Linear equivalence of ideal topologies

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Abstract. It is proved that whenever P is a prime ideal in a commutative Noetherian ring such that the P-adic and the P-symbolic topologies are equivalent, then the two topologies are equivalent linearly. Several explicit examples are calculated, in particular for all prime ideals corresponding to non-torsion points on nonsingular elliptic cubic curves.

There are many examples of prime ideals P in commutative Noetherian rings for which the symbolic Rees algebra $S(P) = \oplus_n P^{(n)}$, where $P^{(n)}$ is the nth symbolic power of P, is not a Noetherian ring. The first such example was found by Rees in [Re4], and later Roberts [Ro1], [Ro2] and Goto, Nishida and Watanabe [GNW] found examples in regular rings. Noetherianness of symbolic Rees algebras has been studied by many other authors, for example Cowsik [Co], Cutkosky [Cu1, Cu2], Eliahou [E], Goto, Nishida, Shimoda [GNS], Huckaba [Hb1], Huneke [Hu1, Hu2, Hu3], Morales [Mo], Schenzel [Sch1, Sch4], Srinivasan [Sr]. Even in the case of space curves, it is not yet known what distinguishes the primes whose symbolic Rees algebras are Noetherian. When a symbolic Rees algebra is Noetherian, then certainly $S(P)$ is generated over the base ring in degrees up to some integer k. It is easy to show that then for all positive integers n, $P^{(kn)} \subseteq P^n$. Thus perhaps this "linear" equivalence of P-symbolic and P-adic topologies distinguishes the primes whose symbolic Rees algebras are Noetherian?

The consequence of the main result of this paper is that this is not at all the case. In fact, I prove that as long as the P-symbolic and P-adic topologies are equivalent, the two topologies are equivalent "linearly". The assumption that the two topologies be equivalent is definitely necessary, yet it is satisfied in great many cases, say if the ring is a regular local domain and P has dimension one (see Schenzel's Theorem 2.2 below). In fact, equivalence of adic and symbolic topologies has been studied by many people, for example by Huckaba, Katz, McAdam, Ratliff, Schenzel, Verma ([Hb2], [Ka], [KR], [Mc], [Ra2], [Sch1], [Sch2], [Sch4], [V1], [V2]).

Throughout all rings will be commutative with identity.

The main result of this paper, Theorem 3.3, says that whenever I and J are two ideals in a Noetherian ring R such that the I-adic topology is equivalent to the topology defined

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by ideals $\{I^n : J^\infty\}_n$, then there exists an integer k such that for all $n \geq 1$, $I^{kn} : J^\infty \subseteq I^n$. Here we use the definition:

Definition 1.1: *For two ideals I and J in a ring R, $I : J^\infty$ denotes $\cup_n(I :_R J^n)$.*

In case when J is the maximal ideal, $I : J^\infty$ is often denoted as I^{sat} and is called *the saturation of I.*

The most common case of the main theorem, and the one which actually implies the main theorem, is the case when R is a Noetherian local ring and J is its maximal ideal.

This paper in particular answers affirmatively the following question of Schenzel's:

Question 1.2: *(Schenzel) Let (R, m) be a commutative Noetherian local ring and I an ideal in R such that the I-adic topology is equivalent to the topology defined by the ideals $\{I^n : m^\infty\}_n$. Is it true that there exists an integer k such that for all $n \geq 1$, $I^{kn} : m^\infty \subseteq I^n$?*

The proof relies on the reduction, done in Section 2, to the case when R is a complete local hypersurface domain of dimension d with maximal ideal generated by $x_1, \ldots, x_{d+1}$, and I is a prime ideal of R of height $d-1$ generated by $x_1, \ldots, x_d$. Here, d is at most the dimension of the original ring.

Equivalence of these adic and symbolic topologies is closely related to the following function:

Definition 1.3: *(See [Sch2, page 144].) Let (R, m) be a Noetherian local ring and let I be an ideal in R whose I-adic topology is equivalent to the topology defined by the ideals $\{I^n : m^\infty\}_n$. Define the function $t_I : \mathbb{N} \to \mathbb{N}$ as follows: for each $n \geq 1$, $t_I(n)$ is the smallest integer such that $I^{t_I(n)} : m^\infty \subseteq I^n$.*

The fact that the I-adic topology is equivalent to the topology defined by the ideals $\{I^n : m^\infty\}_n$ just says that $t_I(n)$ is defined for all n.

Suppose that t_I is bounded above by a linear function. Namely suppose there are integers k and l such that for all $n \geq 1$, $t_I(n) \leq kn + l$. Then necessarily k is nonnegative and for all $n \geq 1$, $t_I(n) \leq kn + l \leq (k + |l|)n$, so $I^{(k+|l|)n} : m^\infty \subseteq I^{t_I(n)} : m^\infty \subseteq I^n$ which gives a positive answer to Schenzel's Question 1.2. Conversely, if Schenzel's Question 1.2 has a positive answer, there exists an integer k such that for all $n \geq 1$, $I^{kn} : m^\infty \subseteq I^n$. Hence $t_I(n) \leq kn$, so t_I is bounded above by a linear function.

Thus Question 1.2 is equivalent to the question whether $t_I(n)$ is bounded above by a linear function in n. I would like to say that this means that the two topologies are equivalent linearly. However, in the literature "linear equivalence" has been used to mean that t_I is bounded above by a linear function of slope one.

This paper proves that t_I, whenever defined, is always bounded above by a linear function.

The paper is organized as follows: Section 2 contains the reductions to the hypersurface domain case and Section 3 contains the main results, all of "linear" flavor. The last section, Section 4, contains explicit calculations of linear upper bounds for primes corresponding to monomial curves studied by Goto, Nishida and Watanabe in [GNW], for primes corresponding to points on nonsingular elliptic cubic curves as first studied by Rees in [Re4], and for the Nagata-Roberts' ideals from [N], [Ro1].

2. Reduction to the hypersurface case

A special case of the true-to-word linear equivalence was proved by Katz and Schenzel independently:

Theorem 2.1: *(Katz [Ka, Theorem 4.1], Schenzel [Sch1, Corollary 2]) Let I be an ideal in an unmixed Noetherian local ring (R, m). Then the analytic spread of I is strictly smaller than the dimension of R if and only if there exists an integer k such that for all $n \geq k$, $I^n : m^\infty \subseteq I^{n-k}$.* ∎

Schenzel also proved the following:

Theorem 2.2: *(Schenzel [Sch1, Theorem 1]) Let (R, m) be a Noetherian local ring and I an ideal in R. Then the following are equivalent:*

(i) $\{I^n : m^\infty\}_{n\in\mathbb{N}}$ is equivalent to the I-adic topology.

(ii) $\cap_{n\in\mathbb{N}} \left(I^n\widehat{R} : m\widehat{R}^\infty\right) = 0$, where $\widehat{R}$ denotes the m-adic completion of R.

(iii) $ht\,(I\widehat{R} + p/p) < dim\,(\widehat{R}/p)$ for all $p \in Ass\,\widehat{R}$. ∎

A good reference for these results is Schenzel [Sch3].

