the zero submodule, or, by reversing this step, to the problem when M is a free R-module.

As for the ideal notion, tight closure of modules also is a closure operation which produces a submodule:

Proposition 4.4: *Let R be a Noetherian ring of positive prime characteristic p, $N, L \subset M$ R-modules, I an ideal in R, $x \in M$.*

(i) N_M^* is a submodule of M containing N.

(ii) *If M is finitely generated, then $(N_M^*)_M^* = N_M^*$.*

(iii) *If $N \subseteq L \subseteq M$, then $N_M^* \subseteq L_M^*$ and $N_L^* \subseteq N_M^*$.*

(iv) $\sqrt{(0)}M \subseteq N_M^*$.

(v) $(N \cap L)^* \subset N^* \cap L^*$.

(vi) $(N + L)^* = (N^* + L^*)^*$.

(vii) $(IN)^* = (I^* N^*)^*$.

(viii) $(N :_M I)^* \subseteq N^* :_M I$, $(N :_R L)^* \subseteq N^* :_R L$. *Thus if N is tightly closed, so is every colon module and ideal of the given forms.*

(xi) $x \in N_M^*$ *if and only if for every minimal prime ideal P of R, the image of x in the R/P-module M/PM lies in the tight closure of the image of $N/PM \cap N$ in M/PM.*

Proof: The basic proofs are omitted here, I only present the proof of (xi) that for modules also one can always safely assume that the ring is an integral domain. Without loss of generality M is a free R-module.

If $x \in N_M^*$, then there exists $c \in R^\circ$ such that for all $q >> 0$, cx^q is in the kernel of $F_R^e(M)$ to $F_R^e(M/N)$. Then certainly cx^q is in the image of $F_R^e(N/(PM \cap N)) \to F_R^e(M/PM)$, and even cx^q is in the image of $F_{R/P}^e(N/(PM \cap N)) \to F_{R/P}^e(M/PM)$, so that $x + PM$ is in the tight closure of the image $N/(PM \cap N)$ of N in M/PM.

Now assume that modulo each minimal prime ideal P of R, the image of x in the R/P-module M/PM lies in the tight closure of the image of N. Thus for each P, there exists $c_P \in R \backslash P$ such that for all $q >> 0$, $c_P x^q$ lies in the image of $F_{R/P}^e(N/(PM \cap N)) \to F_{R/P}^e(M/PM)$, or in other words, $c_P x^q$ lies in the image of $F_R^e(N)$ in $F_{R/P}^e(M/PM) = F_R^e(M)/F_R^e(PM)$. Let d_P be an element of every other prime of R (other than P). Then $d_P c_P x^q$ lies in the image of $F_R^e(N)$ in $F_{R/\sqrt{0}}^e(M/\sqrt{0}M)$. Then by setting $c = \sum_P d_P c_P \in R^\circ$, for all $q >> 0$, cx^q lies in the image of $F_R^e(N)$ in $F_{R/\sqrt{0}}^e(M/\sqrt{0}M)$. In other words, cx^q lies in $N_M^{[q]} + \sqrt{0}F^e(M)$. If $(\sqrt{0})^{[q']} = (0)$, then $c^{q'} x^{qq'}$ lies in $N_M^{[qq']}$, which proves (xi). ■

How do the notions of (weak) F-regularity, which we saw in the context of ideals, translate into modules?

Theorem 4.5: *Let R be a ring of characteristic p such that every ideal is tightly closed. Then every submodule of every finitely generated R-module is tightly closed.*

Proof: Let $N \subseteq M$ be finitely generated R-modules and let $x \in M \setminus N$. We will show that x is not in N_M^*. We may replace N with any maximal submodule of M not containing x. There exists a maximal ideal $\mathfrak{m}$ of R such that x is not in $N_\mathfrak{m}$. Thus by Nakayama's lemma, $x \notin (N + \mathfrak{m}xR)_\mathfrak{m}$, so that by maximality of N then $\mathfrak{m}x \subseteq N$. Thus $(xR+N)/N \cong R/\mathfrak{m}$.

Note that locally at $\mathfrak{m}$ and globally, $x+N$ is in every non-zero submodule of M/N, so that $((x)+N)/N \cong R/\mathfrak{m} \subseteq M/N$ is an essential extension.

If $x \in (N+\mathfrak{m}^k M)_\mathfrak{m}$ for all k, then $x \in N_\mathfrak{m}$, contradiction, so there exists an integer k such that $x \notin (N + \mathfrak{m}^k M)_\mathfrak{m}$. But then again by maximality of N, $(N+\mathfrak{m}^k M)_\mathfrak{m} = N_\mathfrak{m}$, so that by maximality $\mathfrak{m}^k M \subseteq N$.

Recall that by Proposition 2.4, R is a normal ring. Then by a result of Hochster on approximately Gorenstein rings [Ho1], there exists an $\mathfrak{m}$-primary ideal $Q \subseteq \mathfrak{m}^k \subseteq Ann(M/N)$ which is irreducible. The ring R/Q is Artinian Gorenstein, so self-injective, and M/N is an essential extension of $R/\mathfrak{m}$ as an R/Q-module. Thus M/N is a subset of R/Q. Then it suffices to prove that the ideal (0) in R/Q is tightly closed, or that Q is tightly closed in R. But this was the assumption. ∎

In analogy with the ideal version in Proposition 3.1, one can prove also for modules that whenever $N \subseteq M$ is such that some power of some maximal ideal $\mathfrak{m}$ of R annihilates M/N, then $(N_\mathfrak{m})^*_{M_\mathfrak{m}} = (N_M^*)_\mathfrak{m}$. The proof is omitted here (it is easy).

Similarly, if $R \subseteq S$ is a module-finite extension of Noetherian domains, then
$$(N \otimes_R S)^*_{M\otimes_R S} \cap M = N_M^*.$$

Definition 4.6: *Let $N \subseteq M$ be R-modules. The* **finitistic tight closure** *N_M^{*fg} of N in M is the union of all $(L \cap N)_L^*$, as L varies over all the finitely generated R-submodules of M. The* **absolute tight closure** *N_M^{*abs} of N in M is the set of all elements $x \in M$ such that for some module L containing M, $x \in N_L^{*fg}$.*

All these closures of modules are modules. Clearly $N_M^{*fg} \subseteq N_M^{*abs}$, and when M is finitely generated, $N_M^* = N_M^{*fg}$. It is suspected (but not known) that all these closures are the same.

Here is a good place to mention a generalization of the Buchsbaum-Eisenbud criterion for exactness in terms of tight closure. But first a definition:

Definition 4.7: *Let R be a Noetherian ring, and $\mathbf{G}$ a complex of finitely generated R-modules. Then $\mathbf{G}$ has* **phantom homology** *at the ith spot if the module of cycles $Z_i = kernel(G_i \to G_{i-1})$ is contained in the tight closure of the module of the boundaries $B_i = image(G_{i+1} \to G_i)$ in G_i, i.e., if $Z_i \subseteq (B_i)^*_{G_i}$. If $G_i = 0$ for all $i < 0$ and $\mathbf{G}$ has phantom homology at all $i > 0$, then $\mathbf{G}$ is called* **phantom acyclic**.

Now the generalization of the Buchsbaum-Eisenbud exactness criterion to the phantom acyclicity criterion:

Theorem 4.8: *(Hochster-Huneke [HH4]) Let R be a Noetherian ring in positive prime characteristic p, and let*

$$\mathbf{G} : 0 \to G_n \xrightarrow{\varphi_n} G_{n-1} \xrightarrow{\varphi_{n-1}} G_{n-2} \cdots G_1 \xrightarrow{\varphi_1} G_0 \to 0,$$

be a complex of finitely generated free R-modules. Let $(\varphi_i)_{red}$ denote the map $\varphi_i \otimes_R R/\sqrt{(0)}$. Then $\mathbb{G}$ is phantom if and only if for all i, $rank(G_i) = rank((\varphi_i)_{red}) + rank((\varphi_{i+1})_{red})$, and the height of the ideal of minors of φ_i of size $rank(\varphi_i)$ is at least i.

In particular, when the ring is regular and the conditions on the ranks and heights as above are satisfied, then the complex is exact. Also, when the two conditions are satisfied, each module of cycles is contained in the integral closure of the corresponding module of boundaries (see [R2] for definitions). The corresponding result is not known for rings in mixed characteristic 0, but partial results were obtained by Katz in [Ka].

5. Application to symbolic and ordinary powers of ideals

Ein, Lazarsfeld, Smith [ELSm] proved that for smooth affine rings in characteristic 0, for every prime ideal P of height k and every integer n, $P^{(kn)} \subseteq P^n$. This is a remarkable result in itself, but Ein, Lazarsfeld, Smith generalized it further in several ways. By [Sw4] it was known that for any prime ideal P in a polynomial ring (and more generally) there exists an integer h such that $P^{(hn)} \subseteq P^n$ for all n, but the proof in [Sw4] gives no indication that h can be taken to be k and is thus globally bounded. Ein, Lazarsfeld, and Smith used in the proof the theory of multiplier ideals, which are defined only the complex numbers. Hochster and Huneke subsequently in [HH14] used the theory of tight closure to extend these effective results also for rings in characteristic p.

This result is yet another one showing a connection between multiplier ideals and tight closure, and how tight closure has implications in geometry.

Definition 5.1: *Let I be an ideal, and W the set of all non-zerodivisors on R/I. By $I^{(n)}$ we will denote $W^{-1}I^n \cap R$.*

This definition is somewhat non-standard.

