EXISTENCE OF SOLUTIONS TO REACTION CROSS DIFFUSION SYSTEMS

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Abstract. Reaction cross diffusion systems are a two species generalization of the porous media equation. These systems play an important role in the mechanical modelling of living tissues and tumor growth. Due to their mixed parabolic-hyperbolic structure, even proving the existence of solutions to these equations is challenging. In this paper, we exploit the parabolic structure of the system to prove the strong compactness of the pressure gradient in L^2. The key ingredient is the energy dissipation relation, which along with some compensated compactness arguments, allows us to upgrade weak convergence to strong convergence. As a consequence of the pressure compactness, we are able to prove the existence of solutions in a general setting and pass to the Hele-Shaw/incompressible limit in any dimension.

1. Introduction

In this paper, we consider the following two species reaction cross diffusion system

$$
\begin{cases}
\partial_t \rho_1 - \nabla \cdot (\rho_1(\nabla p - V)) = \rho_1 F_{1,1}(p,n) + \rho_2 F_{1,2}(p,n), \\
\partial_t \rho_2 - \nabla \cdot (\rho_2(\nabla p - V)) = \rho_1 F_{2,1}(p,n) + \rho_2 F_{2,2}(p,n), \\
\rho p = z(\rho) + z^*(p), \\
\partial_t n - \alpha \Delta n = -n(c_1\rho_1 + c_2\rho_2),
\end{cases}
\tag{1.1}
$$

on the spacetime domain $Q_\infty := [0,\infty) \times \mathbb{R}^d$. The study of these systems has become extremely important in the modelling of tissue growth and cancer [BKMP03, PT08, RBE$^+$10] and has drawn substantial interest from the mathematical community [PQV14, PV15, GPŚG19, KT20, BCP20, BPPS19, JKT21, AKY14, BM14]. The equations models the growth and death of two populations of cells whose densities are given by ρ_1, ρ_2. The densities are linked through a convex energy z (and its convex dual z^*), which opposes the concentration of the total density $\rho = \rho_1 + \rho_2$. The energy induces a pressure function p, which dissipates energy by pushing the densities down ∇p. In addition, the densities flow along an external vector field V. The source terms that control the growth/death of the two populations depend on both the pressure and a nutrient variable n. The nutrient evolves through a coupled equation that accounts for both diffusion and consumption.

Throughout the paper, we assume that $V \in L^\infty_{\text{loc}}([0,\infty); L^2(\mathbb{R}^d))$ and $\nabla \cdot V \in L^\infty(Q_\infty)$. We will also have the following assumptions on the energy z:

(z1) $z : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, lower semicontinuous, and convex,
(z2) $z(a) = +\infty$ if $a < 0$ and $z(0) = 0$,
(z3) there exists $r > \min(1 - \frac{2}{d}, 0)$ such that $\limsup_{a \to 0^+} a^{-r} z(a) = 0$,

as well as the following assumptions on the source terms:

(F1) the $F_{i,j}$ are continuous on $\mathbb{R} \times [0,\infty)$ and uniformly bounded,
(F2) the cross terms $F_{1,2}, F_{2,1}$ are nonnegative.

In certain cases, we will need the additional assumption:

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(F3) for n fixed, $p \mapsto (F_{1,1}(p,n) + F_{2,1}(p,n))$ and $p \mapsto (F_{1,2}(p,n) + F_{2,2}(p,n))$ are decreasing.

Constructing weak solutions to the system (1.1) is challenging due to the highest order nonlinear terms $\rho_1 \nabla p, \rho_2 \nabla p$. Given a sequence of approximate solutions, one needs either strong convergence of the densities or of the pressure gradient to pass to the limit. Due to the hyperbolic character of the first two equations, the regularity of the individual densities need not improve over time. Furthermore, it is not clear if densities with BV initial data will remain BV in dimensions $d > 1$ (see [CFSS18] and [BPPS19] for results in one dimension). On the other hand, summing the first two equations, one sees that the pressure p and the *total* density ρ satisfy the parabolic equation

$$\partial_t \rho - \nabla \cdot (\rho(\nabla p - V)) = \rho_1 \big(F_{1,1}(p,n) + F_{2,1}(p,n)\big) + \rho_2 \big(F_{1,2}(p,n) + F_{2,2}(p,n)\big), \tag{1.2}$$

(note (1.2) needs to be coupled with the duality relation $\rho p = z(\rho) + z^*(p)$ in order to fully appreciate the parabolic structure). Hence, attacking the problem through the pressure appears to be more promising.

Indeed, recently, several authors have been able to construct solutions to certain cases of (1.1) by exploiting (1.2) to obtain strong convergence of the pressure gradient [GPŚG19, BCP20, LX21]. In [GPŚG19], the authors obtain precompactness of the pressure gradient via regularity, by using the parabolic structure to bound the pressure Laplacian in L^1. Combined with any amount of arbitrarily weak time regularity, this implies gradient compactness via the Aubin-Lions lemma. As it turns out, both space and time regularity can be problematic. It is not clear whether spatial regularity can hold without some structural assumptions on the sources terms $F_{i,j}$ or in the presence of a non-zero vector field V. Time regularity also becomes problematic in the (important) special case where the energy z enforces the incompressibility constraint $\rho \leq 1$. Indeed, in the incompressible case, the coupling between the total density ρ and the pressure p is degenerate and it is not clear how to convert time regularity for ρ (easy) into time regularity for p (hard). In [BCP20, LX21], the authors establish precompactness more directly (including the incompressible limit in [LX21]), but again under certain restrictive structural assumptions on the source terms.

In this paper, we establish the precompactness of the pressure gradient directly by exploiting the energy dissipation relation associated to (1.2). In order to explain our strategy more fully, we need to introduce a change of variables that will make our subsequent analysis easier. Thanks to the duality relation $\rho p = z(\rho) + z^*(p)$, the term $\rho \nabla p$ is equivalent to $\nabla z^*(p)$. This suggests the natural change of variables $q = z^*(p)$. Since the pressure is only relevant on the set $\rho > 0$, we can essentially treat z^* as a strictly increasing function. As a result, we can completely rewrite the system (1.1) and the parabolic equation (1.2) in terms of q instead of p (c.f. Section 2 and 5 for the rigorous justification). Doing so, we get the equivalent system

$$\begin{cases} \partial_t \rho_1 - \nabla \cdot (\frac{\rho_1}{\rho} \nabla q) + \nabla \cdot (\rho_1 V) = \rho_1 F_{1,1}\big((z^*)^{-1}(q), n\big) + \rho_2 F_{1,2}\big((z^*)^{-1}(q), n\big), \\ \partial_t \rho_2 - \nabla \cdot (\frac{\rho_2}{\rho} \nabla q) + \nabla \cdot (\rho_2 V) = \rho_1 F_{2,1}\big((z^*)^{-1}(q), n\big) + \rho_2 F_{2,2}\big((z^*)^{-1}(q), n\big), \\ \rho q = e(\rho) + e^*(q), \\ \partial_t n - \alpha \Delta n = -n(c_1 \rho_1 + c_2 \rho_2), \end{cases} \tag{1.3}$$

where e is the unique convex function such that

$$e(a) = \begin{cases} a z(a) - 2\int_0^a z(s)\, ds & \text{if } z(a) \neq +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

It is worth noting that the change of variables from p to q is essentially the reverse direction of Otto's celebrated interpretation of the porous media equation as a W^2 gradient flow [Ott01]. Indeed, the p variable can be interpreted as a Kantorovich potential for the quadratic optimal transport distance, while the q variable is instead the dual potential for an H^{-1} distance. While the optimal transport interpretation of the system is more physically natural, the linearity of the H^{-1} structure is advantageous for our arguments. Indeed, summing the first two equations of (1.3), we get a more linear analogue of (1.2):

$$\partial_t \rho - \Delta q + \nabla\cdot(\rho V) = \mu, \tag{1.4}$$

where we have defined $\mu := \rho_1\big(F_{1,1}\big((z^*)^{-1}(q), n\big) + F_{2,1}\big((z^*)^{-1}(q), n\big)\big) + \rho_2\big(F_{1,2}\big((z^*)^{-1}(q), n\big) + F_{2,2}\big((z^*)^{-1}(q), n\big)\big)$ for convenience.