Let I be an ideal in a local Noetherian ring (R, m) whose I-adic topology is equivalent to the topology defined by $\{I^n : m^\infty\}_{n\in\mathbb{N}}$. By Theorem 2.2, the $I\widehat{R}$-adic topology is equivalent to the topology defined by $\{I^n\widehat{R} : m^\infty\}_{n\in\mathbb{N}}$, where $\widehat{R}$ denotes the m-adic completion of R. Suppose that the latter equivalence is linear, namely that there exists an integer k such that for all $n \geq 1$, $I^{kn}\widehat{R} : m^\infty \subseteq I^n\widehat{R}$. Then

$$I^{kn} : m^\infty \subseteq \left(I^{kn}\widehat{R} : m^\infty\right) \cap R \subseteq \left(I^n\widehat{R}\right) \cap R \subseteq I^n.$$

Thus in order to answer Schenzel's question affirmatively, it suffices to answer it for complete local rings.

Lemma 2.3: *Suppose that for all complete local rings with only one associated prime ideal, each t_I, whenever defined, is bounded above by a linear function. Then also for I in an arbitrary local ring R, t_I is bounded above by a linear function.*

Proof: (Due to Schenzel) By Theorem 2.2 and the comment after it, we may assume that R is complete with maximal ideal m. Let $0 = q_1 \cap \cdots \cap q_r$ be a primary decomposition of 0. Let $S = R/q_1 \oplus \cdots \oplus R/q_r$. Then S is a module-finite overring of R. By the Artin-Rees Lemma there exists an integer k such that for all $n \geq k$, $I^n S \cap R \subseteq I^{n-k}$.

By Theorem 2.2, for each $i = 1, \ldots, r$, the IR/q_i-adic topology is equivalent to the topology defined by the decreasing sequence $\{I^n R/q_i : m^\infty\}$. Thus by assumption for each $i = 1, \ldots, r$, t_{IR/q_i} is bounded above by a linear function. Let t be a linear function which bounds above all the t_{IR/q_i}. Then

$$\begin{aligned}
I^{t(n)} : m^\infty &\subseteq (I^{t(n)} S : m^\infty) \cap R \\
&= \Big(\oplus_i (I^{t(n)} R/q_i : m^\infty)\Big) \cap R \\
&\subseteq \Big(\oplus_i (I^{t_{IR/q_i}(n)} R/q_i : m^\infty)\Big) \cap R \\
&\subseteq (\oplus_i I^n R/q_i) \cap R \\
&= I^n S \cap R \\
&\subseteq I^{n-k},
\end{aligned}$$

which proves that for all $n \geq 1$, $t_I(n) \leq t(n+k)$, and hence finishes the lemma. ■

So now in order to answer Question 1.2, without loss of generality it is only necessary to consider the case when R is a complete local ring with only one associated prime.

Lemma 2.4: *Suppose that (R, m) is a complete local ring with only one associated prime ideal Q. Let P be any prime ideal in R different from m such that P contains I and such that dim $(R/P) = 1$. Suppose that t_P is bounded above by a linear function. Then t_I is also bounded above by a linear function.*

Proof: If I is nilpotent, then $t_I(n)$ is eventually a constant, so there is nothing to prove. Thus without loss of generality I is not nilpotent.

As P contains I, for all $n \geq 1$,

$$I^{t_P(n)} : m^\infty \subseteq P^{t_P(n)} : m^\infty \subseteq P^n \subseteq m^n.$$

By [Sw, Theorem 3.4] there exists a positive integer l such that for all $n \geq 1$, I^n has a primary decomposition $I^n = q_1 \cap \cdots \cap q_r$ such that for each $i = 1, \ldots, r$, $\sqrt{q_i}^{ln} \subseteq q_i$. In particular, $I^n = (I^n : m^\infty) \cap (I^n + m^{ln})$. Note that $t_P(ln) \geq n$ for all n, for otherwise

$$P^{n-1} \subseteq P^{n-1} : m^\infty \subseteq P^{t_P(ln)} : m^\infty \subseteq P^{ln},$$

which says that P and hence I are nilpotent, contradicting the assumption. Thus for all $n \geq 1$, $t_P(ln) \geq n$ and so

$$I^{t_P(ln)} : m^\infty \subseteq I^{ln} \cap (I^n : m^\infty) \subseteq m^{ln} \cap (I^n : m^\infty) \subseteq I^n,$$

by the choice of l, which proves that $t_I(n) \leq t_P(ln)$. ■

By Theorem 2.2, the equivalence of the I-adic topology and the topology defined by $\{I^n : m^\infty\}_n$ in a complete local ring (R, m) occurs if and only if for all $Q \in Ass(R)$, $I + Q$ is not m-primary. Thus under the assumptions as in Lemma 2.3, when R has only one associated prime Q, there always exist prime ideals different from m which contain $I + Q$. Hence by Lemma 2.4, in order to answer Question 1.2, without loss of generality I may be taken to be a prime ideal of dimension 1 in a complete local ring with only one associated prime ideal.

The next lemma reduces in addition to a complete local integral domain:

Lemma 2.5: *Let (R, m) be a complete Noetherian local ring with only one associated prime ideal Q. Let I be an ideal containing Q. Assume that in all proper quotient rings S of R of the same dimension, whenever the IS-adic topology is equivalent to the topology defined by $\{I^nS : m^\infty\}_n$, then t_{IS} is bounded above by a linear function. It follows that t_I is also bounded above by a linear function.*

Proof: Without loss of generality I is different from Q, for otherwise I is nilpotent so t_I is eventually constant and the conclusion of the lemma is satisfied trivially.

Let p be a nonzero element of Q. Let J be the Q-primary component of pR. Then J is a nonzero ideal. By Theorem 2.2, the $I(S/J)$-adic topology is equivalent to the topology defined by $\{I^nS/J : m^\infty\}$. Thus by assumption $t_{IR/J}$ is bounded above by a linear function. This means that for all $n \geq 1$,

$$\left(I^{t_{IR/J}(n)} + J\right) : m^\infty \subseteq I^n + J.$$

Similarly, $t_{IR/(0:p)}$ is bounded above by a linear function, and for all $n \geq 1$,

$$\left(I^{t_{IR/(0:p)}(n)} + (0:p)\right) : m^\infty \subseteq I^n + (0:p).$$

By the definition of J there exists an element $z \in R$ not in Q such that $zJ \subseteq pR$. Note that z is a nonzerodivisor in R. By the Artin-Rees Lemma there exist integers k and l, such that for all $n \geq k$,

$$I^n : p \subseteq I^{n-k} + (0 : p)$$

and such that for all $n \geq l$,

$$I^n : z \subseteq I^{n-l}.$$

Now let α be an element of $I^{t_{IR/J}(t_{IR/(0:p)}(n+l)+k)} : m^\infty$. Then by assumption α lies in $I^{t_{IR/(0:p)}(n+l)+k} + J$. Write $\alpha = \beta + \gamma$, where β is in $I^{t_{IR/(0:p)}(n+l)+k}$ and γ is in J. Thus $z\gamma = rp$ for some $r \in R$. Hence

$$\begin{aligned} rp = z\gamma = z\alpha - z\beta &\in \left(I^{t_{IR/J}(t_{IR/(0:p)}(n+l)+k)} : m^\infty\right) + I^{t_{IR/(0:p)}(n+l)+k} \\ &\subseteq I^{t_{IR/(0:p)}(n+l)+k} : m^\infty. \end{aligned}$$