Theorem 5.2: *(Hochster-Huneke [HH14]) Let R be a Noetherian ring containing a field. Let I be any ideal of R. Let h be the largest height of any associated prime ideal of I.*

(i) If R is regular, then for all positive n and all non-negative k,

$$I^{((h+k)n)} \subseteq (I^{(k+1)})^n.$$

In particular, $I^{(hn)} \subseteq I^n$ for all n.

(ii) If I has finite projective dimension, then $I^{(hn)} \subseteq (I^n)^*$ for all n.

Proof: Note that all the assumptions are preserved if we pass from R to

$R[t]$ and replace I by $IR[t]$, where t is an indeterminate over R. This passage preserves all the hypotheses and ensures that the residue field of the localization of R at every associated prime of I is infinite.

Let P be an associated prime ideal of I. As the residue field of R_P is infinite, there exists an h-generated reduction J_P of IR_P. Then there exists an integer s_P such that for all $n \geq 0$, $I^{s_P+n} \subseteq J_P^n$. Let s be the maximum of all the s_P. Let $W \subseteq R$ be the set of all the non-zerodivisors on R/I.

Let $u \in I^{((h+k)n)}$. Let $q \geq 1$ and $q = an + r$ with $0 \leq r < n$. Then

$$I_P^{s+(h+k)(n-1)} u^a \subseteq I_P^{s+(h+k)(n-1)+a(h+k)n} \subseteq I_P^{s+(h+k)(an+r)} \subseteq J_P^{(h+k)q}.$$

(The first inclusion holds because localization at P is a further localization of $W^{-1}R$.) But J_P is h-generated, so $J_P^{(h+k)q} \subseteq (J_P^{k+1})^{[q]} \subseteq (I_P^{k+1})^{[q]}$. As this holds for all P, $I^{s+(h+k)(n-1)} u^{\lfloor \frac{q}{n} \rfloor} \subseteq W^{-1}(I^{(k+1)})^{[q]} \cap R$.

When R is regular or when $I^{(k+1)}$ has finite projective dimension, the associated primes of $(I^{(k+1)})^{[q]}$ are the same as the associated primes of $I^{(k+1)}$, so that in these cases we just proved that

$$I^{s+(h+k)(n-1)} u^{\lfloor \frac{q}{n} \rfloor} \subseteq (I^{(k+1)})^{[q]},$$

or $I^{(s+(h+k)(n-1))n} u^q \subseteq (I^{(k+1)})^{n[q]}$. Thus if I contains a non-zerodivisor, this implies the theorem.

In the regular case, R is a direct sum of regular integral domains in which the image of I is either zero or it contains a non-zerodivisor. As the theorem certainly holds for the zero ideal in an integral domain, and by above it holds for non-zero ideals in regular domains, it therefore holds for all regular rings. This proves the first part.

If I has finite projective dimension, then it locally has a finite free resolution, and its rank is the alternating sum of the ranks of the free modules in the finite free resolution. In particular, locally at all the associated primes of I, by the Auslander-Buchsbaum theorem I is free, and of rank either 0 or 1. Thus an ideal of finite projective dimension is locally either (0) or it contains a non-zerodivisor.

In case R is a direct product of rings, if I has finite projective dimension on R, its components have finite projective dimensions and heights of the associated primes of the components can only decrease. Thus it suffices to prove the second part of the theorem under the assumption that $\operatorname{Spec}(R)$ is connected. In that case, as the rank of a module of finite projective dimension is the alternating sum of the ranks of the free module in a free resolution, the rank is well-defined on each local ring. After localizing at each associated prime ideal P of R, the projective dimension of I

is 0 by the Auslander-Buchsbaum formula, i.e., $I_P = 0_P$ or $I_P = R_P$ for all P, independent of P. Thus locally I is either (0) or it contains a non-zerodivisor. The second case has been proved, and the first case follows as always $(0)^{((h+k)n)} \subseteq \sqrt{(0)} = (0)^* = (((0)^{(k+1)})^n)^*$. ∎

Hochster and Huneke also proved the following more general result, using "multipliers" (different from the multiplier ideals mentioned in connection with the result of Ein, Lazarsfeld and Smith at the beginning of this section). In the course of the proof we will make several assumptions – giving motivation for the next section. Namely, in order to prove the nice result below, one needs to prove some technicalities, and the proofs of these and further technicalities are in the next section.

Theorem 5.3: *(Hochster-Huneke [HH14]) Let R be finitely generated, geometrically reduced and equidimensional over a field K. Let I be any ideal of R and h be the largest height of any associated prime ideal of I. Let J be the Jacobian ideal of the extension R/K. Assume that the ideal I is locally either* 0 *or contains a non-zerodivisor. Then for all positive n and all non-negative k,*

$$J^n I^{((h+k)n)} \subseteq ((I^{(k+1)})^n)^*, \text{ and } J^{n+1} I^{((h+k)n)} \subseteq (I^{(k+1)})^n.$$

Recall that a K-algebra R is **geometrically reduced** if $S = \overline{K} \otimes_K R$ is reduced.

Proof: We will prove this here only in the case when R contains a field of characteristic p.

Notice that S is a faithfully flat integral extension of R containing R. The Jacobian ideal of the extension S over $\overline{K}$ is simply JS. The assumptions for I in R also hold for IS in S.

We ASSUME (see next section) that there exists an element $c \in R^\circ$ which satisfies the property that for all ideals I in R or in S , if $x \in I^*$, then $cx^q \in I^{[q]}$ for all $q \geq 0$. (See next section for this.)

If the theorem is proved for IS in S, then

$$J^n I^{((h+k)n)} \subseteq J^n S(IS)^{((h+k)n)} \cap R \subseteq (((IS)^{(k+1)})^n)^* \cap R.$$

Then for all $q \geq 0$ and any $x \in (((IS)^{(k+1)})^n)^* \cap R$,

$$cx^q \in (((IS)^{(k+1)})^n)^{[q]} \cap R = ((I^{(k+1)})^n)^{[q]},$$

so that $x \in ((I^{(k+1)})^n)^*$, as required. Similarly, if the second part of the theorem holds in S, it also holds in R. (For the second part we do not need the existence of the special c.)

Thus without loss of generality we may assume that K is algebraically closed, and so $S = R$. Also, it suffices to prove the theorem on each connected component of $\mathrm{Spec}(S)$.

The Jacobian criterion applies. As R is reduced, J is not contained in any minimal prime ideal of R.

We FURTHERMORE ASSUME that every element $c \in J \cap R^\circ$ satisfies the property that for all ideals I in R or in any of its localizations and completions, if $x \in I^*$, then $cx^q \in I^{[q]}$ for all $q \geq 0$. Furthermore, it is also true for each such c there exists a regular subring A of R such that for all $q \geq 0$, $cR^{1/q} \subseteq A^{1/q}[R]$. It is also true that $A^{1/q}[R]$ is always a flat R-module.

As in the proof of the previous theorem, if $u \in I^{((h+k)n)}$, then for all $q \geq 0$,

$$I^{s+(h+k)(n-1)}u^{\lfloor \frac{q}{n} \rfloor} \subseteq W^{-1}((I^{(k+1)})^{[q]}) \cap R,$$

where W consists of all the non-zerodivisors on R/I.
Let $x \in W^{-1}((I^{(k+1)})^{[q]}) \cap R$. Then there exists $w \in W$ such that $wx \in (I^{(k+1)})^{[q]}$, and thus trivially $w^q x \in (I^{(k+1)})^{[q]}$. Then $wx^{1/q} \in I^{(k+1)}R^{1/q}$, so that by the ASSUMPTION on J, $cwx^{1/q} \subseteq I^{(k+1)}A^{1/q}[R]$. Since $A^{1/q}[R]$ is flat over R and W contains no zerodivisors on $R/I^{(k+1)}$, then also W contains no zerodivisors on $A^{1/q}[R]/I^{(k+1)}A^{1/q}[R]$. It follows that $cx^{1/q} \subseteq I^{(k+1)}A^{1/q}[R]$. Hence $c^q x \subseteq (I^{(k+1)})^{[q]}$. As J is generated by $c \in R^\circ \cap J$, it follows that $J^{[q]}x \subseteq (I^{(k+1)})^{[q]}$. Thus

$$J^{[q]}I^{s+(h+k)(n-1)}u^{\lfloor \frac{q}{n} \rfloor} \subseteq (I^{(k+1)})^{[q]},$$

and so

$$(J^{[q]})^n I^{(s+(h+k)(n-1))n}u^q \subseteq (I^{(k+1)})^{n[q]}.$$

As $\mathrm{Spec}(R)$ is connected, either $I = (0)$ or I contains a non-zerodivisor. The theorem follows for both cases from the last display and the FURTHER ASSUMPTIONS we made in three places. ∎

Thus one can make beautiful statements about symbolic powers of ideals, such as above, if one can prove the existence of elements with special properties. This will be done in the next two lectures. In particular, see Theorem 7.1 for the proofs of the ASSUMPTIONS made in the proof above.

6. Test elements and the persistence of tight closure

In the definition of tight closure, for every element $x \in I^*$, one needs an element $c \in R^\circ$, c depending on x and on I. Actually, as I^* is a finitely generated ideal, one can take c depending only on I and not on x. It is desirable to have $c \in R^\circ$ which does not depend on I either. This points to a recurring theme in the study of tight closure: often it is possible to find uniform data (in some sense).