Now we are ready to give an outline of our strategy. As we mentioned earlier, the key idea is to exploit the energy dissipation relation associated to (1.4). Given any test function $\omega \in W_c^{1,\infty}([0,\infty))$ that depends on time only, the energy dissipation relation states that

$$\int_{\mathbb{R}^d} \omega(0)e(\rho^0(x))\,dx = \int_{Q_\infty} -e(\rho)\partial_t\omega + \omega|\nabla q|^2 + \omega e^*(q)\nabla\cdot V - \omega\mu q. \tag{1.5}$$

where ρ^0 is the initial total density and we recall that $Q_\infty = [0,\infty)\times\mathbb{R}^d$ is the full space-time domain. Suppose we have a sequence (ρ_k, q_k, μ_k) of solutions to equation (1.4) with the same initial data ρ^0 that converges weakly to a limit point $(\bar\rho, \bar q, \bar\mu)$. Thanks to the linearity of (1.4), the limit point $(\bar\rho, \bar q, \bar\mu)$ will also be a solution of (1.4). As a result, if we also know that the relation $\bar\rho\bar q = e(\bar\rho) + e^*(\bar q)$ holds at the limit, then both (ρ_k, q_k, μ_k) and $(\bar\rho, \bar q, \bar\mu)$ satisfy the dissipation relation (1.5). Hence, we could conclude that

$$\begin{aligned}\int_{Q_\infty} &-e(\rho_k)\partial_t\omega + \omega|\nabla q_k|^2 + \omega e^*(q_k)\nabla\cdot V - \omega\mu_k q_k \\ &= \int_{Q_\infty} -e(\bar\rho)\partial_t\omega + \omega|\nabla\bar q|^2 + \omega e^*(\bar q)\nabla\cdot V - \omega\bar\mu\bar q.\end{aligned}$$

If we can prove that $\rho_k q_k, e(\rho_k), e^*(q_k)$ converge weakly to $\rho q, e(\bar\rho), e^*(\bar q)$ respectively and

$$\limsup_{k\to\infty}\int_{Q_\infty}\omega\mu_k q_k \le \int_{Q_\infty}\omega\bar\mu\bar q, \tag{1.6}$$

then we have the upper semicontinuity property

$$\limsup_{k\to\infty}\int_{Q_\infty}\omega|\nabla q_k|^2 \le \int_{Q_\infty}\omega|\nabla\bar q|^2 \tag{1.7}$$

which automatically implies that ∇q_k converges strongly in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$ to $\nabla\bar q$. As a result, the energy dissipation relation gives us a way to upgrade some weak convergence properties into strong gradient convergence.

Of course, in order to exploit this idea, we need:

(i) enough regularity to ensure that the dissipation relation (1.5) is valid,
(ii) enough compactness to prove the weak convergence of the nonlinear terms $\rho_k q_k, e(\rho_k), e^*(q_k)$,
(iii) enough compactness to verify the nonlinear limit (1.6).

The amount of a priori regularity needed for (i) is very low, thus, this point does not pose much of a problem. However, obtaining the compactness needed for points (ii) and (iii) is more delicate. Exploiting convex duality, the weak convergence of the energies $e(\rho_k), e^*(q_k)$ is essentially equivalent to the weak convergence of the product $\rho_k q_k$ (c.f. Proposition 3.2). While we may not know strong convergence of either ρ_k or q_k separately, we can still obtain the weak convergence of the product through compensated compactness arguments (c.f. Lemma 3.5). When e^* is strictly convex, the weak convergence of the energy $e^*(q_k)$ to $e^*(q)$ actually implies that q_k converges to q locally in measure. Thus, in this case, verifying the limit (1.6) becomes trivial. When the strict convexity of e^* fails, we will still be able to verify the limit (1.6) as long as we add the additional structural assumption (F3) on the source terms.

Once we have obtained the strong convergence of the pressure gradient, constructing solutions to the system (1.3) (and hence the system (1.1)) is straightforward via a vanishing viscosity approach (note adding viscosity to the system is compatible with our energy dissipation based argument). Furthermore, the above strategy works even when the energy is allowed to change along the approximating sequence. Hence, we can also use the above arguments to show that solutions to the system (1.1) with the porous media energy $z_m(a) = \frac{1}{m-1}a^m$ converge to the incompressible limit system with the energy $z_\infty(a) = 0$ if $a \in [0,1]$ and $+\infty$ otherwise.

1.1. **Main results.** For the reader's convenience, in this subsection, we collect some of our main results. To prevent the introduction from becoming too bloated, we shall state our results somewhat informally. The rigorous analogues of these results can be found in Section 5.

Our first result concerns the case where the density-pressure coupling is non-degenerate i.e. z is differentiable on $(0,\infty)$.

Theorem 1.1. *Suppose that z is an energy satisfying assumptions (z1-z3) such that $\partial z(a)$ is a singleton for all $a > 0$ and suppose that the source terms satisfy assumptions (F1-F2). Given initial data ρ_1^0, ρ_2^0, n^0 such that $e(\rho_1^0 + \rho_2^0) \in L^1(\mathbb{R}^d)$, there exists a weak solution (ρ_1, ρ_2, p, n) to the system (1.1).*

When the density-pressure coupling becomes degenerate, we need to add the additional assumption (F3) on the source terms.

Theorem 1.2. *Suppose that z is an energy satisfying assumptions (z1-z3) and suppose that the source terms satisfy assumptions (F1-F3). Given initial data ρ_1^0, ρ_2^0, n^0 such that $e(\rho_1^0 + \rho_2^0) \in L^1(\mathbb{R}^d)$, there exists a weak solution (ρ_1, ρ_2, p, n) to the system (1.1).*

In addition to our existence results, we also show that solutions of the system with the porous media energy $z_m(a) := \frac{1}{m-1}a^m$ converge to a solution of the system with the incompressible energy

$$z_\infty(a) := \begin{cases} 0 & \text{if } a \in [0,1] \\ +\infty & \text{otherwise} \end{cases}$$

as $m \to \infty$.

Theorem 1.3. *Let ρ_1^0, ρ_2^0, n^0 be initial data such that $\rho_1^0 + \rho_2^0 \leq 1$ almost everywhere. Suppose that the source terms satisfy (F1-F3). If $(\rho_{1,m}, \rho_{2,m}, p_m, n_m)$ is a sequence of solutions to the system (1.1) with the energy z_m and the fixed initial data $(\rho_1^0, \rho_2^0, n^0)$, then there exists a limit point of the sequence $(\rho_{1,\infty}, \rho_{2,\infty}, p_\infty, n_\infty)$ that solves the system (1.1) with the incompressible energy z_∞. Furthermore, the limiting pressure p_∞ satisfies the so called complementarity condition (c.f. equation (5.13)).*

Theorem 1.3 is just a special case of our more general convergence result, Theorem 5.5, which shows that one can extract limit solutions for essentially any reasonable sequence of energies. Nonetheless, the statement of Theorem 5.5 is a bit too complicated to be cleanly summarized in the introduction, so we leave it to be stated for the first time in Section 5.

1.2. **Limitations and other directions.** Unfortunately, our approach cannot handle the more challenging case where ρ_1, ρ_2 have different mobilities or where ρ_1, ρ_2 flow along different vector fields V_1, V_2. These cases are known to be extremely difficult, however see [KM18] and [KT20] for some partial results. When the mobilities are different, the analogue of (1.4) is a nonlinear parabolic equation with potentially discontinuous coefficients. As a result, one cannot do much with the limiting variables $\bar{\rho}, \bar{q}$. When the densities flow along different vector fields, verifying the upper semicontinuity property (1.7) requires proving the weak convergence of the terms $\rho_{1,k}\nabla q_k$

and $\rho_{2,k}\nabla q_k$. Since this essentially requires knowing strong compactness for ∇q_k in the first place, it completely defeats the purpose of the argument.