The last inclusion used that for all $n \geq 1$, $t_{IR/J}(n) \geq n$; this is satisfied whenever $I + J/J$ is not nilpotent. It follows that

$$\begin{aligned} r &\in \left(I^{t_{IR/(0:p)}(n+l)+k} : m^\infty\right) : p \\ &\subseteq \left(I^{t_{IR/(0:p)}(n+l)+k} : p\right) : m^\infty \\ &\subseteq \left(I^{t_{IR/(0:p)}(n+l)} + (0 : p)\right) : m^\infty \\ &\subseteq I^{n+l} + (0 : p). \end{aligned}$$

Thus $z\gamma = rp \in I^{n+l}$, so $\gamma \in I^{n+l} : z \subseteq I^n$. Finally, $\alpha = \beta + \gamma \in I^n$, so $t_I(n) \leq t_{IR/J}(t_{IR/(0:p)}(n+l)+k)$, which proves the lemma. ■

Noetherian induction and this lemma then imply that in order to find an answer to Schenzel's Question 1.2, it suffices to prove that t_I is bounded above by a linear function whenever I is a prime ideal of dimension one in a complete local domain.

Note that none of the reductions so far have increased the dimension of the ring. In the following we will reduce to the case when R is a d-dimensional complete local hypersurface domain of a special form, where d is again at most the dimension of the original ring.

We first need a definition and some lemmas. The two lemmas enable passing back and forth between finite domain extensions.

Definition 2.6: *Let I, J be ideals in a ring R. For each $n \geq 1$, let $t_{I,J}(n)$ be the least integer such that $I^{t_{I,J}(n)} : J^\infty \subseteq I^n$. If there is no such integer, set $t_{I,J}(n) = \infty$.*

This definition differs from Definition 1.3 in that it allows R to be non-local and J to be different from the maximal ideal of R. The proofs below of the main results necessitate this generalization.

Lemma 2.7: *Let S be a module-finite domain extension of a local Noetherian domain (R, m). Let I and J be ideals in R. Suppose that for all prime ideals Q in S which are minimal over IS, $t_{Q,JS}$ is bounded above by a linear function. Then $t_{I,J}$ is also bounded above by a linear function.*

Proof: Let $Q_1, \ldots, Q_r$ be all the prime ideals in S which are minimal over IS. Let l' be such that $(Q_1 \cdots Q_r)^{l'} \subseteq IS$. Let t be a linear function which bounds above all the $t_{Q_i,mS}$. By [Sw, Theorem 4.1], there exists an integer l such that for all $n \geq 1$,

$$Q_1^{ln} \cap \cdots \cap Q_r^{ln} \subseteq (Q_1 \cdots Q_r)^{l'n}.$$

Hence for all $n \geq 1$, $Q_1^{ln} \cap \cdots \cap Q_r^{ln} \subseteq I^n S$. By the Artin-Rees Lemma there exists an integer k such that for all $n \geq 1$, $I^n S \cap R \subseteq I^{n-k}$. Thus for all $n \geq 1$,

$$\begin{aligned} I^{t(l(n+k))} :_R J^\infty &\subseteq \left(I^{t(l(n+k))} S :_S (JS)^\infty\right) \cap R \\ &\subseteq \left(\cap Q_i^{t(l(n+k))} :_S (JS)^\infty\right) \cap R \\ &\subseteq \left(\cap Q_i^{l(n+k)}\right) \cap R \\ &\subseteq I^{n+k} S \cap R \\ &\subseteq I^n, \end{aligned}$$

which proves that for all $n \geq 1$, $t_{I,J}(n) \leq t(l(n+k))$. ∎

Lemma 2.8: *Let S be a module-finite domain extension of a local Noetherian domain (R, m). Let Q be an ideal in S and I, J ideals in R such that Q is the radical of IS and such that $t_{I,J}$ is bounded above by a linear function. Then $t_{Q,JS}$ is also bounded above by a linear function.*

Proof: By assumption there exists an integer l such that $Q^l \subseteq IS$. S is torsion-free over R, so there exists an integer r such that R^r is an R-submodule of S and S/R^r is a torsion R-module. In other words, there exists a nonzero element c in R such that $cS \subseteq R^r$.

By the Artin-Rees Lemma, there exists an integer k such that for all integers $n \geq k$,

$I^n S :_S c \subseteq I^{n-k} S$. Then

$$\begin{aligned} Q^{lt_{I,J}(n+k)} :_S J^\infty &\subseteq I^{t_{I,J}(n+k)} S :_S J^\infty \\ &\subseteq \left(I^{t_{I,J}(n+k)} R^r :_{R^r} J^\infty \right) :_S c \\ &\subseteq I^{n+k} R^r :_S c \\ &\subseteq I^{n+k} S :_S c \\ &\subseteq I^n S \\ &\subseteq Q^n S. \end{aligned}$$

Hence $t_{Q,JS}(n) \leq lt_{I,J}(n+k)$. ■

Finally, we reduce to the hypersurface case:

We start with an arbitrary complete local domain (R, m) and a prime ideal I in R of dimension one. The goal is to prove that t_I is bounded above by a linear function.

Let V be the coefficient ring of R. So V is either a discrete valuation ring or a field. In case V is a discrete valuation ring, let p be its uniformizing parameter. Choose elements $x_1, \ldots, x_{d-1}$ in I which are part of a system of parameters. In case V is a discrete valuation ring and $p \in I$, let $x_1 = p$. Choose $x_d \in I$ such that the radical of the ideal $(x_1, \ldots, x_d)$ is I. Also, choose u such that $x_1, \ldots, x_{d-1}, u$ are a system of parameters in R. In case V is a discrete valuation ring and p is not an element of I, set $u = p$.

Let $B = V[[x_1, \ldots, x_d, u]]$ be the subring of R generated by the x_i and u. By Cohen's Structure Theorem, R is module-finite over B and B is module-finite over the d-dimensional regular local ring $V[[x_1, \ldots, x_{d-1}, u]]$. Thus B is actually a hypersurface ring of the form $V[[x_1, \ldots, x_{d-1}, X_d, u]]/(g)$, where X_d is a variable and g is a monic polynomial in X_d (and a power series in the rest of the variables).

Let $J = (x_1, \ldots, x_d)B$. Then J is a prime ideal in B as $J = I \cap B$. As I is the radical of JR, by Lemma 2.8, in order to answer Schenzel's question, it suffices to prove that t_J is bounded above by a linear function.