In fact, there are contexts in which the element c does not seem to play a role in tight closure, such as in the following tight closure version of Rees' multiplicity theorem:

Theorem 6.1: *(Hochster-Huneke [HH4]) Let $(R, \mathfrak{m})$ be a local ring, module-finite, torsion-free and generically smooth over a regular local ring. Let I be any $\mathfrak{m}$-primary ideal in R and $x \in R$. Then*

$$\lim_{q \to \infty} \frac{\lambda\left(\frac{I^{[q]}+(x^q)}{I^{[q]}}\right)}{q^{\dim R}} = 0 \textit{ if and only if } x \in I^*.$$

Compare this theorem to Rees' theorem [Re1]: if $(R, \mathfrak{m})$ is a formally equidimensional Noetherian local ring, then $\lim\limits_{n \to \infty} \frac{\lambda\left(\frac{I^n+(x^n)}{I^n}\right)}{n^{\dim R}} = 0$ if and only if $x \in \overline{I}$. (The proof of the displayed Theorem 6.1 appears in Section 9.) That theorem shows that no c appears in this definition of tight closure. However, there is no way to remove c! In fact, if in the definition of tight closure we always take $c = 1$, that defines the so called **Frobenius closure**, which is in general strictly contained in the tight closure. For example, if $R = \mathbb{Z}/p\mathbb{Z}[x, y, z]/(x^3 + y^3 + z^3)$ and p is a prime integer congruent to 1 modulo 3, then for all $q \geq 0$, $x^{2q} \notin (y, z)^q$, but we have seen that $x \in (y, z)^*$. For more on the Frobenius closure, see [Mc3].

Thus in general some c is needed. The theory of tight closure is much nicer over rings in which one element c suffices for all tight closure tests (sompare with results and assumptions of the previous section):

Definition 6.2: *An element $c \in R^\circ$ is said to be a* **test element** *if for any finitely generated module M, any submodule N, and any $x \in N_M^*$, for all $q \geq 1$, $cx^q \in N_M^{[q]}$.*

Lemma 6.3: *An element c is a test element for ideals if and only if for all ideals I, $cI^* \subseteq I$.*

Proof: If c is a test element, then for any I and $x \in I^*$, when $q = 1$, $cx \in I$, which proves that for all I, $cI^* \subseteq I$.

Now assume that for all I, $cI^* \subseteq I$. Let $x \in I^*$. Then for all $q \geq 1$, $x^q \in (I^*)^{[q]} \subseteq (I^{[q]})^*$, so by assumption $cx^q \in I^{[q]}$. Thus c is a test element for ideals. ∎

It is not clear if the test elements for ideals are also test elements, but that has been proved to be true under some extra assumptions.

It is not known if test elements exist in every domain, or in every excellent reduced rings.

An application of the existence: if $R \subseteq S$ is an integral extension of Noetherian integral domains with common test element c, then for any ideal I of R, $(IS)^* \cap R \subseteq I^*$. Here is a proof: $c((IS)^* \cap R)^{[q]} \subseteq I^{[q]}S \cap R$. Then by Theorem 1.6, $I^{[q]}S \cap R \subseteq (I^{[q]})^*$. Thus for all $q >> 0$,

$$c^2((IS)^* \cap R)^{[q]} \subseteq c(I^{[q]})^* \subseteq I^{[q]},$$

so that $(IS)^* \cap R \subseteq I^*$.

In this application above, something weaker than a test element was needed:

Definition 6.4: *An element $c \in R^\circ$ is said to be a* **weak test element** *if there exists q' such that for all $q \geq q'$, all finitely generated R-modules $N \subseteq M$ and all $x \in N_M^*$, $cx^q \in N_M^{[q]}$. Such an element is also called a* **q′-weak test element**.

The proof above has to be modified quite a bit with the assumption of the existence of weak test element rather than the existence of a test element: for a weak test element c we do not have that $cI^* \subseteq I$ for all ideals I. Instead, for all $q \geq q'$ and all ideals I, $c(I^*)^{[q]} \subseteq I^{[q]}$.

Proposition 6.5: *Let $R \subseteq S$ be an integral extension of Noetherian integral domains with a common q'-weak test element c. Then for any ideal I of R, $(IS)^* \cap R \subseteq I^*$.*

Proof: As in the proof above, we can reduce to

$$c((IS)^* \cap R)^{[q]} \subseteq I^{[q]}S \cap R \subseteq (I^{[q]})^*$$

for all $q >> 0$. Then for all $q >> 0$,

$$(c)^{q'}((IS)^* \cap R)^{[qq']} \subseteq ((I^{[q]})^*)^{[q']},$$

so that

$$c(c)^{q'}((IS)^* \cap R)^{[qq']} \subseteq c((I^{[q]})^*)^{[q']} \subseteq I^{[qq']},$$

and as $c^{1+q'} \in R^\circ$, this proves the proposition. ∎

Proposition 6.6: *If R has a weak test element then the tight closure of $I \subseteq R$ is the intersection of tightly closed ideals containing I which are primary to a maximal ideal.*

Proof: (Compare with Corollary 3.2.) Suppose $x \in R \setminus I^*$. Let $c \in R$ be a q'-weak test element. As $x \notin I^*$, there exists $q \geq q'$ such that $cx^q \notin I^{[q]}$. Then there exists a maximal ideal $\mathfrak{m}$ in R such that $\frac{cx^q}{1} \notin I^{[q]}_{\mathfrak{m}}$. Hence there exists an integer n such that $\frac{cx^q}{1} \notin (I+\mathfrak{m}^n)^{[q]}_{\mathfrak{m}}$. Hence $cx^q \notin (I+\mathfrak{m}^n)^{[q]}$. Thus by the weak test element assumption, $x \notin (I+\mathfrak{m}^n)^*$, which proves that x is not in the intersection of all the tightly closed ideals containing I which are primary to some maximal ideal. ∎

When do test elements exist? Perhaps surprisingly, they exist not so infrequently, and the test elements are often better than "ordinary" test elements:

Definition 6.7: *An element $c \in R^\circ$ is called a* **locally stable (q-weak) test element** *if c is a (q-weak) test element in every localization of R, and it is called a* **completely stable (q-weak) test element** *if c is locally stable (q-weak) and also a (q-weak) test element in the completion of each R_P as P varies over all the prime ideals of R.*

The names in parentheses get omitted when $q = 1$.

Theorem 4.5 proved that if every ideal in R is tightly closed, then every submodule of a finitely generated R-module is tightly closed. A similar proof (here omitted) shows the following:

Proposition 6.8: *If R is locally approximately Gorenstein, then $c \in R^\circ$ is a (completely stable, weak) test element for R if and only if it is a (completely stable, weak) test element for all tests of tight closure for ideals of R.* ■

Theorem 6.9: *(Hochster-Huneke) Let $(R, \mathfrak{m})$ be a reduced local ring in characteristic p for which $R^{1/p}$ is module-finite over R. If c is any element of R° such that R_c is regular, then c has a power which is a completely stable test element for R.*

The assumption that $R^{1/p}$ is module-finite over R imply that R is an excellent ring (Kunz [Ku2]).

(Recall that a ring R is **excellent** if it is universally catenary, $Reg(A)$ is an open subset of $\mathrm{Spec}(A)$ for every finitely generated R-algebra A, and is a G-ring. A ring is a **G-ring** if for every prime ideal P in R, $R_P \to \widehat{R_P}$ is regular. A map $A \to S$ is **regular** if it is flat, and all the fibers are geometrically regular (over the base). A map $k \to A$ is **geometrically regular** if k is a field and $A \otimes_k \overline{k}$ is regular.)

First a naive approach to prove Theorem 6.9: let I be an ideal in R, $x \in I^*$. Then $x \in (I_c)^*$, so as R_c is regular, $x \in I_c$. Hence there exists an integer k such that $c^k x \in I$. This integer k depends on x and I, or actually one can choose k which depends only on I, but it is not clear that there is an upper bound on these k as I varies over all the ideals of R. So this attempt at a proof results in a dead end.

The theory of tight closure raises, and answers, many such "asymptotic" questions about Noetherian rings.

But Theorem 6.9 can be proved in greater generality:

Theorem 6.10: *(Hochster-Huneke [HH9]) Let $(R, \mathfrak{m})$ be a reduced local ring in characteristic p for which $R^{1/p}$ is module-finite over R. If c is any element of R° such that R_c is Gorenstein and weakly F-regular, then c has a power which is a completely stable test element for R.*

Clearly this implies Theorem 6.9 as every regular ring is Gorenstein and weakly F-regular.

Proof: By Theorem 3.4, R_c is strongly F-regular. Thus for every $d \in R^\circ$ there exists q such that the R_c-module map $R_c^{1/q} \to R_c$ taking $d^{1/q}$ to 1 splits. Thus there exists an R-module map $R^{1/q} \to R$ taking $d^{1/q}$ to a power of c. When $d = 1$, we may assume that $q = p$, so then by replacing c by a power of itself, we may assume that the R-linear map $\psi : R^{1/p} \to R$ takes 1 to c.

The domain of the map ψ for $d = 1$ is $R^{1/p}$. But more is true: for every $q \geq 1$, there exists an R-linear map $\psi_q : R^{1/q} \to R$ taking 1 to c^2 as follows. The case $q = p$ has been established. If $q > p$, define the map ψ_q as follows:

$$R^{1/q} \xrightarrow{\psi_{q/p}^{1/p}} R^{1/p} \xrightarrow{c^{\frac{p-2}{p}}} R^{1/p} \xrightarrow{\psi_1} R,$$

where the middle map is multiplication. Note that the composition above takes 1 to

$$1 \mapsto c^{2/p} \mapsto c \mapsto c^2,$$

as desired.