Nonetheless, it would be interesting to see if this strategy could be applied to other systems of equations that have some parabolic structure. For instance, if $\{L_{i,j}\}_{i,j\in\{1,2\}}$ are linear operators whose symbols are dominated by $(-\Delta)^{1/2}$ i.e. their Fourier transforms satisfy $\limsup_{|\xi|\to\infty}\frac{|\hat{L}_{i,j}(\xi)|}{|\xi|}=0$, then it should be possible to extend our arguments to the more general system

(1.8)
$$\begin{cases}\partial_t\rho_1-\nabla\cdot(\frac{\rho_1}{\rho}\nabla q)+\nabla\cdot(\rho_1 V)+L_{1,1}\rho_1+L_{1,2}\rho_2=\rho_1F_{1,1}\big((z^*)^{-1}(q),n\big)+\rho_2F_{1,2}\big((z^*)^{-1}(q),n\big),\\ \partial_t\rho_2-\nabla\cdot(\frac{\rho_2}{\rho}\nabla q)+\nabla\cdot(\rho_2 V)+L_{2,1}\rho_1+L_{2,2}\rho_2=\rho_1F_{2,1}\big((z^*)^{-1}(q),n\big)+\rho_2F_{2,2}\big((z^*)^{-1}(q),n\big),\\ \rho q=e(\rho)+e^*(q),\\ \partial_t n-\alpha\Delta n=-n(c_1\rho_1+c_2\rho_2),\end{cases}$$

(perhaps with some other mild requirements on the $L_{i,j}$). However, it is not so clear that this more general system models physically relevant phenomena, and hence, we will not pursue this line of inquiry further in this work.

1.3. **Paper outline.** The rest of the paper is organized as follows. In Section 2, we explore some of the consequences of the change of variables $q=z^*(p)$. After this Section, we will focus only on the transformed system (1.3) until Section 5. In Section 3, we provide some generic convex analysis and compensated compactness arguments needed for the weak convergence of the primal and dual energies. In Section 4, we analyze parabolic PDEs, establishing basic estimates and the energy dissipation relation. Finally, in Section 5, we combine our work to prove the main results of the paper.

2. The transformation $q=z^*(p)$

In this section, we will explore some of the consequences of the transformation $q=z^*(p)$. Note that the full verification of the equivalence between the systems (1.1) and (1.3) will not occur until the final section, Section 5. Before we begin our work in this section, let us give a bit more motivation for introducing this change of variables. First of all, the spatial derivative in the parabolic equation (1.4) is linear with respect to q, whereas the spatial derivative in parabolic equation for the p variable (1.2) is not. As a result, establishing the strong L^2 gradient compactness for q is simpler than for p. Furthermore, the q variable is always nonnegative, while certain choices of z will lead to a p variable that is not bounded from below. The lack of lower bounds on p leads to some very annoying integrability issues that are completely absent when one works with q instead.

We begin by establishing the fundamental properties of the transformation $q=z^*(p)$. In particular, we will show that the transformation is essentially invertible.

Lemma 2.1. *If z is an energy satisfying (z1-z3), then z^* is nonnegative, nondecreasing, and $(z^*)^{-1}$ is well defined and Lipschitz on $z^*(\mathbb{R})\cap(0,\infty)$. Furthermore, for any $\delta>0$, $(z^*)^{-1}$ is uniformly Lipschitz on $[\delta,\infty)$*

Proof. Given any $b\in\mathbb{R}$, we have
$$z^*(b)=\sup_{a\in\mathbb{R}} ab-z(a)\geq 0-z(0)=0.$$
It is also clear that $\inf\partial z^*(b)\geq 0$ since $z(a)=+\infty$ for any $a<0$. If $b_1<b_2$, then $z^*(b_2)-z^*(b_1)\geq a_1(b_2-b_1)\geq 0$ where a_1 is any element of $\partial z^*(b_1)$. Thus, z^* is both nonnegative and nondecreasing.

Since z is proper, we know that $z(a) \neq -\infty$ for all a. Thus given some $a_0 > 0$, there must exist some $b_0 \in \mathbb{R}$ such that $b_0 \leq \frac{z(a_0)}{a_0}$. It then follows that for all $a \geq a_0$

$$ab_0 - z(a) \leq ab_0 - z(a_0) - (a-a_0)\frac{z(a_0)}{a_0} = a(b_0 - \frac{z(a_0)}{a_0}) \leq 0.$$

Therefore, for all $b \leq b_0$

$$\sup_{a\in\mathbb{R}} ab - z(a) = \sup_{a\in[0,a_0]} ab - z(a).$$

Fix $\epsilon > 0$ and let $a_n \in [0, a_0]$ be a decreasing sequence such that $z^*(-n) \leq \epsilon - na_n - z(a_n)$ (note that from the above logic such choices of a_n must exist once n is sufficiently large). Since a_n is decreasing and bounded from below, it must converge to a limit point $\bar{a}$ as $n \to \infty$. Thus,

$$0 \leq \liminf_{n\to\infty} z^*(-n) \leq \epsilon - z(\bar{a}) - \limsup_{n\to\infty} na_n,$$

which immediately implies that $\bar{a} = 0$. We can then rewrite the above as

$$\liminf_{n\to\infty} z^*(-n) \leq \epsilon - \limsup_{n\to\infty} na_n \leq \epsilon.$$

Therefore, $\liminf_{n\to\infty} z^*(-n) = 0$.

It now follows that if $z^*(b) \in (0,\infty)$, then there must exist some $b_0 < b$ such that $2z^*(b_0) \leq z^*(b)$. We then have

$$\inf \partial z^*(b) \geq \frac{z^*(b)}{2(b-b_0)} > 0.$$

Thus, z^* is strictly increasing at b whenever $z^*(b) \in (0,\infty)$. Since z^* is convex, it follows that for any $\delta > 0$ $(z^*)^{-1}$ is uniformly Lipschitz on $[\delta,\infty)$. □

Perhaps the most significant aspect of the change of variables $q = z^*(p)$ is the change in the energy controlling the primal and dual coupling. Recall that we defined the new energy e through the formula

$$e(a) = \begin{cases} az(a) - 2\int_0^a z(s)\,ds & \text{if } \; z(a) \neq +\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

While this formula appears somewhat mysterious, e is the unique (up to an irrelevant constant factor) convex function such that $\partial e(a) = z^* \circ \partial z(a)$ when $\partial z(a) \neq \varnothing$. Thus, when $p \in \partial z(\rho)$ we will know that $q \in \partial e(\rho)$. Note that the monotonicity of z^* is key, otherwise e would fail to be convex. The following Lemma records the properties that e inherits from z.

Lemma 2.2. *Suppose that z is an energy satisfying (z1-z3). If we define $e : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ according to (2.1), then e satisfies the following properties*

(e1) $e : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous.

(e2) $e(a) = +\infty$ if $a < 0$, $e(0) = 0$, and e is increasing on $e^{-1}(\mathbb{R})$.

(e3) $\limsup_{a\to0^+} \frac{e(a)}{a} = 0$, $\liminf_{b\to\infty} \frac{e^*(b)}{b} > 0$ and there exists $\alpha > \max(1-\frac{2}{d}, 0)$ such that $\limsup_{a\to0^+} a^{-\alpha-1}e(a) = 0$.

Furthermore, if $a > 0$, we have

$$\partial e(a) = \{ab - z(a) : b \in \partial z(a)\} = \{z^*(b) : b \in \partial z(a)\},$$

and so $\partial e(a)$ is a singleton if and only if $\partial z(a)$ is a singleton.

Proof. It is clear that $e(0) = 0$ and $e(a) = +\infty$ if $z(a) = +\infty$.

Given any two points $a_0, a_1 \in z^{-1}(\mathbb{R})$, convexity implies that

$$2(a_1 - a_0)z(\frac{a_1+a_0}{2}) \leq 2\int_{a_0}^{a_1} z(s)\,ds \leq (a_1-a_0)(z(a_0)+z(a_1)). \tag{2.2}$$

Thus, if $z(a) \neq +\infty$, then

$$0 \leq e(a) \leq az(a) - 2az(\frac{a}{2}) < \infty.$$

Therefore $e(a) = +\infty$ if and only if $z(a) = +\infty$. Thus, the set $e^{-1}(\mathbb{R})$ is an interval. Furthermore, the above inequalities combined with $(z3)$ clearly imply that $\limsup_{a\to 0^+} a^{-\alpha-1}e(a) = 0$.