The preceeding discussion just proved:

Theorem 2.9: *Assume that $t_{IA/(g)}$ is bounded above by a linear function whenever*

(i) $(A, (x_1, \ldots, x_d, u))$ is a complete regular local ring of dimension $d+1$,

(ii) g is a prime element of the ideal $I = (x_1, \ldots, x_d)$,

(iii) g is, as a function of x_d, a monic polynomial,

(iv) the prime ideal $IA/(g)$ is minimal over $(x_1, \ldots, x_{d-1})A/(g)$.

Then whenever I is an ideal in an arbitrary Noetherian local ring (R, m) with the *property that the I-adic topology is equivalent to the topology defined by the ideals $\{I^n : m^\infty\}_n$, then t_I is also bounded above by a linear function.* ■

Note that in this theorem, $IA/(g)$ is minimal over the ideal $(x_1, \ldots, x_{d-1})$, where the elements $x_1, \ldots, x_{d-1}$ generate a regular sequence in the ring.

This last lemma was not completely necessary, but it provides a neater setup in the proof of the main theorem in the next section.

Note that the existence of an ideal I in (R, m) whose I-adic topology is equivalent to the topology defined by $\{I^n : m^\infty\}$ implies that I is not m-primary and hence the dimension of the ring is at least one. If the dimension is one, then I must have height zero and by Theorem 2.2 it must be contained in all the associated primes of R. But then I is nilpotent, hence t_I is eventually a constant function. Thus it suffices to analyze rings of dimension at least two.

3. The main results

This section affirmatively answers Schenzel's question 1.2 (see Theorem 3.1), and also contains some generalizations. Theorem 3.4 contains a related "linear" result.

Theorem 3.1: *Let (R, m) be a Noetherian local ring (R, m) and I an ideal in R such that the I-adic topology is equivalent to the topology defined by the ideals $\{I^n : m^\infty\}_n$. Then there exists an integer k such that for all $n \geq 1$,*

$$I^{kn} : m^\infty \subseteq I^n.$$

Proof: By the definition of t_I (see Definitions 1.3 and 2.6), it suffices to prove that t_I is bounded above by a linear function. By Theorem 2.9 and the discussion preceeding it, we may assume that $R = A/(g)$, where

(i) A is a complete regular local ring of dimension $d+1$ and with maximal ideal $(x_1, \ldots, x_d, u)$,

(ii) g is a prime element of the prime ideal $(x_1, \ldots, x_d)A$,

(iii) g is, as a function of x_d, a monic polynomial,

(iv) $I = (x_1, \ldots, x_d)R$ is a prime ideal in R minimal over the complete intersection ideal $(x_1, \ldots, x_{d-1})R$,

(v) R is module-finite over the regular local ring $B = k[[x_1, \ldots, x_{d-1}, u]]$, where k is a coefficient subring of R. In case k is discrete valuation ring with uniformizing parameter p, then p is either x_1 or u.

With this notation, t_I equals $t_{I,uR}$. Set J to be the ideal $(x_1, \ldots, x_{d-1})B$.

Let R' be the integral closure of R inside the normal closure of the quotient field of R over the quotient field of B. Then R' is still a complete local Noetherian ring, module-finite over R and B. Every prime ideal in R' minimal over IR' is also minimal over JR'. By Lemma 2.7, it suffices to prove that $t_{Q,uR'}$ is bounded above by a linear function for all Q minimal over JR'.

Let R'' be the integral closure of B in the separable closure of the quotient field of B inside the quotient field of R'. Then R'' is also a complete Noetherian local ring, module finite and Galois over B, and R'' is contained in R'. Moreover, the extension $R'' \subseteq R'$ is module-finite and purely inseparable, thus there is a one-to-one correspondence between prime ideals in R'' and prime ideals in R'. Let Q be a prime in R' minimal over JR' and let $P = Q \cap R''$. Then Q is the radical of PR'. By Lemma 2.8, if $t_{P,uR''}$ is bounded above by a linear function, then so is $t_{Q,uR'}$.

Thus without loss of generality it suffices to prove that whenever R'' is an integrally closed complete local ring with maximal ideal m, which is a module-finite Galois extension of a complete d-dimensional regular local ring $(B, (x_1, \ldots, x_{d-1}, u))$, and P is a prime ideal in R'' minimal over JR'', where $J = (x_1, \ldots, x_{d-1})B$, then $t_{P,uR''}$ is bounded above by a linear function.

Let $P = P_1, P_2, \ldots, P_s$ be all the primes in R'' minimal over JR''. Let $p_{i,n}$ be the P_i-primary component of $J^n R''$. Then $p_{1,n} \cap \cdots \cap p_{s,n} = J^n R'' : m^\infty$. By Theorem 2.1, there exists an integer q such that for all $n \geq q$, $J^n R'' : m^\infty \subseteq J^{n-q} R''$. Thus for all $n \geq q$,

$$p_{1,n} \cdots p_{s,n} \subseteq p_{1,n} \cap \cdots \cap p_{s,n} \subseteq J^{n-q} R''.$$

If v is an arbitrary valuation on the quotient field of R'' which is non-negative on R'', then

$$v(p_{1,n} \cdots p_{s,n}) \geq v(J^{n-q}) = (n-q)v(J),$$

so that there exists an integer $m \in \{1, \ldots, s\}$ such that

$$v(p_{m,n}) \geq \frac{n-q}{s} v(J).$$

Now we need some basic facts about Rees valuations. More details can be found in [Re1]. For the ideal $M = (x_1, \ldots, x_{d-1}, u)B$ there exist finitely many valuations $v_1, \ldots, v_r$ on the quotient field of R'', all centered on the maximal ideal m of R'', such that for all integers $n \geq 1$, an element x of R'' lies in the integral closure of $M^n R''$ if and only if for all $i = 1, \dots, r$, $v_i(x) \geq nv_i(M)$. We also need the following remarkable result of David Rees (see (E), page 409 in [Re2]), based on Lipman's [L]: for all $i, j \in \{1, \dots, r\}$, there exist positive integers C_{ij} such that for all nonzero x in R'',

$$v_i(x) \leq C_{ij} v_j(x).$$

Let C be the maximum of all the C_{ij}. Then by above for each positive integer n and each $i = 1, \dots, r$, there exists an integer $m \in \{1, \dots, s\}$, such that

$$v_i(p_{m,sn+q}) \geq nv_i(J) \geq nv_i(M).$$

Thus for all $j \in \{1, \dots, r\}$,

$$v_j(p_{m,sn+q}) \geq \frac{1}{C} v_i(p_{m,sn+q}) \geq \frac{n}{C} v_i(M) \geq \frac{n}{C^2} v_j(M).$$

Since $v_1, \dots, v_r$ are all the Rees valuations of MR'', this means that $p_{m,sn+q}$ is contained in the integral closure of $M^{[\frac{n}{C^2}]}$. As R'' is analytically unramified, by Rees [Re3] there exists an integer l such that for all $n \geq 1$, the integral closure of M^n is contained in M^{n-l}. Thus the above proves that for all $n \geq 1$, there exists an integer m such that $p_{m,sn+q}$ is contained in $M^{[\frac{n}{C^2}]-l}$.