Now let I be an ideal in R and $x \in I^*$. Then there exists $d \in R^\circ$ such that for all $q >> 0$, $dx^q \in I^{[q]}$. As above, we may choose a sufficiently large q' such that there is an R-linear map $\varphi : R^{1/q'} \to R$ taking $d^{1/q'}$ to some power $c^{q''}$ of c. Then for all large q, $dx^{qq'q''} \in I^{[qq'q'']}$, hence $d^{1/qq'q''}x \in IR^{1/qq'q''}$. By applying $\varphi^{1/qq''}$ we get

$$c^{1/q}x = \varphi^{1/qq''}(d^{1/qq'q''}x) \in IR^{1/qq''}.$$

It follows that

$$c^{3/q}x = c^{1/q}x\psi_{q''}^{1/q}(1) = \psi_{q''}^{1/q}(c^{1/q}x) \in \psi_{q''}^{1/q}(IR^{1/qq''}) \subseteq IR^{1/q}.$$

Thus $c^3x^q \in I^{[q]}$, which proves that c^3 is a test element for ideals in R. The same argument also works for all tight closure tests for $N \subseteq R^n$, whence c^3 is a test element for R.

The maps ψ_q also localize to any localization of R, and can be tensored with the completion of any localization of R at a prime ideal. Then c^3 is a test element also in every localization and in every such completion. ∎

A stronger theorem about the existence of test elements is the following (the proof uses the proved theorem above):

Theorem 6.11: *Let $(R, \mathfrak{m})$ be a local ring in characteristic p for which $R \to \hat{R}$ has regular fibers. If c is any element of R° such that $(R_{red})_c$ is regular, then c has a power which is a completely stable weak test element for R. If R is reduced, then c has a power which is a completely stable test element for R.*

In particular, if R is an excellent local ring, then R has a completely stable test element.

Note that the theorem above does not assume F-finiteness. F-finiteness is a rather strong assumption: even when R is a field, F-finiteness is rare.

We will prove the theorem above after establishing several lemmas.

Lemma 6.12: *Let $R \to S$ be a faithfully flat extension. Let $c \in R^\circ$.*

(i) If c is a (q'-weak) test element in S, then c is a (q'-weak) test element in R.

(ii) If c is a completely stable (q'-weak) test element in S, then c is a completely stable (q'-weak) test element in R.

In particular this holds when R is local and S is its completion.

Proof: (i) Let I be an ideal in R and $x \in I^*$. As $R \to S$ is flat, it follows that $R^\circ \subseteq S^\circ$, so that $x \in (IS)^*$. By assumption on c, for all $q \geq q'$, $cx^q \in (IS)^{[q]} = I^{[q]}S$, hence by faithful flatness $cx^q \in I^{[q]}S \cap R = I^{[q]}$. As x and I were arbitrary, this proves the first part for ideals. A similar proof works for all submodules $N \subseteq R^n$, which proves the first part in general.

(ii) Let P be a prime ideal in R. We need to show that c is a completely stable (q'-weak) test element in R_P, and by the first part it suffices to prove that c is a (q'-weak) test element in $\widehat{R_P}$.

Let Q be a prime ideal in S lying over P. By assumption c is a (q'-weak) test element in $\widehat{S_Q}$. But $\widehat{S_Q}$ is faithfully flat over $\widehat{R_P}$, so we are done by the first part. ∎

Lemma 6.13: *Let R be a Noetherian ring, $c \in R^\circ$ a completely stable q'-weak test element for R_{red}. Then for some $q'' \geq q'$, some power of c is a completely stable q''-weak test element for R.*

Proof: Let N be the nilradical of R. Let q'' be such that $N^{[q'']} = (0)$. We will prove that $c^{q''}$ is a completely stable $q'q''$-weak test element for R. By the previous lemma it suffices to prove that for every prime ideal P of R, $c^{q''}$ is a $q'q''$-weak test element for $\widehat{R_P}$. We know that c is a q'-weak test element for $\widehat{(R_{red})_P} = \widehat{R_P}/N\widehat{R_P}$. Let I be an ideal of $\widehat{R_P}$ and $x \in I^*$. As the images of the elements of $\widehat{R_P}^\circ$ lie in $\widehat{(R_{red})_P}^\circ$, then also $x \in (I\widehat{(R_{red})_P})^*$. Thus for all $q \geq q'$,

$$cx^q + P \in (I\widehat{(R_{red})_P})^{[q]}, \text{ or } cx^q \in (I^{[q]} + N)\widehat{R_P}.$$

Thus

$$c^{q''}x^{qq''} \in (I^{[qq'']} + N^{[q'']})\widehat{R_P} = I^{[qq'']}\widehat{R_P}.$$

A similar proof works for all submodules of finitely generated modules, which proves the lemma. ∎

Thus in order to prove Theorem 6.11, we may assume that R is a reduced local ring. By lemma 6.12 it then suffices to prove that c has a power which is a completely stable (weak) test element for $\hat{R}$. Thus we have reduced the proof of Theorem 6.11 to proving the following:

Theorem 6.14: *(Hochster-Huneke [HH9]) Let $(R, \mathfrak{m})$ be a reduced local ring in characteristic p for which $R \to \hat{R}$ has regular fibers. If c is any element of R° such that R_c is regular, then c has a power which is a completely stable test element for $\hat{R}$.*

Here is another lemma which helps make further reductions to prove the theorem:

Lemma 6.15: *Let $R \to S$ be a ring homomorphism of Noetherian rings.*

(i) If all the fibers are regular and R is regular (resp., Gorenstein, Cohen-Macaulay, reduced), then S is regular (resp., Gorenstein, Cohen-Macaulay, reduced).

(ii) If S is (weakly) F-regular, and $R \to S$ is faithfully flat, then R is (weakly) F-regular.

(iii) If R is (weakly) F-regular, $R \to S$ is purely inseparable, and for every maximal ideal $\mathfrak{m}$ of R, $\mathfrak{m}S$ is a maximal ideal in S, then S is (weakly) F-regular.

(iv) If $(R, \mathfrak{m})$ is weakly F-regular and all the tight closure tests in S can be performed by elements of R°, then S is weakly F-regular.

Proof: The proof of the first part is standard commutative algebra, and the proof of the second statement is easy, and not given here. We next prove the third part. Note that it suffices to prove that every ideal in S which is primary to a maximal ideal is tightly closed. Let I be such an ideal, primary to $\mathfrak{n}$. Then there exists an integer k such that $\mathfrak{n}^k \subseteq I$. Let $\mathfrak{m} = \mathfrak{n} \cap R$. Then $\mathfrak{m}$ is a maximal ideal of R. By the (weak) F-regularity assumption, R is normal, so locally approximately Gorenstein (Hochster [Ho1]), so there exists an $\mathfrak{m}$-primary irreducible ideal J in R contained in $\mathfrak{m}^k$. Note that by the assumptions, JS is irreducible, $\mathfrak{n}$-primary, and contained in $\mathfrak{m}^k S \subseteq \mathfrak{n}^k \subseteq I$. As $I = JS :_S (JS :_S I)$, it suffices to prove that JS is tightly closed. For this, it suffices to prove that a socle element of JS is not in the tight closure of JS. Let $x \in J :_R \mathfrak{m}$ be a socle element of J, i.e., $x \notin J$ and $x\mathfrak{m} \subseteq J$. Then $x \notin JS$ and $x\mathfrak{n} \subseteq JS$, so it suffices to prove that

x is not in $(JS)^*$. Otherwise there exists $c \in S^\circ$ such that for all $q >> 0$, $cx^q \in I^{[q]}S$. By replacing c by some power of itself, we may assume that $c \in R^\circ$. Hence $cx^q \in I^{[q]}S \cap R = I^{[q]}$, so $x \in J^* = J$, contradiction. This proves the third part.

Similarly for the fourth part also it suffices to prove that for every irreducible 0-dimensional ideal J of R, if $x \in (J :_R \mathfrak{m}) \setminus J$, then $x \notin (JS)^*$. As the testing c for this tight closure can be taken from R, the rest follows as for the third part. ∎

Thus to prove the last stated theorem and therefore also Theorem 6.11 one may pass to the completion $\hat{R}$ of R: $\hat{R}$ is reduced $\hat{R}_c$ is regular. Thus we may assume that R is complete. Thus by the Cohen Structure Theorem the proofs of Theorems 6.14 and Theorem 6.11 reduces to:

Theorem 6.16: *(Hochster-Huneke [HH9]) Let R be a reduced ring finitely generated over a complete local ring in characteristic p. If c is any element of R° such that R_c is regular, then c has a power which is a test element.*

Before we embark on the proof, we will digress for the so-called **Gamma construction** ([HH9]) needed for this theorem and for its generalizations:

Let $(R, \mathfrak{m}, K)$ be a complete local ring in characteristic p, K a coefficient field of R. Let Σ be a **p-base** of K. This means that $K = K^p(\Sigma)$, and for every finite subset Γ of Σ of cardinality s, $[K^p[\Gamma] : K^p] = p^s$. It is standard field theory that a p-base always exists.

For each subset Γ of Σ, define $K_q^\Gamma = K[\lambda^{1/q} | \lambda \in \Gamma]$ and $K^\Gamma = \cup_q K_q^\Gamma$. For this, we fix some algebraic closure $\overline{K}$ of K so that all $K_q^\Gamma \subseteq \overline{K}$.

Then for each $\Gamma \subseteq \Sigma$, K_q^Γ is a subfield of $\overline{K}$ containing K.

For any field L containing K, define $L[[R]]$ to be the completion of $L \otimes_K R$ at $\mathfrak{m}(L \otimes_K R)$. Note that if R is regular, then $R \cong K[[x_1, \ldots, x_d]]$ for some variables $x_1, \ldots, x_d$ over K, and then $L[[R]] \cong L[[x_1, \ldots, x_d]]$, which is also a complete regular ring. Furthermore, $L[[R]]$ is faithfully flat, Noetherian and Henselian. If in addition L is finite over K, so is $L[[R]]$ over R, and if L is purely inseparable over K, so is $L[[R]]$ over R.