Again using (2.2),

$$e(a_1)-e(a_0) = a_0(z(a_1)-z(a_0))+(a_1-a_0)z(a_1)-2\int_{a_0}^{a_1} z(s)\,ds \geq a_0(z(a_1)-z(a_0))-(a_1-a_0)z(a_0)$$

If $b_0 \in \partial z(a_0)$, then

$$e(a_1) - e(a_0) \geq (a_1 - a_0)\big(a_0 b_0 - z(a_0)\big).$$

Thus, $b \in \partial z(a)$ implies that $ab - z(a) \in \partial e(a)$ whenever $a \in e^{-1}(\mathbb{R})$. Thus, the subdifferential of e is nonempty whenever the subdifferential of z is nonempty. Combining this with the equality $z^{-1}(\mathbb{R}) = e^{-1}(\mathbb{R})$, it follows that e is convex, lower semicontinuous and proper.

Note that $b \in \partial z(a)$ implies that $z^*(b) = ab - z(a)$. Therefore, $\{ab - z(a) : b \in \partial z(a)\} = \{z^*(b) : b \in \partial z(a)\}$. Since $\int_0^a z(s)\,ds$ is everywhere differentiable on the interior of $z^{-1}(\mathbb{R})$, every element of $\partial e(a)$ must have the form $ab - z(a)$ for $b \in \partial z(a)$. Convexity implies that $ab - z(a) \geq -z(0) = 0$, thus e is increasing on the interior $e^{-1}(\mathbb{R})$.

It remains to show that $\lim_{b\to\infty} \frac{e^*(b)}{b} > 0$. Since $\limsup_{a\to 0^+} \frac{e(a)}{a} = 0$, there must exist some $a_0 > 0$ such that $e(a_0) < \infty$. Thus,

$$\liminf_{b\to\infty} \frac{e^*(b)}{b} \geq \liminf_{b\to\infty} a_0 - \frac{e(a_0)}{b} = a_0.$$

□

Parameter	z energy $a \in [0,\infty)$	z^* energy $b \in \mathbb{R}$	e energy $a \in [0,\infty)$	e^* energy $b \in \mathbb{R}$
$m \in (0,\infty] \setminus \{1\}$	$\frac{1}{m-1}(a^m - a)$	$\max(\frac{(m-1)b+1}{m}, 0)^{m/(m-1)}$	$\frac{1}{m+1}a^{m+1}$	$\frac{m}{m+1}\max(b,0)^{\frac{m+1}{m}}$
$m \to 1$	$a\log(a) - a$	$\exp(b)$	$\frac{1}{2}a^2$	$\frac{1}{2}\max(b,0)^2$

Table 1. Some examples of the transformation from z to e.

Now that we have established properties of the transformation $q = z^*(p)$ we can temporarily forget about the original system (1.1) and focus on (1.3). We will eventually return to (1.1) in the final section, where we show that solutions to (1.3) can be transformed into solutions to (1.1). Until then, our efforts will be concentrated on establishing the energy dissipation strategy described in the introduction.

3. Convex analysis and compensated compactness

In this section, we collect some results that we will need to establish the weak convergence of the primal and dual energy terms. We begin by defining some convex spaces that we will work with throughout the paper.

Definition 3.1. Given an energy e satisfying (e1-e3), we define

$$X(e) := \{\rho \in L^\infty_{\mathrm{loc}}(Q_\infty) : e(\rho) \in L^\infty_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))\},$$

$$Y(e^*) := \{q \in L^2_{\mathrm{loc}}(Q_\infty) : e^*(q) \in L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))\}.$$

We are now ready to introduce a result that is one of the cornerstones of our argument.

Proposition 3.2. *Let $e : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ be an energy satisfying (e1−e3). Let $e_k : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ be a sequence of energies satisfying (e1-e3) such that e_k converges pointwise everywhere to e. Suppose we have a sequence of nonnegative density and pressure functions $\rho_k \in X(e_k)$, $q_k \in Y(e_k^*)$ such that $\rho_k q_k = e_k(\rho_k) + e_k^*(q_k)$ almost everywhere and ρ_k, q_k converge weakly in $L^1_{\mathrm{loc}}(Q_\infty)$ to limits $\rho, q \in L^1_{\mathrm{loc}}(Q_\infty)$ respectively. If $\rho q \in L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))$ and for every nonnegative $\varphi \in C_c^\infty(Q_\infty)$*

$$\limsup_{k\to\infty} \int_{Q_\infty} \varphi\rho_k q_k \leq \int_{Q_\infty} \varphi\rho q,$$

then $\rho \in X(e), q \in Y(e^*)$, $\rho q = e(\rho) + e^*(q)$ almost everywhere, and $\rho_k q_k, e_k(\rho_k), e_k^*(q_k)$ converge weakly in $L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))$ to $\rho q, e(\rho), e^*(q)$ respectively.

Proof. Given some nonnegative $\varphi \in C_c^\infty(Q_\infty)$, let D be a compact set containing the support of φ. From our assumptions, we have

$$\int_{Q_\infty} \varphi\rho q \geq \limsup_{k\to\infty} \int_{Q_\infty} \varphi\rho_k q_k = \limsup_{k\to\infty} \int_{Q_\infty} \varphi e_k(\rho_k) + \varphi e_k^*(q_k).$$

Fix some simple functions $g_1, g_2 \in L^\infty(D)$ such that every value of g_1 is a value where e_k^* converges to e^* (c.f. Lemma A.1). It then follows that

$$\limsup_{k\to\infty} \int_{Q_\infty} \varphi\big(e_k(\rho_k)+e_k^*(q_k)\big) \geq \limsup_{k\to\infty} \int_{Q_\infty} \varphi\big(g_1\rho_k - e_k^*(g_1) + g_2 q_k - e_k(g_2)\big) = \int_{Q_\infty} \varphi\big(g_1\rho - e^*(g_1) + g_2 q - e(g_2)\big).$$

Taking a supremum over g_1, g_2, we can conclude that

$$\int_{Q_\infty} \varphi\rho q \geq \limsup_{k\to\infty} \int_{Q_\infty} \varphi\big(e_k(\rho_k) + e_k^*(q_k)\big) \geq \int_{Q_\infty} \varphi\big(e(\rho) + e^*(q)\big).$$

On the other hand, Young's inequality immediately implies that

$$\rho q \leq e(\rho) + e^*(q)$$

almost everywhere. Thus, $\rho q = e(\rho) + e^*(q)$ almost everywhere. This also now implies that $\rho \in X(e)$ and $q \in Y(e^*)$.

The previous calculation shows that $e_k(\rho_k)+e_k^*(q_k)$ is uniformly bounded in $L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))$. Thus, for any time $T > 0$, there exists $w_1, w_2 \in C(Q_T)^*$ such that $e_k(\rho_k), e_k^*(q_k)$ converge (along a subsequence that we will not relabel) to w_1, w_2 respectively. Arguing as in the first paragraph, it follows that

$$\int_{Q_T} \varphi w_1 = \liminf_{k\to\infty} \int_{Q_T} \varphi e_k(\rho_k) \geq \int_{Q_T} \varphi e(\rho), \quad \int_{Q_T} \varphi w_2 = \liminf_{k\to\infty} \int_{Q_T} \varphi e_k^*(q_k) \geq \int_{Q_T} \varphi e^*(q).$$

Hence,

$$\int_{Q_T} \varphi|w_1 - e(\rho)| + \varphi|w_2 - e^*(q)| = \int_{Q_T} \varphi\big(w_1 - e(\rho) + w_2 - e^*(q)\big) =$$

$$\limsup_{k\to\infty} \int_{Q_T} \varphi\big(e_k(\rho_k) + e_k^*(q_k) - e(\rho) - e^*(q)\big) = \limsup_{k\to\infty} \int_{Q_T} \varphi\big(\rho_k q_k - \rho q\big) \leq 0.$$

Thus, $w_1 = e(\rho)$ and $w_2 = e^*(q)$. Since w_1, w_2 and $T > 0$ were arbitrary, it follows that $e(\rho), e^*(q)$ are the only weak limit points of $e_k(\rho_k), e_k^*(q_k)$ in $L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))$. Thus, the full sequences $e_k(\rho_k), e_k^*(q_k)$ must converge weakly in $L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))$ to $e(\rho)$ and $e^*(q)$ respectively. The weak $L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))$ convergence of $\rho_k q_k$ to ρq is an immediate consequence.