Let G be the Galois group of the extension $B \subseteq R''$. Note that G acts transitively on each of the sets $\{p_{1n}, \dots, p_{sn}\}$, where $n = 1, 2, \dots$. Let σ be an element in the Galois group which maps $p_{m,sn+q}$ isomorphically to $p_{1,sn+t}$. As M is generated by elements of B, σ leaves any power of M unchanged. This means that for all integers $n \geq 1$, $p_{1,sn+t} \subseteq M^{[\frac{n}{C^2}]-l}$. In other words, for all $n \geq 1$, p_{1,sC^2n+t} is contained in M^{n-l}, or better yet, $p_{1,sC^2(n+l)+t}$ is contained in M^n.

Set $P = P_1$. Let p be a positive integer such that P_1^p is contained in $p_{1,1}$. Then for all integers $n \geq 1$, P^{pn} is contained in $p_{1,n}$ and hence also $P^{pn} : (uR'')^\infty$ is contained in $p_{1,n}$. Thus for all $n \geq 1$,

$$P^{p(sC^2(n+l)+q)} : (uR'')^\infty \subseteq p_{1,sC^2(n+l)+q} \subseteq M^n.$$

Finally, by [Sw, Theorem 3.4], there exists an integer m such that for all $n \geq 1$, $P^n = (P^n : (uR'')^\infty) \cap (P^n + M^{nm})$. Here, $P^n : (uR'')^\infty$ is the P-primary component of P^n and $P^n + M^{nm}$ is the (possibly redundant and definitely non-unique) zero-dimensional primary component. Hence for all $n \geq 1$,

$$P^{p(sC^2(mn+l)+q)} : (uR'')^\infty \subseteq (P^n : (uR'')^\infty) \cap M^{mn} \subseteq P^n,$$

which proves the theorem. ■

Thus Theorem 2.2 and Theorem 3.1 combine to give

Theorem 3.2: *Let (R, m) be a Noetherian local ring and I an ideal in R. Then the following are equivalent:*

(i) $\{I^n : m^\infty\}_{n\in\mathbb{N}}$ *is equivalent to the I-adic topology.*

(ii) $\cap_{n\in\mathbb{N}}\left(I^n\widehat{R} : m\widehat{R}^\infty\right) = 0$, *where $\widehat{R}$ denotes the m-adic completion of R.*

(iii) $ht\,(I\widehat{R} + p/p) < dim\,(\widehat{R}/p)$ *for all $p \in Ass\,\widehat{R}$.*

(iv) There exists an integer k such that for all $n \geq 1$, $I^{kn} : m^\infty \subseteq I^n$. ■

By localizing at appropriate finitely many primes we also get more generally:

Main theorem 3.3: *(cf. [Sch2, Theorem 3.2]) Let R be a Noetherian local ring and I and J ideals in R. Then the following are equivalent:*

(i) $\{I^n : J^\infty\}_{n\in\mathbb{N}}$ *is equivalent to the I-adic topology.*

(ii) for all $P \in \cup_j Ass\,(R/I^j))$, $\{I^nR_P : JR_P^\infty\}_n$ *is equivalent to the IR_P-adic topology.*

(iii) for all $P \in V(J) \cap \left(\cup_j Ass\,(R/I^j)\right)$, $\{I^nR_P : JR_P^\infty\}_n$ *is equivalent to the IR_P-adic topology.*

(iv) for all $P \in V(J) \cap \left(\cup_j Ass\,(R/I^j)\right)$, $\{I^nR_P : PR_P^\infty\}_n$ *is equivalent to the IR_P-adic topology.*

(v) for all $P \in V(J) \cap \left(\cup_j Ass\,(R/I^j)\right)$, $\cap_{n\in\mathbb{N}}\left(I^n\widehat{R_P} : P\widehat{R_P}^{\infty}\right) = 0$, *where $\widehat{R_P}$ denotes the P-adic completion of R_P.*

(vi) for all $P \in V(J) \cap \left(\cup_j Ass\,(R/I^j)\right)$, $ht\,(I\widehat{R_P} + p/p) < dim\,(\widehat{R_P}/p)$ *for all $p \in Ass\,\widehat{R_P}$.*

(vii) there exists an integer k such that for all $n \geq 1$ and all $P \in V(J) \cap \left(\cup_j Ass\,(R/I^j)\right)$, $I^{kn}R_P : P^\infty \subseteq I^nR_P$.

(viii) there exists an integer k such that for all $n \geq 1$ and all $P \in V(J) \cap \left(\cup_j Ass\,(R/I^j)\right)$, $I^{kn}R_P : J^\infty \subseteq I^nR_P$.

(ix) there exists an integer k such that for all $n \geq 1$, $I^{kn} : J^\infty \subseteq I^n$.

Proof: (ix) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ (iv) as $J \subseteq P$. (iv), (v), (vi) and (vii) are equivalent by the previous theorem and by Ratliff's result [Ra1] that $\cup_j \mathrm{Ass}\,(R/I^j)$ is a finite set.

(vii) $\Rightarrow$ (viii) is obvious if P is minimal in the set $V(J) \cap \left(\cup_j \mathrm{Ass}\,(R/I^j)\right)$. In that case $I^nR_P : P^\infty = I^nR_P : J^\infty$ for all $n \geq 1$. Now assume that P is not minimal in $V(J) \cap \left(\cup_j \mathrm{Ass}\,(R/I^j)\right)$. Without loss of generality we may localize at P to assume that P is the only maximal ideal of R. By induction on the length of a chain of primes in $V(J) \cap \left(\cup_j \mathrm{Ass}\,(R/I^j)\right)$, we may assume that for all Q in $\left(V(J) \cap \left(\cup_j \mathrm{Ass}\,(R/I^j)\right)\right) \setminus \{P\}$

there exists an integer k' such that for all $n \geq 1$, $I^{k'n} R_Q : J^\infty \subseteq I^n R_Q$. Then

$$\begin{aligned} I^{k'kn} : J^\infty &\subseteq \bigcap_Q \left(I^{kn} R_Q \cap R\right) \\ &= \text{intersection of all the non-}P\text{-primary components of } I^{kn} \\ &= I^{kn} : P^\infty \\ &\subseteq I^n, \end{aligned}$$

which proves (vii) $\Rightarrow$ (viii).

To prove (viii) $\Rightarrow$ (ix), it suffices to prove that $I^{kn} : J^\infty \subseteq I^n$ after localization at all $P \in \cup_j \text{Ass}\,(R/I^j)$. If J is not contained in P, this holds trivially, and if J is contained in P, then it holds by assumption. ■

This last theorem obviously implies Theorem 3.1.

The following result is also in the same "linear" spirit. It implies, vaguely speaking, that the degrees of the zero-divisors in the graded ring $\oplus_n \frac{m^n}{m^{n+1}}$ behave rather nicely.