A general complete local ring R is module-finite over a regular local subring A. By module-finiteness, $L[[R]] \cong L[[A]] \otimes_A R$. Then it is clear that for any purely inseparable finite extension L of K, $L[[R]]$ is faithfully flat and purely inseparable over R, is Noetherian and Henselian.

For each $\Gamma \subseteq \Sigma$, define $R^\Gamma = \cup_q K_q^\Gamma[[R]]$. This is a union/direct limit of Henselian rings and local homomorphisms of purely inseparable extensions, with maximal ideals extending to maximal ideals. Thus R^Γ is a Noetherian Henselian local ring, faithfully flat and purely inseparable over R, its residue field is K^Γ, and its completion is $K^\Gamma[[R]]$.

If R is module-finite over a complete regular ring (A, n, K), then $R^\Gamma = A^\Gamma \otimes_A R$. Each A^Γ is a regular ring. For every **cofinite** subset Γ of Σ, $(A^\Gamma)^{1/p}$ is a finite A^Γ-module. Thus A^Γ is a finite module over $(A^\Gamma)^p$. Thus also for every cofinite subset Γ of Σ, the ring R^Γ is a finite module over $(R^\Gamma)^p$.

More generally, if R is a finitely generated algebra over A, define $R^\Gamma = A^\Gamma \otimes_A R$. Then R^Γ is Noetherian, and faithfully flat and purely inseparable over R. Furthermore, the ring R^Γ is a finite module over $(R^\Gamma)^p$ whenever the ring A^Γ is a finite module over $(A^\Gamma)^p$. We know this holds for all cofinite subsets Γ of Σ.

If P is a prime ideal of R, then

$$\begin{aligned}(R/P)^\Gamma &= \bigcup_q K_q^\Gamma[[R/P]] = \bigcup_q (\mathfrak{m}\text{-completion of } K_q^\Gamma \otimes_K R/P) \\ &= \bigcup_q \frac{\mathfrak{m}\text{-completion of } K_q^\Gamma \otimes_K R}{P(\mathfrak{m}\text{-completion of } K_q^\Gamma \otimes_K R)} = \bigcup_q \frac{K_q^\Gamma[[R]]}{PK_q^\Gamma[[R]]} = \frac{R^\Gamma}{PR^\Gamma}.\end{aligned}$$

Let $Q = P \cap A$. The fiber of $R \to R^\Gamma$ at P is independent of the complete local subring $A' \subseteq A/Q$ when this extension is module-finite. Thus without loss of generality $A/Q = A'$ is a complete regular local ring, $Q = (0)A'$. The fiber of $R \to R^\Gamma$ at P equals

$$\bigcup_q (R \setminus P)^{-1} K_q^\Gamma[[R/P]] = \bigcup_q (A' \setminus (0))^{-1} K_q^\Gamma[[A']] \otimes_{k(0A')} k(P).$$

As $k(0A') \subseteq k(P)$ is a finitely generated extension of fields, $k(P)$ equals a polynomial ring over $k(0A')$ modulo a special regular sequence: if the variables are $x_1, \ldots, x_n$, then the regular sequence can be taken to be $f_1, \ldots, f_n$ with $f_i = f_i(x_1, \ldots, x_i)$ monic in x_i. Also, $(A' \setminus \{0\})^{-1} K_q^\Gamma[[A']]$ is a localization of a regular ring, so that each fiber above is Gorenstein. Hence their (special) direct limit is Gorenstein, which proves that whenever R is finitely generated over a complete local ring, then all the fibers of the map (correspondingly defined) $R \to R^\Gamma$ are Gorenstein.

Lemma 6.17: *Let L be a field, $\{F_j\}_{j \in J}$ a family of subfields directed by reverse inclusion. Let F be a subfield of $\cap_j F_j$ and M a subfield of L which is a finite algebraic extension of F. If $\cap_j F_j$ and M are linearly disjoint in L over F, then there exists an element $j \in J$ such that each $F_i \subseteq F_j$ is linearly disjoint in L over F.*

Proof is straightforward field theory/commutative algebra.

Lemma 6.18: *Let $(A, \mathfrak{m}, K)$ be a complete local ring with coefficient field K of characteristic p. Let Σ be a p-base for K. For any finitely generated A-algebra R and any subset Γ of Σ, define $R^\Gamma = A^\Gamma \otimes_A R$.*

(i) If R is an integral domain (resp., reduced), then there exists a cofinite subset Γ_0 of Σ such that for all $\Gamma \subseteq \Gamma_0$, R^Γ is an integral domain (resp., reduced).

(ii) If P is a prime ideal in R, then there exists a cofinite subset Γ_0 of Σ such that for all $\Gamma \subseteq \Gamma_0$, PR^Γ is a prime ideal.

Proof: Nothing changes if A is replaced first by its homomorphic image which is contained in R and then by a complete regular local subring (of itself and of R) with the same coefficient field. Thus we may assume that A is a complete regular local ring contained in R.

First assume that R is an integral domain. We want to show that R^Γ is an integral domain. As $R \subseteq R^\Gamma$ is purely inseparable, it suffices to prove that R^Γ is reduced.

The field of fractions M of R is finitely generated over the field of fractions F of A. For each cofinite subset Γ of Σ, let F_Γ be the field of fractions of A^Γ. As $R^\Gamma = A^\Gamma \otimes_A R \subseteq A^\Gamma \otimes_A M \subseteq F_\Gamma \otimes_F M$, it suffices to find a cofinite subset Γ_0 of Σ such that for all $\Gamma \subseteq \Gamma_0$, $F_\Gamma \otimes_F M$ is reduced. It is clear that if for some $\Gamma \subseteq \Sigma$, $F_\Gamma \otimes_F M$ is reduced, then the same holds for all smaller Γ. So it suffices to find a cofinite $\Gamma \subseteq \Sigma$ such that $F_\Gamma \otimes_F M$ is reduced.

Then it suffices to prove that for any larger overfield M' of M, $F_\Gamma \otimes_F M'$ is reduced. In particular, we enlarge M to a finite extension M' such that M' is obtained from F by first making a purely transcendental extension $F(y_1, \ldots, y_s)$ followed by a purely inseparable extension M'' and then by a separable finite extension. If $F_\Gamma \otimes_F M''$ is reduced, then $F_\Gamma \otimes_F M' = F_\Gamma \otimes_F M'' \otimes_{M''} M'$ is reduced as M'/M'' is separable. Thus we may assume that $M = M''$. Furthermore, M is a subfield of $F'(y_1^{1/q}, \ldots, y_s^{1/q})$ for some q and some purely inseparable extension F' of F. We may then even assume that $M = F'(y_1^{1/q}, \ldots, y_s^{1/q}) = F'(\underline{y}^{1/q})$.

Then $F_\Gamma \otimes_F M$ equals the localization of a polynomial ring over $F_\Gamma \otimes_F F'$, so that it suffices to prove that $F_\Gamma \otimes_F F'$ is reduced for some cofinite subset $\Gamma \subseteq \Sigma$.

Now observe that the intersection of all the F_Γ is F, and that F and F' are linearly disjoint over F. Then by Lemma 6.17, (i) is proved when R

is a domain. For any prime ideal P of R, as $R^\Gamma/PR^\Gamma = (R/P)^\Gamma$, this also proves part (ii).

Now suppose that R is reduced. For every minimal prime P of R there exists a cofinite subset Γ_P of Σ such that for all smaller Γ, PR^Γ is a prime ideal. Thus for all $\Gamma \subseteq \cap_P \Gamma_P$, $\cap_P PR^\Gamma = (\cap_P P)R^\Gamma$ (flatness!) $= (0)$, so R^Γ is reduced. ∎

Armed with this Gamma construction, we get back to proving Theorem 6.16. But a sneak preview of the proof shows that we have to – and also can – prove more:

Theorem 6.19: *(Hochster-Huneke [HH9]) Let R be a finitely generated algebra over an excellent local ring in characteristic p. If c is any element of R° such that R_c is weakly F-regular and Gorenstein, then c has a power which is a completely stable weak test element.*

Proof: As for any excellent local ring A the fibers of $A \to \hat{A}$ are regular, by Lemma 6.15 we may assume that R is a finitely generated algebra over a complete local ring.

Note that $R_c = (R_{red})_c$, so by Lemma 6.13, we may assume that R is reduced. We will prove that a power of c is a completely stable test element.

By assumption there exists a complete local ring inside R so that R is finitely generated over it, and by Cohen Structure theorem we may assume that R is finitely generated over a complete regular local ring A in R. Let Σ be a p-base of the coefficient field of A. We use the Gamma construction, and define $R^\Gamma = A^\Gamma \otimes_A R$.

By Lemma 6.18, for every cofinite subset Γ of some cofinite subset Γ_0 of Σ, R^Γ is reduced. Furthermore, R^Γ is faithfully flat over R, $(R^\Gamma)^{1/p}$ is module-finite over R^Γ, and the fibers of the map $R \to R^\Gamma$ are Gorenstein. Thus the assumption that R_c is Gorenstein implies that R_c^Γ is Gorenstein. By Lemma 6.15, R_c^Γ is also weakly F-regular. Then by Theorem 6.10, some power of c is a completely stable test element for R^Γ. But then by Lemma 6.12, that power of c is a completely stable test element for R. ∎

This finishes the proof of the existence of test elements in many contexts. An immediate corollary is the very important and powerful property of tight closure called "persistence":

Theorem 6.20: *(Persistence of tight closure) Let $\varphi : R \to S$ be a homomorphism of Noetherian rings of characteristic p. If R is a finitely generated algebra over an excellent local ring or if R_{red} is F-finite, then for any ideal*

I of R, $\varphi(I^*) \subseteq (IS)^*$ (and the corresponding statement for modules).