□

Knowing the weak convergence of the energy terms actually implies a certain limited strong convergence property that can be deduced from the convexity of e, e^*. When the energies are strictly convex, we will in fact have convergence in measure of ρ_k to ρ and q_k to q. When strict convexity fails, the convergence property will be weaker, but nonetheless will still be quite useful in our subsequent analysis.

Lemma 3.3. *Let $e : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be an energy satisfying (e1-e3). Let $e_k : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a sequence of energies satisfying (e1-e3) such that e_k converges pointwise everywhere to e. Suppose we have a sequence of uniformly bounded nonnegative density and pressure functions $\rho_k \in X(e_k)$, $q_k \in Y(e_k^*)$ such that $\rho_k q_k = e_k(\rho_k) + e_k^*(q_k)$ almost everywhere and ρ_k, q_k converge weakly in $L^1_{\text{loc}}(Q_\infty)$ to limits $\rho, q \in L^1_{\text{loc}}(Q_\infty)$ respectively. Given $\epsilon > 0$, we define*

$$A_{k,\epsilon} = \{(t, x) \in Q_\infty : [\rho_k(t, x) - \epsilon, \rho_k(t, x) + \epsilon] \cap \partial e^*(q(t, x)) = \varnothing\}$$

and

$$B_{k,\epsilon} = \{(t, x) \in Q_\infty : [q_k(t, x) - \epsilon, q_k(t, x) + \epsilon] \cap \partial e(\rho(t, x)) = \varnothing\}$$

If for every nonnegative $\varphi \in C_c^\infty(Q_\infty)$ we have $\limsup_{k\to\infty} \varphi\rho_k q_k \leq \int_{Q_\infty} \varphi\rho q$, then

$$\limsup_{k\to\infty} |D \cap A_{k,\epsilon}| + |D \cap B_{k,\epsilon}| = 0 \tag{3.1}$$

for any compact set $D \subset Q_\infty$.

Remark 3.4. If ∂e^* is always a singleton (i.e. e^* is everywhere differentiable, equivalently e is strictly convex), then the conditions $|\rho - \rho_k| > \epsilon$ and $[\rho_k - \epsilon, \rho_k + \epsilon] \cap \partial e^*(q) = \varnothing$ are equivalent. Thus, in this case, the vanishing of $A_{k,\epsilon}$ would imply that ρ_k converges in measure to ρ. The same holds for the convergence of q_k to q with the roles of the energies swapped (i.e. q_k will converge strongly in measure to q if e is everywhere differentiable or equivalently if e^* is strictly convex). Even when we do not have full convergence in measure, the above result can be useful to show that certain compositions $f \circ \rho_k, g \circ q_k$ converge to the correct limits $f \circ \rho, g \circ q$ (for well chosen functions $f, g : \mathbb{R} \to \mathbb{R}$).

Proof. Fix a compact set $D \subset Q_\infty$ and $\epsilon > 0$. From our assumptions and Proposition 3.2, it follows that

$$\limsup_{k\to\infty} \int_D e_k(\rho_k) - e(\rho) - q(\rho_k - \rho) + e_k^*(q_k) - e^*(q) - \rho(q_k - q) = 0.$$

Note that this line is nearly the sum of two Bregman divergences. Thus, we begin by making some manipulations to transform the quantity into a Bregman divergence.

Let $a_\infty = \sup\{a > 0 : e(a) < \infty\}$ and $b_\infty = \sup\{b > 0 : e^*(b) < \infty\}$. By Lemma A.1, for any $\delta > 0$, e_k converges uniformly to e on $[0, a_\infty - \delta]$ and e_k^* converges uniformly to e^* on $[0, b_\infty - \delta]$. If we define $\rho_{k,\delta} := \min\big(\rho_k, a_\infty - \delta\big)$, $q_{k,\delta} := \min\big(q_k, b_\infty - \delta\big)$, then from the above considerations, we have

$$\begin{aligned}\limsup_{k\to\infty} &\int_D e(\rho_{k,\delta}) - e(\rho) - q(\rho_{k,\delta} - \rho) + e^*(q_{k,\delta}) - e^*(q) - \rho(q_{k,\delta} - q) \\ &+ \int_D e_k(\rho_k) - e_k(\rho_{k,\delta}) - q(\rho_k - \rho_{k,\delta}) + e_k^*(q_k) - e_k^*(q_{k,\delta}) - \rho(q_k - q_{k,\delta}) = 0.\end{aligned} \tag{3.2}$$

The first line in (3.2) is now the sum of two Bregman divergences.

The Bregman divergence associated to any convex function is a premetric, i.e. it takes two points and returns a nonnegative number, however all the other metric axioms may fail. Nonetheless, Lemma A.3 guarantees the existence of strictly positive functions λ_e, λ_{e^*} such that

$$\int_D e(\rho_{k,\delta}) - e(\rho) - q(\rho_{k,\delta} - \rho) + e^*(q_{k,\delta}) - e^*(q) - \rho(q_{k,\delta} - q)$$
$$\geq \int_{D\cap A_{k,\epsilon,\delta}} \epsilon\lambda_e(\rho, q, \epsilon) + \int_{D\cap B_{k,\epsilon,\delta}} \epsilon\lambda_{e^*}(q, \rho, \epsilon),$$

where $A_{k,\epsilon,\delta} = \{(t,x) \in Q_\infty : \ [\rho_{k,\delta}(t,x) - \epsilon, \rho_{k.\delta}(t,x) + \epsilon] \cap \partial e^*(q(t,x)) = \varnothing\}$ and $B_{k,\epsilon,\delta} = \{(t,x) \in Q_\infty : \ [q_{k,\delta}(t,x) - \epsilon, q_{k,\delta}(t,x) + \epsilon] \cap \partial e(\rho(t,x)) = \varnothing\}$. Combining this with (3.2) we have

$$\limsup_{k\to\infty} \ \epsilon\Big(\int_{A_{k,\epsilon,\delta}\cap D} \lambda_e(\rho, q, \epsilon) + \int_{B_{k,\epsilon,\delta}\cap D} \lambda_{e^*}(q, \rho, \epsilon)\Big) + \int_D e_k(\rho_k) - e_k(\rho_{k,\delta}) + e_k^*(q_k) - e_k^*(q_{k,\delta})$$
$$\leq \limsup_{k\to\infty} \int_D q(\rho_k - \rho_{k,\delta}) + \rho(q_k - q_{k,\delta}).$$

We now want to control the right hand side of the inequality.

Let $S_{k,\delta} = \{(t,x) \in D : \rho_k > \rho_{k,\delta} + 2\delta\}$ and let $S^*_{k,\delta} = \{(t,x) \in D : q_k > q_{k,\delta} + 2\delta\}$. Note that

$$\int_D e_k(\rho_k) + e_k^*(q_k) \geq |S_{k,\delta}| e_k(a_\infty + \delta)| + |S^*_{k,\delta}| e_k^*(b_\infty + \delta).$$

Since $e_k(\rho_k) + e_k^*(q_k)$ is uniformly bounded in $L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))$, and for any fixed $\delta > 0$

$$\lim_{k\to\infty} e_k(a_\infty + \delta) = \infty, \quad \lim_{k\to\infty} e_k^*(b_\infty + \delta) = \infty,$$

it follows that $\limsup_{k\to\infty} |S_{k,\delta}| + |S^*_{k,\delta}| = 0$. Hence, using the inequality

$$\int_D q(\rho_k - \rho_{k,\delta}) + \rho(q_k - q_{k,\delta}) \leq 2\delta\|q+\rho\|_{L^1(D)} + |S_{k,\delta}|^{1/2}\|q\|_{L^2(D)}\|\rho_k\|_{L^\infty(D)} + |S^*_{k,\delta}|^{1/2}\|\rho\|_{L^\infty(D)}\|q_k\|_{L^2(D)},$$

we obtain

$$\lim_{\delta\to 0^+} \limsup_{k\to\infty} \ \epsilon\Big(\int_{A_{k,\epsilon,\delta}\cap D} \lambda_e(\rho, q, \epsilon) + \int_{B_{k,\epsilon,\delta}\cap D} \lambda_{e^*}(q, \rho, \epsilon)\Big) + \int_D e_k(\rho_k) - e_k(\rho_{k,\delta}) + e_k^*(q_k) - e_k^*(q_{k,\delta}) \leq 0$$

Since e, e^* are increasing, the last integral is nonnegative, and so we can conclude from the previous line that

$$\lim_{\delta\to 0^+} \limsup_{k\to\infty} |A_{k,\epsilon,\delta} \cap D| + |B_{k,\epsilon,\delta} \cap D| = 0,$$

for any $\epsilon > 0$. Finally, we can conclude by noting that $A_{k,\epsilon} \subset A_{k,\epsilon/4,\delta} \cup S_{k,\delta}$ and $B_{k,\epsilon} \subset B_{k,\epsilon/4,\delta} \cup S^*_{k,\delta}$ whenever $\delta < \epsilon/4$.