Theorem 3.4: *Let (R, m) be an analytically irreducible Noetherian local ring. Then there exists an integer k such that for all positive integers n and all elements α and β of R whose product lies in m^{kn}, either α or β lies in m^n.*

Proof: Without loss of generality we may pass to the completion of R. Let $v_1, \ldots, v_r$ be the Rees valuations of m. As in the proof of Theorem 3.1, by Rees' [Re2, page 409] there exists a positive integer C such that for all nonzero x in R, and all $i, j \in \{1, \ldots, r\}$,

$$v_i(x) \leq C v_j(x).$$

Also, by Rees [Re3] there exists an integer l such that for all $n \geq 1$, the integral closure of m^n is contained in m^{n-l}. Now let $k = 2C^2(l+1)$.

By assumption $v_1(\alpha) + v_1(\beta) = v_1(\alpha\beta) \geq knv_1(m) = 2C^2(l+1)nv_1(m)$. It follows that, say, $v_1(\alpha) \geq C^2(l+1)nv_1(m)$. Hence for all $i \in \{1, \ldots, r\}$,

$$v_i(\alpha) \geq \frac{1}{C} v_1(\alpha) \geq C(l+1)nv_1(m) \geq (l+1)nv_i(m).$$

This means that α lies in the integral closure of $m^{(l+1)n}$, and by the choice of l, α lies in m^n. ■

With some work, this last theorem and Theorem 2.9 imply the Main Theorem 3.3. The main ingredient in the proofs of the theorem above and the main theorem was Rees' result on linear relations between the various valuations centered on the maximal ideal.

4. Explicit calculations of linear equivalence

In this section I calculate for various prime ideals P an explicit integer k such that for all $n \geq 1$, the knth symbolic power $P^{(kn)}$ of P is contained in P^n. In general it is difficult to calculate this k following the proof of existence from the previous section, as in the process one may need to find some of the following: rank of a module-finite extension, torsion elements, Artin-Rees constants, a primary decomposition constant from [Sw,Theorem 3.4], and possibly more. All these calculations tend to be difficult.

Explicit calculations are given here for certain primes corresponding to monomial curves (Example 1), for primes corresponding to points on nonsingular elliptic cubic curves in the projective plane (Example 2), and for the Nagata-Roberts ideals (Example 3). All these prime ideals were chosen because their symbolic Rees algebras $\oplus_n P^{(n)}$ are NOT Noetherian. The point is that for a prime ideal whose symbolic Rees algebra IS Noetherian, the existence of k (but possibly not the calculation) is a trivial matter: take k to be the largest integer such that $\oplus_n P^{(n)}$ has an algebra generator over R in that degree.

Example 4.1: The first example analyzes some binomial primes in $F[[X,Y,Z]]$, where F is a field and X, Y and Z are variables over F corresponding to some monomial curves. Namely, let m be an arbitrary integer bigger than or equal to 4 and not divisible by 3, and let P be the kernel of the homomorphism from $F[[X,Y,Z]]$ to $F[[T]]$ which takes X to T^{7m-3}, Y to $T^{(5m-2)m}$, and Z to T^{8m-3}. Goto, Nishida and Watanabe proved in [GNW] that the symbolic Rees algebra $\oplus_n P^{(n)}$ is a Noetherian ring if and only if the characteristic of F is positive.

I prove that, independently of the characteristic and of m, for all $n \geq 1$, $P^{(3n)} \subseteq P^n$. All the hard work for this proof has already been done by Herzog in [He] and Kunz in [Ku]. Namely, Herzog proved that P equals $(X^{3m-1} - YZ^{2m-1}, Y^3 - X^m Z^m, Z^{3m-1} - X^{2m-1}Y^2)$ and that P is a set-theoretic complete intersection. Kunz showed in [Ku, page 139] more specifically that there exists a polynomial f in $F[X,Y,Z]$ such that $P = \sqrt{(f, Y^3 - X^m Z^m)}$ and that

$$(X^{3m-1} - YZ^{2m-1})^3 \equiv -X^{m+1} f \bmod (Y^3 - X^m Z^m),$$
$$(Z^{3m-1} - X^{2m-1}Y^2)^3 \equiv Z^{3m-1} f \bmod (Y^3 - X^m Z^m).$$

Note that $f, Y^3 - X^m Z^m, Z$ and $f, Y^3 - X^m Z^m, Y$ are two regular sequences. Then it is easy to deduce from the relations

$$Z^m(X^{3m-1} - YZ^{2m-1}) + X^{2m-1}(Y^3 - X^m Z^m) + Y(Z^{3m-1} - X^{2m-1}Y^2) = 0,$$
$$Y^2(X^{3m-1} - YZ^{2m-1}) + Z^{2m-1}(Y^3 - X^m Z^m) + X^m(Z^{3m-1} - X^{2m-1}Y^2) = 0,$$

(by multiplying the first one by $(Z^{3m-1} - X^{2m-1}Y^2)^2$, the second one by $(X^{3m-1} - YZ^{2m-1})^2$) that actually $P^3 \subseteq (f, Y^3 - X^m Z^m)$. Hence for all $n \geq 1$,

$$P^{(3n)} = P^{3n} : (X,Y,Z)^\infty \subseteq (f, Y^3 - X^m Z^m)^n : (X,Y,Z)^\infty = (f, Y^3 - X^m Z^m)^n \subseteq P^n. \quad \blacksquare$$

Example 4.2: The second example is about prime ideals corresponding to points on nonsingular elliptic cubic curves in the projective plane. Rees proved in [Re4] that when a prime ideal corresponds to a non-torsion point on such a cubic, the corresponding symbolic Rees algebra is not Noetherian. I prove that for every such prime ideal P and all $n \geq 1$, $P^{(18n)} \subseteq P^n$. I suspect that 18 is overly generous. In fact, the proof below shows that in many cases 3 or 9 work. I did some explicit calculations on the computer algebra system Macaulay, but I quickly ran into computer limitations with the degrees of polynomials in higher symbolic powers.

So let C be a nonsingular elliptic cubic curve in the projective plane and $R = \mathbb{C}[x,y,z]/(f)$ the corresponding graded affine ring. Then for any point $p = (x_0 : y_0 : z_0)$ on C we naturally associate a prime ideal P in R of height one generated by the linear forms $x_0 y - y_0 x, x_0 z - z_0 x, y_0 z - z_0 y$, two of which suffice. For example, Schenzel [Sch2, page 145] gives the example $R = \mathbb{C}[x,y,z]/(x^2 z + xz^2 + yz^2 - y^3)$, for which the point $p = (0:0:1)$ corresponds to the prime ideal $P = (x,y)$.

Now let p be an arbitrary point of C and let P be the corresponding prime ideal in R generated by two linear forms. By a linear change of coordinates we may assume that $P = (x,y)$. Similarly, by the Noether Normalization Lemma (or even the Cohen Structure Theorem for graded affine rings), we may assume that two linear forms, x and z, satisfy that $\mathbb{C}[x,z] \subseteq R$ is a module-finite graded extension. The element $y \in R$ then satisfies an irreducible integral equation over $\mathbb{C}[x,z]$ with the constant coefficient lying in the ideal (x). This integral equation necessarily equals f. Note that as f has degree three, xR has one, two, or three primary components, all minimal over xR, and one of them being P-primary.