Proof: Both assumptions pass to homomorphic integral domain images and imply that R is excellent.

If the conclusion does not hold, then for some ideal I of R, $\varphi(I^*) \not\subseteq (IS)^*$. By mapping further to S/P for some minimal prime ideal P of S, we get that $\varphi(I^*) + P \not\subseteq (IS/P)^*$. Thus we may assume that S is an integral domain. Let P be the kernel of φ. If the image of I^* in R/P lies in $(IR/P)^*$, then as $R/P \subseteq S$, the theorem is proved. So it suffices to assume that $S = R/P$.

Let $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = P$ be a saturated chain of prime ideals in R. We will prove that whenever Q is a prime ideal in an excellent ring A of height at most 1, then for any ideal I of A, the image of I^* in A/Q lies in $(IA/Q)^*$. Repeated application of this then proves the theorem.

Thus we may assume that $S = R/P$, with P a prime ideal in R of height at most 1. If the height is 0, this has been proved, so we may assume that the height of P is exactly 1 and that R is an integral domain.

As R is excellent, the integral closure $\overline{R}$ of R is module-finite over R. Let Q be a prime ideal in $\overline{R}$ which lies over P. As $\overline{R}$ is excellent and normal and Q has height one, there exists $c \in \overline{R} \setminus Q$ such that $\overline{R}_c$ is regular. Then by Theorem 6.19, c has a power c^k which is a test element for $\overline{R}$. Then $I^* \subseteq I^*\overline{R} \subseteq (I\overline{R})^*$, so that for all $q \geq 1$, $c^k(I^*)^{[q]} \subseteq I^{[q]}\overline{R}$. As $c^k \notin Q$, the image of I^* in $\overline{R}/Q$ lies in the tight closure of the image of I. But $\overline{R}/Q$ is module-finite over R/P, so that the image of I^* in R/P lies in the tight closure of the image of I. This proves the theorem. ∎

Note that if the existence of test elements is proved for more rings, then the persistence of tight closure would hold for more rings. Is there a counterexample to persistence? In what further rings do test elements exist? Aberbach and MacCrimmon proved in [AM] that $\mathbb{Q}$-Gorenstein can replace Gorenstein in the existence statement of test elements. But do test elements exist in greater generality?

7. More on test elements, or what is needed in Section 5

The following theorem finishes the proof of Hochster-Huneke's Theorem 5.3:

Theorem 7.1: *Let K be a field of characteristic p and R a finitely generated geometrically reduced equidimensional ring over K. Let J be the Jacobian ideal of R over K. Then J is generated by completely stable test elements.*

Proof: By tensoring with $K(t)$, t an indeterminate, if necessary, and by using the reductions of the previous section, we may without loss of generality assume that K is an infinite field. Write $R = K[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Let d be the dimension of R. Then J is generated by the $(n-d) \times (n-d)$ minors of the Jacobian matrix $(\frac{\partial g_j}{\partial x_i})$. There is a non-empty Zariski-open subset $U \subseteq K^{n^2}$ such that whenever $(u_{ij}) \in U$, then with $y_i = \sum_j u_{ij} x_i$, any d of the y_i generate a Noether normalization A of R. As R is reduced and equidimensional, then R is module-finite and torsion-free over each such A. Furthermore, by possibly restricting U further, R is generically smooth over A. (Recall: a ring R is **generically smooth** over an integral domain A means that $R \otimes_A Q(A)$ is smooth over $Q(A)$. A ring S is **smooth** over a field L if S is a finite product of fields each of which is a finite separable extension of L.)

Without loss of generality we may assume that the identity matrix is in U (by possibly replacing the x_i others). The relative Jacobian of R over A is then generated by the $(n-d) \times (n-d)$ minors of $(\frac{\partial g_j}{\partial x_i})$ using the columns corresponding to the x_j not appearing in A. Thus it suffices to prove the following:

Theorem 7.2: *Let K be a field of characteristic p and A a regular domain* *which is finitely generated over K. Let R be module-finite, torsion-free and generically smooth over A. Let $c \in R^\circ$ be an element of the Jacobian ideal for R over K, and the relative Jacobian of R over A. Then c is a completely stable test elements for R.*

Proof: Let q be a power of p. As A is a regular ring, $A^{1/q}$ is flat over A. As R is generically smooth, then $A^{1/q} \otimes_A R = A^{1/q}[R]$ inside $R^{1/q}$. Separability also ensures that $R^{1/q}$ is in the field of fractions of $A^{1/q}[R]$, thus $R^{1/q}$ is in the integral closure of $A^{1/q}[R]$. According to Lipman and Sathaye [LS], if R were an integral domain, $cR^{1/q} \subseteq A^{1/q}[R]$. But Hochster showed in [Ho10] that essentially the same proof of the Lipman-Sathaye result works even with the given more general assumptions, so that $cR^{1/q} \subseteq A^{1/q}[R]$.

Then c is a test element for R. Namely, if I is an ideal in R and $x \in I^*$, then there exists $d \in R^\circ$ such that $dx^q \in I^{[q]}$ for all $q >> 1$. Then $d^{1/q}x \in IR^{1/q}$, so that $cd^{1/q}x \in IA^{1/q}[R]$, or $d^{1/q} \in (IA^{1/q}[R] :_{IA^{1/q}[R]} cx)$. As $A^{1/q}[R]$ is faithfully flat over R by base change, then $d^{1/q} \in (I :_R cx)A^{1/q}[R]$, so that $d \in (I :_R cx)^{[q]}$ for all $q >> 1$. But then necessarily $I :_R cx) = R$, so that $cx \in I$. Thus c is a test element in R.

Note that the property $cR^{1/q} \subseteq A^{1/q}[R]$ localizes. Then it is easy to show c is also a test element in every localization of R.

Now let P be a localization of R and $S = \widehat{R_P}$. Let $Q = P \cap A$. The Q-adic completions A' and R' of A_Q and $R_{A \setminus Q}$ still satisfy the property $cR'^{1/q} \subseteq A'^{1/q}[R']$ (think Cauchy sequences). But R' is the direct sum of finitely many complete local rings, one of which is S. Thus $cS^{1/q} \subseteq A'^{1/q}[S]$. Thus by the same proof as before, c is a test element also for the completion S of a local ring of R. ∎

8. Tight closure in characteristic 0

Tight closure is defined in all rings of positive prime characteristic p, and can be extended to some rings of characteristic 0. Most of this section is taken from [HH15]:

Definition 8.1: *Let R be a finitely generated algebra over a field K of characteristic 0. Let $N \subseteq M$ be finitely generated R-modules, $u \in M$. The quintuple (D, R_D, M_D, N_D, u_D) is called* **descent data** *for (K, R, M, N, u) if*

(i) D is a finitely generated $\mathbb{Z}$-subalgebra of K,

(ii) R_D is a finitely generated D-subalgebra of R,

(iii) $N_D \subseteq M_D$ are finitely generated R_D-submodules of M with $N_D \subseteq N$,

(iv) N_D, M_D and M_D/N_D are free over D,

(v) The canonical map $K \otimes_D R_D \to R$ is a K-algebra isomorphism, and the canonical map $K \otimes_D M_D \to M$ is a R-module isomorphism,

(vi) the element $u \in M$ is in M_D and $u_D = u$.

The descent data always exist! Namely, let $R = K[X_1, \ldots, X_n]/J$ for some variables X_i over K and some ideal $J = (f_1, \ldots, f_m)$ in $K[X_1, \ldots, X_n]$. In the first approximation, set D to be the subalgebra of K generated over $\mathbb{Z}$ by all the finitely many coefficients appearing in the polynomials f_i. Let $R_D = D[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$.

Recall the Generic Freeness theorem: let A be a Noetherian integral domain, R a finitely generated A-algebra and S a finitely generated R-algebra. Let M be a finitely generated S-module, N a finitely generated R-submodule of M, and W a finitely generated A-submodule of M. Then there exists a non-zero element a in A such that after localizing at a, the A-module $M/(N + W)$ is free over A_a. This version of the Generic Freeness theorem was proved by Hochster and Roberts [HR1], and prior to that a simpler version was proved by Grothendieck [Gr].

By this version of the Generic Freeness theorem then there exists a non-zero element $a \in D$ such that after localizing at a, R_D is free over the localized D. Then after inverting a, R_D injects into $R_D \otimes_D QF(D)$, which injects into $R_D \otimes_D K \cong R$, so that the localized R_D injects into R. The next approximation is to enlarge D to $D[\frac{1}{a}]$ and R_D to $R_D[\frac{1}{a}]$.

We will have to enlarge D and correspondingly R_D finitely many more times by elements of K. In this way this operation preserves the first two desired properties of descent data.

Next we tackle M. As M is a finitely generated R-module, it can be written as the cokernel of an $m \times n$ matrix (r_{ij}), $r_{ij} \in R$. So each r_{ij} is the image of some polynomial in $K[X_1, \ldots, X_n]$. By adding to D all the finitely many coefficients appearing in these finitely many polynomials, we may then assume that there exist $r_{ij} \in R_D$. Set M_D to be the cokernel of the matrix (r_{ij}) over R_D. By generic freeness, by inverting another element of D we may assume that M_D is free over A. Hence also $M_D \otimes_{R_D} K = M$.