□

Of course, to even be able to use Proposition 3.2, we somehow need to know an upper semicontinuity type property for the product $\rho_k q_k$. In practice, this seems to require establishing the weak convergence of $\rho_k q_k$ to ρq. Luckily, the following "compensated compactness"-type Lemma shows that the weak convergence of the product can hold even when the strong convergence of both ρ_k and q_k is unknown. Unlike typical compensated compactness arguments that decompose the codomain of the function, the following compensated compactness argument is based on a decomposition of the domain of the functions. Indeed, we show that if ρ_k has some time regularity and q_k has some space regularity then their product weakly converges. This argument was inspired by the proof of the main Theorem in [MRCS10], although we would not be surprised if this result was already established in an earlier work. We state the lemma in the following generic form.

Lemma 3.5. *Fix some $r \in (1,\infty)$ and let r' be the Holder conjugate of r. Let $Z_r = L^r_{\mathrm{loc}}(Q_\infty) \times L^{r'}_{\mathrm{loc}}(Q_\infty)$ and let η be a spatial mollifier. Suppose that $(u_k, v_k) \in Z_r$ is a sequence that converges weakly in Z_r to a limit point $(u,v) \in Z_r$. If u_k is equicontinuous with respect to space in $L^r_{\mathrm{loc}}(Q_\infty)$ and for any $\epsilon > 0$, $\eta_\epsilon * v_k$ is equicontinuous with respect to space and time in $L^{r'}_{\mathrm{loc}}(Q_\infty)$, then $u_k v_k$ converges weakly in $(C_c(Q_\infty))^*$ to uv.*

Proof. Define $v_{k,\epsilon} := \eta_\epsilon * v_k$ and $v_\epsilon := \eta_\epsilon * v$. For $\epsilon > 0$ fixed and any compact set $D \subset Q_\infty$, the Riesz-Frechet-Kolmogorov compactness theorem implies that $v_{k,\epsilon}$ converges strongly in $L^{r'}(D)$ to v_ϵ as $k \to \infty$.

Given $\varphi \in C^\infty_c(Q_\infty)$, we must have

$$\lim_{\epsilon\to 0} \int_{Q_\infty} \varphi(v - v_\epsilon)u = 0,$$

and

$$\lim_{k\to\infty} \int_{Q_\infty} \varphi(v_{k,\epsilon} - v_\epsilon)u_k + v_\epsilon(u - u_k) = 0.$$

Thus, to prove the weak convergence of $u_k v_k$ to uv, it will suffice to show that

$$\lim_{\epsilon\to 0} \lim_{k\to\infty} \int_{Q_\infty} \varphi(v_k - v_{k,\epsilon})u_k = 0.$$

Rearranging the convolution, this is equivalent to showing

$$\lim_{\epsilon\to 0} \lim_{k\to\infty} \int_{Q_\infty} v_k\big(\eta_\epsilon * \varphi u_k - \varphi u_k\big) = 0.$$

Choose some compact set $D \subset Q_\infty$ such that for any ϵ sufficiently small, the support of $\varphi, \eta_\epsilon * \varphi$ is contained in D. We then have the estimate

$$\Big|\int_{Q_\infty} v_k\big(\eta_\epsilon * \varphi u_k - \varphi u_k\big)\Big| \lesssim \|v_k\|_{L^{r'}(D)}\big(\|\varphi\|_{L^\infty(Q_\infty)}\|u_k - \eta_\epsilon * u_k\|_{L^r(D)} + \epsilon\|u_k\|_{L^r(D)}\|\nabla\varphi\|_{L^\infty(Q_\infty)}\big).$$

The weak convergence of (u_k, v_k) to (u,v) in Z_r implies that $\|u_k\|_{L^r(D)} + \|v_k\|_{L^{r'}(D)}$ is bounded with respect to k. Spatial equicontinuity gives us

$$\lim_{\epsilon\to 0} \sup_k \|u_k - \eta_\epsilon * u_k\|_{L^r(D)} = 0.$$

Thus, it follows that

$$\lim_{\epsilon\to 0} \sup_k \Big|\int_{Q_\infty} v_k\big(\eta_\epsilon * \varphi u_k - \varphi u_k\big)\Big| = 0,$$

and so we can conclude that $u_k v_k$ converges in $(C_c(Q_\infty))^*$ to uv. ☐

4. Energy dissipation and estimates

We will now begin to analyze the parabolic structure of the equation (1.4). In order to do this, we will need to upgrade the spaces $X(e), Y(e^*)$ into spaces that are more appropriate for solving PDEs

Definition 4.1. Given an energy e satisfying (e1-e3), we define

$$\mathcal{X}(e) := \{\rho \in X(e) : \rho \in L^\infty_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \cap H^1_{\mathrm{loc}}([0,\infty); H^{-1}(\mathbb{R}^d))\},$$

$$\mathcal{Y}(e^*) := \{q \in Y(e^*) : q \in L^2_{\mathrm{loc}}([0,\infty); \dot{H}^1(\mathbb{R}^d))\},$$

and

$$\mathcal{D}(e, e^*) = \{(\rho, q) \in \mathcal{X}(e) \times \mathcal{Y}(e^*) : \rho q \in L^2_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))\}.$$

We begin by proving the energy dissipation relation in a form that is localized in both space and time.

Proposition 4.2. *Given an energy $e : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ satisfying (e1-e3), suppose that $e(\rho^0) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\rho^0 \in L^1(\mathbb{R}^d)$. Let $(\rho, q) \in \mathcal{D}(e, e^*)$ be a density-pressure pair that satisfy the duality relation $\rho q = e(\rho) + e^*(q)$ almost everywhere. Suppose that $\mu \in L^\infty(\frac{1}{\rho})$ is a growth rate and $V \in L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$ is a vector field such that $\nabla \cdot V \in L^\infty(Q_\infty)$. If for every $\psi \in W^{1,1}_c([0,\infty); L^1(\rho) \cap \dot{H}^1(\mathbb{R}^d))$, ρ, q are weak solutions of the parabolic equation*

$$\int_{\mathbb{R}^d} \psi(0,x)\rho^0(x)\, dx = \int_{Q_\infty} \nabla q \cdot \nabla \psi - \rho \partial_t \psi - \rho V \cdot \nabla \psi - \mu \psi, \tag{4.1}$$

then for any $\varphi \in W^{1,\infty}_c([0,\infty); W^{1,\infty}(\mathbb{R}^d) \cap \dot{H}^1(\mathbb{R}^d))$, we have the dissipation relation

$$\int_{\mathbb{R}^d} \varphi(0,x) e(\rho^0(x))\, dx = \int_{Q_\infty} -e(\rho)\partial_t \varphi + \varphi |\nabla q|^2 + q \nabla q \cdot \nabla \varphi - \rho q V \cdot \nabla \varphi + e^*(q) \nabla \cdot (V\varphi) - \varphi \mu q. \tag{4.2}$$

In particular, if $\varphi(t,x) = \omega(t)$ where $\omega \in W^{1,\infty}_c([0,\infty))$, then the relation simplifies to

$$\int_{\mathbb{R}^d} \omega(0) e(\rho^0(x))\, dx = \int_{Q_\infty} -e(\rho)\partial_t \omega + \omega |\nabla q|^2 + \omega e^*(q) \nabla \cdot V - \omega \mu q. \tag{4.3}$$