If the number of components is one, necessarily f is of the form $y^3 +$ multiple of x, so $P^3 \subseteq xR$. Thus by Theorem 2.1 for all $n \geq 1$,

$$P^{(3n)} = P^{3n} : m^\infty \subseteq x^n R : m^\infty = x^n R \subseteq P^n R.$$

Now assume that there is more than one component of xR. Let y_2 be a root of f other than y. Write $f = y^3 + a_1 y^2 + a_2 y + a_3$, where each a_i is a form in $\mathbb{C}[x,z]$ of degree i and a_3 is a multiple of x. It is easy to see that y_2 and the third root of f satisfy the monic polynomial $W^2 + (a_1 + y)W + a_2 + a_1 y + y^2$ in indeterminate W over R. By the nonsingularity assumption, R is integrally closed in its field of fractions, so that if y_2 satisfies a nontrivial polynomial over R of order one, y_2 necessarily lies in R. It follows that either y_2 lies in R or else it satisfies no polynomials over R of order one. Set S to be the ring $R[y_2]$. Then by the quadratic formula S contains also the third root of f. When y_2 is not in R, by above then S is of the form $S = \mathbb{C}[x, y, z, y_2]/(f, g)$, where $g = y_2^2 + (a_1 + y)y_2 + a_2 + a_1 y + y^2$ and, by abuse of notation, we now think of x, y, z and y_2 as variables over $\mathbb{C}$.

Thus regardless of whether y_2 lies in R, S is a Cohen-Macaulay ring with x being a nonzerodivisor. Hence all the associated prime ideals of powers of xS are the primes minimal over xS.

We first assume that $S = R[y_2]$ is different from R. Then by the structure of S,

$$\begin{aligned}\frac{S}{xS} &= \frac{\mathbb{C}[x, y, z, y_2]}{(x, f, g)} \\ &\cong \frac{\mathbb{C}[y, z, y_2]}{(y^3 + \overline{a_1} y^2 + \overline{a_2} y, y_2^2 + (\overline{a_1} + y)y_2 + \overline{a_2} + \overline{a_1} y + y^2)},\end{aligned}$$

where $\overline{a_i}$ denotes the image of a_i in $\mathbb{C}[x, z]/(x) \cong \mathbb{C}[z]$. Similarly,

$$\frac{S}{PS} = \frac{\mathbb{C}[x, y, z, y_2]}{(x, y, f, g)} \cong \frac{\mathbb{C}[z, y_2]}{(\text{image of } g)} = \frac{\mathbb{C}[z, y_2]}{(y_2^2 + \overline{a_1} y_2 + \overline{a_2})},$$

so there are at most two prime ideals in S lying over P.

Let Q be a prime ideal in S lying over P. We first analyze the case when $\overline{a_2}$ is nonzero. Then $y(y^2 + a_1 y + a_2) \in xS$ implies that the Q-primary component of xS contains y. Note that then Q is the Q-primary component of xS if and only if $y_2^2 + \overline{a_1} y_2 + \overline{a_2}$ factors into two distinct linear factors over $\mathbb{C}[z, y_2]$. In the remaining case when $y_2^2 + \overline{a_1} y_2 + \overline{a_2}$ is the square of a linear form in $\mathbb{C}[z, y_2]$, necessarily $\overline{a_1}^2 = 4\overline{a_2}$ and the linear factor is $y_2 + \overline{a_1}/2$. Then P has only one prime ideal $Q = (x, y, y_2 + a_1/2)$ in S lying over it and Q^2 is contained in the Q-primary component $(x, y, (y_2 + a_1/2)^2)$ of xS. By assumption that xR have at least two primary components, necessarily $\overline{a_1} \neq 0$ and

$$(x, f, g)\mathbb{C}[x, y, z, y_2] = (x, y(y + a_1/2)^2, y_2^2 + (a_1 + y)y_2 + {a_1}^2/4 + a_1 y + y^2),$$

so that the primary decomposition of xS equals

$$xS = (x, y, (y_2 + a_1/2)^2) \cap (x, (y + a_1/2)^2, y_2) \cap (x, (y + a_1/2)^2, y_2 + a_1 + y).$$

In particular, the number of primary components of xS is three.

If $\overline{a_2}$ is zero, however, the assumption that f have at least two factors modulo (x) implies that $\overline{a_1}$ is nonzero. Hence $(x,f,g)\mathbb{C}[x,y,z,y_2] = (x, y^2(y+a_1), y_2^2 + (a_1+y)y_2 + a_1y + y^2)$. It follows easily that y^2 is contained in the Q-primary component of xS and that Q is either the prime $Q = (x,y,y_2)$ or $Q = (x,y,y_2+a_1)$. From

$$(y+y_2)(y_2+a_1) = y_2^2 + (a_1+y)y_2 + a_1y \in (g,x,y^2)\mathbb{C}[x,y,z,y_2]$$

it follows that

$$xS = (x,y^2,y+y_2) \cap (x,y^2,y_2+a_1) \cap (x,y+a_1,y_2^2)$$

is the irredundant primary decomposition.

Thus in all the cases above under the assumption that xR have more than one component, for each prime Q in S lying over P, either Q^2 lies in the Q-primary component of xS and the number of primary components of xS is three or else Q is the Q-primary component of xS.

A similar analysis of the case when $y_2 \in R$ shows that when $\overline{a_2} \neq 0$, xR has at most three primary components and P is the P-primary component of xR. By the quadratic formula, y_2 equals one half of $-(a_1+y) + \sqrt{(a_1+y)^2 - 4(a_2+a_1y+y^2)}$. Under the assumption that $y_2 \in R$, $(a_1+y)^2 - 4(a_2+a_1y+y^2) = a_1^2 - 2a_1y - 4a_2$ is a perfect square in R. If we further assume that $\overline{a_2} = 0$ and that $\overline{a_1} \neq 0$, we get a contradiction. Thus $y_2 \in R$ and $\overline{a_2} = 0$ is impossible.

Let G be the Galois group of f, where f is regarded as a polynomial in y with coefficients in $\mathbb{C}[x,z]$. As G fixes S, it permutes the primary components of xS. Thus the number of components of xS is at most the order of the Galois group, and that is at most six. Say $xS = p_1 \cap \cdots \cap p_t$ is an irreducible primary decomposition. Then for all $n \geq 1$, $x^nS = p_1^{(n)} \cap \cdots \cap p_t^{(n)}$ is an irreducible primary decomposition, where $p_i^{(n)} = p_i^n S_{\sqrt{p_i}} \cap S$. Thus for all forms $\alpha \in p_i^{(n)}$, $\cup_{\sigma \in G}\sigma(\alpha)$ lies in x^nS. Note that each element of the Galois group preserves the degrees of elements so that α has degree at least $\lfloor \frac{n}{|G|} \rfloor \geq \lfloor \frac{n}{6} \rfloor$. Moreover, when $t = 3$, then actually α has degree at least $\lfloor \frac{n}{3} \rfloor$. In particular, if y_2 is in R, R is Galois over $\mathbb{C}[x,z]$ and the Galois group has order three.