In the future, when we add further finitely many elements of K to D, we will preserve all the properties obtained so far: R_D and M_D are free over D.

Now to incorporate the construction for N, there exist $s_{ij} \in R$ such that the image of the $m \times k$ matrix (s_{ij}) in M is N. Then the cokernel of the $m \times k$ matrix $(s_{ij}|r_{ij})$ is M/N. Via the same construction as for M_D we may assume that the $s_{ij} \in R_D$ and that $(s_{ij}|r_{ij})$ determines an R_D-module $(M/N)_D$, which tensored with K over D equals M/N and which is free over D. Set N_D to be the kernel of the natural map $M_D \to (M/N)_D$. By further additions to D we may assume that N_D is free over D.

Similarly we can construct an element $u_D = u$ in M_D. Then all the properties of descent data work out. Furthermore, the following is clear:

Proposition 8.2: *Let R be a finitely generated algebra over a field K of characteristic 0. Let $N \subseteq M$ be finitely generated R-modules, $x \in M$, and $c \in R$. Then descent data for (K, R, M, N, u) always exist. If (D, R_D, M_D, N_D, u_D) are* **descent data** *for (K, R, M, N, u), then so is every $(D \otimes_D B, R_D \otimes_D B, M_D \otimes_D B, N_D \otimes_D B, u_D \otimes_D B)$ whenever B is a finitely generated D-algebra contained in K.*

Clearly the descent data are not uniquely determined. As R_D and D in the descent data are finitely generated algebras over $\mathbb{Z}$, for every maximal ideal $\mathfrak{m}$ of D of of R_D, the corresponding residue field k is finitely generated over $\mathbb{Z}/\mathfrak{m} \cap \mathbb{Z}$, which forces $\mathbb{Z}/\mathfrak{m} \cap \mathbb{Z}$ to be a field. Thus k is a field of positive prime characteristic.

Definition 8.3: *Let R be a finitely generated algebra over a field K of characteristic 0. Let $I \subseteq R$, $x \in R$. We say that x is in the* **tight closure** *I^* of I if there exist descent data (D, R_D, I_D) for R, I, x such that for every maximal ideal $\mathfrak{m}$ of D, the image of x_D in $R_D/\mathfrak{m}R_D$ lies in the (usual positive prime characteristic) tight closure of the image of I_D in $I_D R_D/\mathfrak{m}R_D$.*

If $I = I^*$, then I is called tightly closed.

The definition for the tight closure N_M^* of a submodule N of a finitely generated R-module M is similar: $x \in M$ is in the **tight closure** *N_M^* if there exist descent data (D, R_D, M_D, N_D, x_D) for (K, R, M, N, x) such that for every maximal ideal $\mathfrak{m}$ of D, the image of x_D in $M_D/\mathfrak{m}M_D$ lies in the tight closure of the image of N_D in $M_D/\mathfrak{m}M_D$.*

Notice that the phrase "for every maximal ideal $\mathfrak{m}$ of D" in the definition above can be replaced with the phrase "for all except finitely many maximal ideals $\mathfrak{m}$ of D". For if the second condition is satisfied, there exists a non-zero element a in D which is in all of the given finitely many maximal ideals. Then D can be replaced by $D[\frac{1}{a}]$, in which "for every maximal ideal $\mathfrak{m}$ of $D[\frac{1}{a}]$" holds!

It is a very hard theorem of Hochster and Huneke that this definition is equivalent to the following definition:

Definition 8.4: *Let R be a finitely generated algebra over a field K of characteristic 0. Let $N \subseteq M$ be finitely generated R-modules, $x \in M$. We say that x is in the* **tight closure** *N_M^* of N in M if there exists an element $c \in R^\circ$ and descent data (D, R_D, M_D, N_D, x_D) for (K, R, M, N, x) and c such that for every maximal ideal $\mathfrak{m}$ of D, if p is the characteristic of $D/\mathfrak{m}$, then for all $q >> 0$, the image of cx_D^q in $M_D/\mathfrak{m}M_D$ lies in $(N_D/\mathfrak{m}M_D \cap N_D)^{[q]}$.*

The proof that the two definitions are equivalent uses some results of Lipman-Sathaye [LS]. Note that in the original definition, for each $R_D/\mathfrak{m}R_D$ one could find a distinct c not in any minimal prime ideal, but in the second definition one c works for all $\mathfrak{m}$! Thus tight closure again forces one to think about – and prove – "uniform"-type results (independent of $\mathfrak{m}$).

With either definition, it is clear that if the quintuples (D, R_D, M_D, N_D, x_D) and $(D', R'_D, M'_D, N'_D, x'_D)$ are descent data for (K, R, M, N, x), then one can obtain new descent data (E, R_E, M_E, N_E, x_E) incorporating the information of the two given descent data. Namely, this can be done say by first setting E to be the subring of K generated by the finitely many generators of D and D' over $\mathbb{Z}$, R_E the image in R of the E-algebra generated by all the finitely many generators of R_D and R'_D over D and D' respectively, and similarly for the modules. To ensure the freeness over E, one may need to modify E by inverting some non-zero element in it.

This process enables one to prove that N^*_M is a submodule of M, and that it is tightly closed in M.

And how does one prove other elementary properties of tight closure (which were relatively easy to prove in characteristic p)? For this we need to establish what properties of R and M "descend" to descent data R_D and M_D:

(i) Finite generation of modules descends. (Clear from the construction.)

(ii) Freeness of modules descends. (Clear from the construction.)

(iii) Module containment descends. (Clear from the construction.)

(iv) Similarly as in the construction, the kernel, image, and cokernel of maps descend as well (first make them free over D).

(v) Thus surjectivity and injectivity of maps also descend.

(vi) Commutation of diagrams descends.

(vii) Bounded complexes and bounded exact sequences of finitely generated modules descend. In other words, homologies descend: $(H_i(\mathbb{G}))_D = H_i((\mathbb{G})_D)$. Furthermore, split-exactness descends.

(viii) If N and N' are submodules of a finitely generated R-module M, there exist descent data such that $N_D \cap N'_D = (N \cap N')_D$ and $N_D + N'_D = (N + N')_D$.

(ix) If $\varphi : R \to S$ is a ring homomorphism of affine K-algebras, there exist descent data $\varphi_D : R_D \to S_D$ such that $\varphi_D \otimes_D K = \varphi$.

(x) If $x_1, \ldots, x_n \in R$ is an M-regular sequence, there exist descent data such that $(x_1)_D, \ldots, (x_n)_D \in R_D$ form an M_D-regular sequence. (Descend the Koszul complex $K(x_1, \ldots, x_n; M)$.)

(xi) If $K \to R$ is regular (or reduced), there exist descent data such that R_D is regular (or reduced). For this one uses the Jacobian criterion of regularity and/or reducedness.

The list of properties which descend is much longer. See [HH15] for more details.

Proposition 8.5: *(Briançon-Skoda theorem) Let R be an affine K-algebra, where K is a field of characteristic 0. Let I be an ideal generated by at most n elements. Then $\overline{I^n} \subseteq I^*$.*

Proof: One can find descent data such that I_D is generated by at most n elements. As this theorem holds in characteristic p, the proposition then follows by the definition of tight closure in characteristic 0. ∎

This seems almost too easy! Again this shows the power of tight closure: once you set it up, it quickly produces results.

But tight closure can be defined in greater generality as well:

Definition 8.6: *Let S be a Noetherian K-algebra, where K is a field of characteristic 0. Let $N \subseteq M$ be finitely generated S-modules, $x \in M$. Then x is in* **the direct K-tight closure $\mathbf{N^{>*K}}$** *of N in M if*

- *(i) there exists an affine K-algebra R, finitely generated R-modules N_R and M_R, and an element $x_R \in M_R$,*
- *(ii) there exists a K-algebra homomorphism $\varphi : R \to S$,*
- *(iii) there exists an R-linear map $\beta : M_R \to M$ with the induced map $S \otimes_R M_R \to M$ taking $1 \otimes_R x_R$ to x being an isomorphism,*
- *(iv) there exists an R-linear homomorphism $\eta : N_R \to M_R$ such that the composition $N_R \otimes_R S \to M_R \otimes_R S \to M$ maps onto N,*
- *(v) x_R is in the tight closure of the image of N_R in M_R.*

Similarly as for affine K-algebras, also the direct K-tight closure of N in M is a submodule of M containing N.

Note that a Noetherian algebra containing $\mathbb{Q}$ may be a K-algebra for uncountably many subfields K, so in principle there are uncountably many notions of tight closure of submodules of finitely generated S-modules. However, it is conjectured that all of these notions are the same. (So once again, tight closure raises a uniform-type question. This one has not been answered.)

The proof of the Briançon-Skoda theorem as above is equally easy for this notion: one collects the generators $x_1, \ldots, x_n$ of I, the finitely many generators $y_1, \ldots, y_m$ of $\overline{I^n}$ and all the finitely many coefficients in the m

equations of integral dependence of the y_i over I and adjoins these to a finitely generated K-algebra R. As this form of the Briançon-Skoda theorem is known for affine K-algebra, the result also follows for S.