Proof. Let $\tilde{q} \in C^\infty_c(\mathbb{R}^d)$ such that $e^*(\tilde{q}) \in L^1(\mathbb{R}^d)$. Extend q backwards in time by defining $q(-t,x) = \tilde{q}(x)$ for all $t \in (0,\infty)$. Fix $\epsilon > 0$, and define

$$q_\epsilon(t,x) := \frac{1}{\epsilon} \int_{t-\epsilon}^t q(s,x)\, ds$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$. By Jensen's inequality, $q_\epsilon \in \mathcal{Y}(e^*)$ and a direct computation shows that $\partial_t q_\epsilon$ is the linear combination of two $\mathcal{Y}(e^*)$ functions for any $\epsilon > 0$. Assumption (e3) implies that there exists $\alpha_0 > 0$ such that e is bounded on $[0, \alpha_0]$. Hence, there must exist a point $\alpha_1 \in (0, \alpha_0)$ such that e is differentiable at α_1. Hence, we can split

$$q = q_1 + q_0$$

where $q_1 = \max(q - e'(\alpha_1), 0)$ and $q_0 = \min(q, e'(\alpha_1))$. Combining this decomposition with the duality relation $e(\rho) + e^*(q) = \rho q$ and the condition $\rho q \in L^2_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))$, it follows that for any $T \geq 0$

$$\int_{Q_T} q_1 + q_1^2 \leq \frac{1}{\min(\alpha_1, \alpha_1^2)} \int_{Q_T} \rho q_1 + (\rho q_1)^2 < \infty.$$

Thus, it follows that

$$q_0 \in L^\infty(Q_\infty) \cap L^2_{\mathrm{loc}}([0,\infty); \dot{H}^1(\mathbb{R}^d)), \quad q_1 \in L^2_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)).$$

Clearly, such a decomposition must hold for q_ϵ as well. As a result, given any nonnegative $\varphi \in W^{1,\infty}_c([0,\infty); W^{1,\infty}(\mathbb{R}^d) \cap \dot{H}^1(\mathbb{R}^d))$, it now follows that $q_\epsilon \varphi$ is a valid test function for the weak equation (4.1). Thus, we have

$$\int_{\mathbb{R}^d} q_\epsilon(0,x)\varphi(0,x)\rho^0(x)\, dx = \int_{Q_\infty} -\rho \partial_t(\varphi q_\epsilon) + (\nabla q - \rho V) \cdot \nabla(q_\epsilon \varphi) - \mu \varphi q_\epsilon, \tag{4.4}$$

Note that for almost every $(t,x) \in Q_\infty$

$$\rho \partial_t(\varphi q_\epsilon) = \rho(t,x) q_\epsilon(t,x) \partial_t \varphi(t,x) + \varphi(t,x) \frac{q(t,x) - q(t-\epsilon,x)}{\epsilon} \rho(t,x).$$

Hence, we can apply Young's inequality to deduce that

$$\left(\frac{q(t,x) - q(t-\epsilon,x)}{\epsilon}\right)\rho(t,x) \geq \frac{e^*(q(t,x)) - e^*(q(t-\epsilon,x))}{\epsilon} \tag{4.5}$$

By defining

$$(e^*(q))_\epsilon := \frac{1}{\epsilon}\int_{t-\epsilon}^{t} e^*(q(s,x))\,ds$$

we can write the above inequality in the more compact form

$$\rho\partial_t q_\epsilon \geq \partial_t (e^*(q))_\epsilon$$

Plugging this into (4.4), we get the inequality

$$\int_{\mathbb{R}^d} q_\epsilon(0,x)\varphi(0,x)\rho^0(x)\,dx \leq \int_{Q_\infty} -\rho q_\epsilon \partial_t\varphi - \varphi\partial_t(e^*(q))_\epsilon + (\nabla q - \rho V)\cdot\nabla(q_\epsilon\varphi) - \mu\varphi q_\epsilon,$$

Moving time derivatives back on to φ, we get the equivalent inequality

$$\int_{\mathbb{R}^d} \varphi(0,x)\Big(q_\epsilon(0,x)\rho^0(x) - \big(e^*(q)\big)_\epsilon(0,x)\Big)\,dx \\ \leq \int_{Q_\infty} \partial_t\varphi((e^*(q))_\epsilon - \rho q_\epsilon) + (\nabla q - \rho V)\cdot\nabla(q_\epsilon\varphi) - \mu\varphi q_\epsilon. \tag{4.6}$$

Note that we also have

$$\int_{\mathbb{R}^d} \varphi(0,x)\Big(q_\epsilon(0,x)\rho^0(x) - \big(e^*(q)\big)_\epsilon(0,x)\Big)\,dx = \int_{\mathbb{R}^d} \varphi(0,x)\Big(\tilde q(x)\rho^0(x) - e^*\big(\tilde q(x)\big)\Big)\,dx$$

thanks to our construction of q_ϵ.

Since all of the time derivatives are now on φ, we can safely send $\epsilon \to 0$. Thus, it follows that

$$\int_{\mathbb{R}^d} \varphi(0,x)\Big(\tilde q(x)\rho^0(x) - e^*\big(\tilde q(x)\big)\Big)\,dx \\ \leq \int_{Q_\infty} \partial_t\varphi(e^*(q) - \rho q) + (\nabla q - \rho V)\cdot\nabla(q\varphi) - \mu\varphi q.$$

Exploiting the duality relation $\rho q = e(\rho) + e^*(q)$, we have arrived at the inequality

$$\int_{\mathbb{R}^d} \omega(0)\Big(\tilde q(x)\rho^0(x) - e^*\big(\tilde q(x)\big)\Big) \leq \int_{Q_\infty} -e(\rho)\partial_t\varphi + (\nabla q - \rho V)\cdot\nabla(q\varphi) - \mu\varphi q.$$

$\tilde q$ was arbitrary, thus, taking a supremum over $\tilde q$ we obtain

$$\int_{\mathbb{R}^d} \omega(0)e(\rho^0(x)) \leq \int_{Q_\infty} -e(\rho)\partial_t\varphi + (\nabla q - \rho V)\cdot\nabla(q\varphi) - \mu\varphi q. \tag{4.7}$$

Expanding the right hand side, using the identity $\rho\nabla q = \nabla e^*(q))$, and integrating by parts, we obtain one direction of (4.2).

To get the other direction, we instead smooth q forwards in time by defining

$$\bar q_\epsilon := \frac{1}{\epsilon}\int_t^{t+\epsilon} q(s,x).$$

The argument will then proceed identically to the above except that the forward-in-time smoothing does not allow us to conclude that $\bar q_\epsilon(0,x) = \tilde q$. Luckily, Jensen's inequality and Young's inequality are now in our favor, and so we can just estimate

$$\int_{\mathbb{R}^d} \varphi(0,x)\Big(\bar q_\epsilon(0,x)\rho^0(x) - \frac{1}{\epsilon}\int_0^\epsilon \big(e^*\big(q(s,x)\big)\,ds\big)\Big)\,dx \leq \int_{\mathbb{R}^d} \varphi(0,x)\Big(\bar q_\epsilon(0,x)\rho^0(x) - e^*\big(\bar q_\epsilon(0,x)\big)\Big)\,dx \\ \leq \int_{\mathbb{R}^d} \varphi(0,x)e(\rho^0(x))dx.$$

We can extend the formula to general functions $\varphi \in W_c^{1,\infty}([0,\infty); W^{1,\infty}(\mathbb{R}^d) \cap \dot H^1(\mathbb{R}^d))$ by noting that the relation is linear in φ and any function can be written as the difference of two nonnegative functions.

□

In the next proposition, we will focus on collecting a priori estimates for solutions to (1.4). In fact, we will consider a slightly modified equation where we add an additional viscosity term $-\gamma\Delta\rho$ for some $\gamma > 0$. As we will see, the estimates will give us uniform control independent of γ when we consider sequences of solutions.