First assume that Q is the Q-primary component of xS. Then $P^{(6n)}$ lies in $Q^{(6n)}$ which is the Q-primary component of $x^{6n}S$. Hence by above for all $n \geq 1$, elements of $P^{(6n)}$ have degree at least n.

Now assume that Q is not the Q-primary component of xS. Then Q^2 is contained in the Q-primary component and the number of primary components is three. Thus $P^{(6n)}$

lies in $Q^{(6n)}$ which lies in the Q-primary component of $x^{3n}S$ which by above lies in degrees at least n. Hence here also for all $n \geq 1$, elements of $P^{(6n)}$ have degree at least n.

Similarly, if y_2 is in R, for all $n \geq 1$, elements of $P^{(3n)}$ have degree at least n.

We are almost done with this example. First observe that for all $n \geq 1$,

$$P^{(n)} = P^n : m^\infty = P^n : z^\infty = P^n : z^{2n}.$$

The only equality to be proved is the last one. This follows from the observation that the ring $\oplus_n P^n/P^{n+1}$ is isomorphic to $\mathbb{C}[X,Y,Z]/(f^*)$, where X,Y,Z are indeterminates over $\mathbb{C}$ and f^* is the part of $f = f(x,y,z)$ of least degree in the two variables x and y (for details and more on hypersurface rings see for example [Hb2]). As $f \in P$, f^* has degree at least one in X,Y. Thus f^* is divisible by Z^l for l at most 2. The equality above then follows by examining zero divisors modulo higher powers of P.

Finally,

$$\begin{aligned}
P^{(18n)} &\subseteq P^{(n)} \cap (x,y,z)^{3n} \\
&\subseteq P^{(n)} \cap \left((x,y)^n + (z)^{2n}\right) \\
&= P^n + P^{(n)} \cap (z)^{2n} \\
&= P^n + z^{2n}\left(P^{(n)} : z^{2n}\right) \\
&= P^n + z^{2n}\left(P^n : z^{2n}\right) \\
&= P^n + P^n \cap (z)^{2n} \\
&= P^n.
\end{aligned}$$

Similarly, if y_2 is in R, for all $n \geq 1$, $P^{(9n)} \subseteq P^n$. ■

Example 4.3: The last examples I analyze are the Nagata-Roberts examples. Let F be a field, X,Y,Z indeterminates over F. Let I be the ideal in $R = F[X,Y,Z]$ corresponding to m generic lines through the origin, i.e., I is the intersection of m ideals, each of which is generated by two generic linear forms. Let $M = (X,Y,Z)$, and for each $n \geq 0$, define $I^{(n)} = I^n : M^\infty$. Nagata proved in [N] that if F is a sufficiently large field of characteristic zero, say if F is the complex numbers, and m is a perfect square bigger than or equal to 16, then $\oplus I^{(n)}$ is not Noetherian.

I prove below that for all I of the form above, namely for all $m \geq 1$, we have $I^{(mn)} \subseteq I^n$ for all $n \geq 1$. An analysis of the proof then implies that for Roberts' prime P, constructed in [Ro1], and for all $n \geq 1$, $P^{(mn)} \subseteq P^n$.

Lemma 4.4: *Let m be a positive integer and $I_1, I_2, \ldots, I_m$ ideals in R each of which is generated by two generic linear forms. Let $J = I_1 I_2 \cdots I_m$. Then for all $n \geq 1$,*

$$M^{mn} \cap I_1^n \cap I_2^n \cap \cdots \cap I_m^n \subseteq J^n.$$

Proof: If $m = 1$, there is nothing to show. Now let $m = 2$. By a change of coordinates we may assume that $I_1 = (X, Y)$, $I_2 = (Y, Z)$. Then

$$\begin{aligned} M^{2n} \cap I_1^n \cap I_2^n &= M^n I_1^n \cap M^n I_2^n \quad \text{by grading} \\ &= (X^a Y^b Z^c : a + b + c = 2n, a + b \geq n) \\ &\qquad \cap (X^a Y^b Z^c : a + b + c = 2n, b + c \geq n) \\ &= (X^a Y^b Z^c : a + b + c = 2n, a \leq n, c \leq n) \\ &\subseteq (X, Y)^n (Y, Z)^n \\ &= J^n. \end{aligned}$$

Now let $m > 2$. We introduce some notation: $L = I_3 I_4 \cdots I_m$, $J_1 = I_1 L$, $J_2 = I_2 L$. Then by induction on m,

$$M^{mn} \cap I_1^n \cap I_2^n \cap \cdots \cap I_m^n \subseteq M^{mn} \cap J_1^n \cap J_2^n.$$

Again by grading, this is contained in $M^n J_1^n \cap M^n J_2^n = M^n I_1^n L^n \cap M^n I_2^n L^n$. By a change of coordinates we may assume again that $I_1 = (X, Y)$, $I_2 = (Y, Z)$. Let $L^n = (a_1, \ldots, a_t)$, where the a_i are forms of degree $(m-2)n$. Let r be an element of $M^{mn} \cap I_1^n \cap I_2^n \cap \cdots \cap I_m^n$. Then we can write $r = \sum r_i a_i$, for some $r_i \in M^n I_1^n = (X, Y, Z)^n (X, Y)^n$. So $r_i = \sum r_{iabc} X^a Y^b Z^c$, where the sum is over all a, b, c such that $a + b + c = 2n$ and $a + b \geq n$. We now combine all terms in r_i for which $a \leq n$ into c_i and the rest equals $r_i - c_i = d_i X^n$. Then $r = \sum c_i a_i + X^n \sum d_i a_i$. Note that $c_i \in (X, Y)^n \cap (Y, Z)^n \cap (X, Y, Z)^{2n}$, so that by the case $m = 2$, $c_i \in (X, Y)^n (Y, Z)^n = I_1^n I_2^n$, hence $\sum c_i a_i \in J^n$. Thus it suffices to prove that $X^n \sum d_i a_i$ is an element of J^n. By construction, $X^n \sum d_i a_i = r - \sum c_i a_i$ lies in $M^n I_1^n L^n \cap M^n I_2^n L^n$. Hence $X^n \sum d_i a_i$ lies in $I_2^n = (Y, Z)^n$, so

$$\begin{aligned} \sum d_i a_i &\in I_2^n \cap L^n \cap M^{(m-1)n} \\ &\subseteq I_2^n \cap I_3^n \cap \cdots \cap I_m^n \cap M^{(m-1)n} \\ &\subseteq J_2^n \quad \text{by induction,} \end{aligned}$$

so that $X^n \sum d_i a_i$ lies in $X^n J_2^n \subseteq J^n$. ∎

With notation as in this lemma, Nagata's ideal I is just $I_1 \cap \cdots \cap I_m$. Then for all $n \geq 1$,

$$\begin{aligned} I^{(mn)} &= I_1^{mn} \cap \cdots \cap I_m^{mn} \\ &\subseteq M^{mn} \cap I_1^n \cap \cdots \cap I_m^n \\ &\subseteq (I_1 I_2 \cdots I_m)^n \\ &\subseteq I^n. \qquad \blacksquare \end{aligned}$$

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