It is somewhat more involved to prove that in a regular ring, all submodules of finitely generated modules are tightly closed. The sketch of a proof below is not complete as we have not developed enough theory of descent from Noetherian K-algebras to affine K-algebras. More details can be found in [HH15]:

Theorem 8.7: *Let K be a field of characteristic* 0 *and S a regular K-algebra. Let N be a submodule of a finitely generated S-module M. Then $N^{>*K} = N$.*

Proof: Here are some reductions for proving this theorem. Suppose there exists $x \in N^{>*K} \setminus N$. We may choose a prime ideal P such that x is not in N_P. But the definition shows that x is still in $N_P^{>*K}$. Thus we may assume that S is a regular local domain. Similarly, we may complete, so without loss of generality S is a complete regular local ring, with maximal ideal $\mathfrak{m}$. Also, we can replace N by a maximal submodule of M not containing x. There exists an integer t such that $x \notin N + \mathfrak{m}^t M$, so without loss of generality $N = N + \mathfrak{m}^t M$. Furthermore, by maximality of N, M/N has finite length, the image of u generates its socle, and M/N is an essential extension of Ku.

Let $x_1, \ldots, x_n$ be a regular system of parameters of S. Then by self-injectivity of $S/(x_1^t, \ldots, x_n^t)$, M/N embeds in $S/(x_1^t, \ldots, x_n^t)$ for t sufficiently large. Now the goal is to show that $(x_1 \cdots x_n)^{t-1}$ is not in the direct K-tight closure of the ideal $(x_1^t, \ldots, x_n^t)$, which would prove the theorem. Then it suffices to prove that $(x_1 \cdots x_n)^{t-1}$ is not in the direct L-tight closure of $(x_1^t, \ldots, x_n^t)$, where L is a coefficient field of S containing K.

If $(x_1 \cdots x_n)^{t-1}$ is in the direct L-tight closure of $(x_1^t, \ldots, x_n^t)$, then there exists an affine L-algebra R as in Definition 8.6 which contains all the x_i and in which $(x_1 \cdots x_n)^{t-1}$ is in the L-tight closure of $(x_1^t, \ldots, x_n^t)R$. One can further replace R by a larger affine L-algebra, and then one may assume that R is regular with the x_i forming a regular sequence (details of this are omitted here). But as regularity of an affine algebra and regularity of a sequence descend to descent data, then this is impossible: $(x_1 \cdots x_n)^{t-1}$ is not in the direct L-tight closure of $(x_1^t, \ldots, x_n^t)$. ∎

9. A bit on the Hilbert-Kunz function

Let $(R, \mathfrak{m})$ be a Noetherian local ring in positive prime characteristic p. For any $\mathfrak{m}$-primary ideal I, define **the Hilbert Kunz function** $HK_I : \mathbb{N} \to \mathbb{N}$ to be

$$HK_I(q) = \lambda\left(\frac{R}{I^{[q]}}\right).$$

This interesting and strange function has been studied by Kunz, Monsky, Han, Seibert, Buchweitz-Chen, Chiang-Hung, Chang, Watanabe-Yoshida, Conca, and so on. It has been proved that there exists a positive real number α such that $HK_I(q) = \alpha q^{\dim R} + o(q^{\dim R - 1})$. When α is known, it does not seem to reflect on the geometric properties of I and of R very much!

Theorem 9.1: *(Hochster-Huneke: length criterion for tight closure, [HH4]) Let R be a Noetherian ring of characteristic p, and let $N \subseteq L \subseteq M$ be finitely generated R-modules such that L/N is annihilated by a power of some maximal ideal $\mathfrak{m}$ of R. Let d be the height of $\mathfrak{m}$.*

(i) If $L \subseteq N_M^*$, then there exists a constant C such that for all $q \geq 0$, $\lambda(L_M^{[q]}/N_M^{[q]}) \leq Cq^{d-1}$.

(ii) If $R_{\mathfrak{m}}$ is analytically unramified and formally equidimensional and has a completely stable q'-weak test element, and if

$$\liminf_{e\to\infty} \frac{\lambda\left(L_M^{[q]}/N_M^{[q]}\right)}{q^d} = 0,$$

then $L \subseteq N_M^*$.

Proof: As L is finitely generated, there exists $c \in R^\circ$ such that for all $q >> 0$, $cL_M^{[q]} \subseteq N_M^{[q]}$. Let J be an $\mathfrak{m}$-primary ideal such that $JL \subseteq N$. The number of the generators of $L_M^{[q]}/N_M^{[q]}$ is at most the number b of generators of L/N, and $L_M^{[q]}/N_M^{[q]}$ is annihilated by $J^{[q]} + cR$. Thus

$$\lambda\left(L_M^{[q]}/N_M^{[q]}\right) \leq b\lambda\left(\frac{R}{J^{[q]} + cR}\right).$$

Note that the latter module has support only at $\mathfrak{m}$. But R_m/cR_m has dimension $d-1$, so there exists $x_1, \ldots, x_{d-1}$ in R such that $(x_1, \ldots, x_{d-1}, c)$ is an $\mathfrak{m}$-primary ideal in $R_\mathfrak{m}$ contained in $JR_\mathfrak{m} + cR_\mathfrak{m}$. Then

$$\lambda\left(\frac{R}{J^{[q]} + cR}\right) \leq \lambda\left(\frac{R}{(x_1^q, \ldots, x_{d-1}^q, c)}\right) \leq q^{d-1}\lambda\left(\frac{R}{(x_1, \ldots, x_{d-1}, c)}\right),$$

which proves the first part.

Now assume the hypotheses in the second part, and also (without loss of generality) that M is a free R-module and $L = N + yR$. These hypotheses still hold after localizing at $\mathfrak{m}$. Let $x \in L \setminus N_M^*$. As L/N is annihilated by a power of $\mathfrak{m}$, then also locally at $\mathfrak{m}$, $x \in L \setminus N_M^*$. This will bring to a contradiction. Without loss of generality we may assume that $R = R_\mathfrak{m}$ is reduced, local and equidimensional with a completely stable q'-weak test element c. Then there exists $q \geq q'$ such that $cx^q \notin N_M^{[q]}$. Then also $cx^q \notin N^{[q]}\hat{R}_{M\hat{R}}$, so x is not in $(N\hat{R})^*_{M\hat{R}}$. The assumptions are thus also satisfied after passing to the completion $\hat{R}$ of R, so without loss of generality we may assume that R is in addition a complete ring.

Then R is module-finite and torsion-free over a complete regular local ring A. As in Section 7, there exists a power q'' of p such that $R[A^{1/q}]$ is separable over $A^{1/q}$, and there exists $c \in A^\circ$ such that $cR1/q \subseteq R[A^{1/q}]$ for all $q >> 0$.

For each $q >> 0$, define $K_q = N_M^{[q]} :_A L_M^{[q]}$. Then by assumption

$$\liminf_{q\to\infty} \frac{\lambda\left(R/K_q\right)}{q^d} = 0.$$

Let $a \in K_q$. Then $ay^q \in N_M^{[q]}$, so that as M is a free module, this says that $a^{1/q}y \in NR^{1/q}$. Then by the choice of c, $ca^{1/q}y \in NR[A^{1/q}]$, or $a^{1/q} \in NR[A^{1/q}] :_{R[A^{1/q}]} cy$. As $R[A^{1/q}]$ is flat over R, this says that $a^{1/q} \in (N :_R cy)R[A^{1/q}]$. Thus $a \in (N :_R cy)^{[q]}$.

If $cy \notin N$, then $a \in \mathfrak{m}^{[q]}$, so that $K_q \subseteq \mathfrak{m}^{[q]} \cap A$ for all $q >> 1$. By the Artin-Rees lemma, then $\liminf_{q\to\infty} \frac{\lambda(R/K_q)}{q^d}$ is not zero.

So necessarily $cy \in N$, or $cL \subseteq N$. Then similarly $cL^{[q]} \subseteq N^{[q]}$ for all q, so that $L \subseteq N^*$. ■

10. Summary of research in tight closure

This section is only a haphazard naming of big areas of research related to tight closure, with no theorems and no proofs. This is meant as an outline of a reference, and more information can be found in the bibliography.

(i) (Hochster, McDermott) Define a closure operation in mixed characteristic which would also give proofs of the Briançon-Skoda thoerem, Cohen-Macaulayness of direct summands of regular rings, homological conjectures, etc.

(ii) (Smith, Watanabe, Hara) Classification of geometric singularities in algebraic geometry using tight closure.

(iii) (Hochster, Huneke, Smith) Vanishing theorems: of Tor, Kodaira, etc.

(iv) (Aberbach, Hochster, Huneke) Phantom homology.

(v) (Aberbach, Hochster, Huneke, Smith) Plus closure (absolute integral closure).

(vi) (Aberbach, Hochster, Huneke, MacCrimmon, Singh, Smith) Deformations of various F-properties: F-rationality deforms, but F-purity, F-regularity, strong F-regularity, and weak F-regularity do not in general. Under further assumptions on the ring they do deform.

(vii) (Aberbach, Huneke, Swanson) Briançon-Skoda theorems, with coefficients, and with joint reductions.

(viii) (Aberbach, Huneke, Smith) Arithmetic Macaulayfication, i.e. the problem of determining when for a given ring R there exists an ideal I in R such that $R[It]$ is Cohen-Macaulay. Non-tight closure related work is due to Brodmann, Faltings, Goto-Yamagishi...

(ix) (Aberbach, Hochster, Huneke, MacCrimmon, Singh, Smith) Test elements and test ideals.

(x) (Aberbach, Hochster, Huneke, Katzman, Smith, Swanson) Uniform-type results.

(xi) (Smith, Hara-Watanabe, Lyubeznik) Characterization of the tight closure of special ideals via local cohomology.

(xii) (Hochster, Vraciu) Tight integral closure of a set of ideals.

(xiii) (Katzman, Smith, Hermiller-Swanson) Computational aspects.

(xiv) (Lyubeznik, Smith) Graded modules.

(xv) (Han, Monsky, Seibert, ...) The Hilbert-Kunz function.

(xvi) (Smith, ...) Multiplier ideals are related to tight closure.

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