Proposition 4.3. *Let e be an energy function satisfying (e1-e3), let $V \in L^\infty_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$ be a vector field such that $\nabla\cdot V \in L^\infty_{\mathrm{loc}}([0,\infty); L^\infty(\mathbb{R}^d))$, let $\frac{\mu}{\rho} \in L^\infty(Q_\infty)$ and let γ be a positive constant. Suppose that $\rho \in \mathcal{X}(e) \cap L^2_{\mathrm{loc}}([0,\infty); H^1(\mathbb{R}^d))$, $q \in \mathcal{Y}(e^*)$ and ρ, q satisfy the duality relation $\rho q = e(\rho) + e^*(q)$ almost everywhere. If $e(\rho^0) \in L^1(\mathbb{R}^d)$ and the variables satisfy the weak equation*

$$\int_{\mathbb{R}^d} \psi(0,x)\rho^0(x)\,dx = \int_{Q_\infty} \gamma\nabla\rho\cdot\nabla\psi + \nabla q\cdot\nabla\psi - \rho\partial_t\psi - \rho V\cdot\nabla\psi - \mu\psi, \tag{4.8}$$

for every test function $\psi \in W^{1,1}_c([0,\infty); L^1(\rho)\cap\dot{H}^1(\mathbb{R}^d))$, then for any nonnegative $\omega \in W^{1,\infty}_c([0,\infty))$ we have the dissipation properties

$$\int_{Q_\infty} -e(\rho)\partial_t\omega + \omega|\nabla q|^2 + \omega e^*(q)\nabla\cdot V - \omega\mu q \le \int_{\mathbb{R}^d} \omega(0) e(\rho^0(x))\,dx, \tag{4.9}$$

$$\int_{Q_\infty} \omega\gamma(m-1)\rho^{m-2}|\nabla\rho|^2 - \rho^m\big(\frac{1}{m}\partial_t\omega + \omega(\frac{\mu}{\rho} - \frac{m-1}{m}\nabla\cdot V)\big) \le \int_{\mathbb{R}^d} \frac{\omega(0)}{m}(\rho^0)^m\,dx, \tag{4.10}$$

and setting $B = \|\frac{\mu}{\rho}\|_{L^\infty(Q_T)} + \|\nabla\cdot V\|_{L^\infty(Q_T)}$ we have the estimates

$$\gamma\|\nabla\rho\|^2_{L^2(Q_T)} \le \|\rho^0\|^2_{L^2(\mathbb{R}^d)} + B\|\rho\|^2_{L^2(Q_T)}, \tag{4.11}$$

$$\|\rho(T,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|\rho^0\|_{L^1(\mathbb{R}^d)}\exp(BT) \tag{4.12}$$

$$\|\partial_t\rho\|_{L^2([0,T];H^{-1}(\mathbb{R}^d))} \le \gamma\|\nabla\rho\|_{L^2(Q_T)} + \|\nabla q\|_{L^2(Q_T)} + \|\mu\|_{L^2(Q_T)} + \|\rho V\|_{L^2(Q_T)} \tag{4.13}$$

$$\|\rho(T,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le \|\rho^0\|_{L^\infty(\mathbb{R}^d)}\exp\big(2TB\big). \tag{4.14}$$

Finally, if we choose some $\alpha_0 > 0$ such that e is differentiable at α_0 and we set $\beta = e'(\alpha_0)$ then the following estimates hold where the unspecified constants depend only on $B, T, \beta, \alpha_0^{-1}$, and d:

$$\|\nabla q\|^2_{L^2(Q_T)} \lesssim \|\rho\|^{\frac{2(d+1)}{d}}_{L^\infty([0,T];L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d))} + \|\rho\|_{L^2([0,T];L^1(\mathbb{R}^d))} + \int_{\mathbb{R}^d} e(\rho^0)\,dx \tag{4.15}$$

$$\|\rho q\|_{L^2([0,T];L^1(\mathbb{R}^d))} \lesssim \|\rho\|_{L^2([0,T];L^1(\mathbb{R}^d))} + \|\rho\|^{\frac{d+1}{d}}_{L^\infty([0,T];L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d))}\|\nabla q\|_{L^2(Q_T)} \tag{4.16}$$

$$\|\rho q\|_{L^2(Q_T)} \lesssim \|\rho\|_{L^2(Q_T)} + \|\rho\|_{L^\infty(Q_T)}\|\rho q\|^{\frac{2}{d+2}}_{L^2([0,T];L^1(\mathbb{R}^d))}\|\nabla q\|^{\frac{d}{d+2}}_{L^2(Q_T)}, \tag{4.17}$$

for any set $K \subset Q_T$ with finite measure

$$\|q\|_{L^2(K)} \lesssim |K| + \|\rho q\|_{L^2(Q_T)}, \tag{4.18}$$

and there exists $r \in (\min(1-\frac{2}{d}, 0), 1]$ such that

$$\|\rho\log(1+|x|)\|_{L^\infty([0,T];L^1(\mathbb{R}^d))} \lesssim \tag{4.19}$$

$$e^{BT}\Big(\|\rho\|_{L^2(Q_T)}\|V\|_{L^2(Q_T)} + \gamma\|\rho\|_{L^1(Q_T)} + 1 + (\|\rho q\|_{L^2(Q_T)\cap L^1(Q_T)} + \|\rho\|^r_{L^1(Q_T)} + \int_{\mathbb{R}^d}\log(1+|x|)\rho^0(x)\,dx\Big).$$

Proof. The dissipation inequalities (4.9) and (4.10) follow from choosing the test functions $q\varphi$ and $\rho^{m-1}\omega$ respectively. These test functions do not have the required time regularity, however, by following an identical argument to Proposition 4.2, this technicality can be overcome. In addition, note that in both inequalities we have dropped a term involving $\nabla\rho\cdot\nabla q$, which is nonnegative thanks to the duality relation.

Estimates (4.12) and (4.13) are straightforward consequences of the weak equation (4.8). Estimate (4.11) follows from (4.10) with $m=2$. Estimate (4.14) follows from applying a Gronwall argument to (4.10) and then sending $m\to\infty$.

The estimates (4.15-4.18) are all linked. Fix a time $T>0$, and consider $\|\rho q\|_{L^2([0,T];L^1(\mathbb{R}^d))}$. We begin by splitting $q=q_0+q_1$ where $q_0=\min(q,\beta)$ and $q_1=\max(q-\beta,0)$. From our choice of β we know that $\rho\geq\alpha_0>0$ on the support of q_1. Therefore, q_1 must have finite support. Combining this with the coercivity of e^* and the bound $e^*(q)\in L^1(Q_T)$ it follows that $q_1\in L^1(Q_T)$. Thus,

$$\|\rho q\|_{L^2([0,T];L^1(\mathbb{R}^d))}\leq\beta\|\rho\|_{L^2([0,T];L^1(\mathbb{R}^d))}+\|\rho q_1\|_{L^2([0,T];L^1(\mathbb{R}^d)}\tag{4.20}$$

and it is now at least clear that the quantity is finite.

Working in Fourier space, we have

$$\|\rho q_1\|^2_{L^2([0,T];L^1(\mathbb{R}^d))}\leq\int_0^T\Big(\int_{\mathbb{R}^d}|\hat\rho(t,\xi)\hat q_1(t,\xi)|\,d\xi\Big)^2dt$$

$$\leq\int_0^T\Big(|B_R|\|\rho(t,\cdot)\|_{L^1(\mathbb{R}^d)}\|q_1(t,\cdot)\|_{L^1(\mathbb{R}^d)}+\int_{|\xi|>R}|\hat\rho(t,\xi)\hat q_1(t,\xi)|\,d\xi\Big)^2dt$$

where $R>0$ and B_R is the ball of radius R. Using the estimate

$$\int_{|\xi|>R}|\hat\rho(t,\xi)\hat q_1(t,\xi)|\,d\xi\leq(2\pi R)^{-1}\|\rho(t,\cdot)\|_{L^2(\mathbb{R}^d)}\|\nabla q_1(t,\cdot)\|_{L^2(\mathbb{R}^d)},$$

optimizing over R and dropping dimensional constants, it follows that

$$\|\rho q_1\|^2_{L^2([0,T];L^1(\mathbb{R}^d))}\lesssim_d\int_0^T\|\rho(t,\cdot)\|^{\frac{2}{d+1}}_{L^1(\mathbb{R}^d)}\|q_1(t,\cdot)\|^{\frac{2}{d+1}}_{L^1(\mathbb{R}^d)}\|\rho(t,\cdot)\|^{\frac{2d}{d+1}}_{L^2(\mathbb{R}^d)}\|\nabla q(t,\cdot)\|^{\frac{2d}{d+1}}_{L^2(\mathbb{R}^d)}.$$

$$\leq\|\rho\|^2_{L^\infty([0,T];L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d))}\|\nabla q\|^{\frac{2d}{d+1}}_{L^2(Q_T)}\|q_1\|^{\frac{2}{d+1}}_{L^2([0,T];L^1(\mathbb{R}^d))}.$$

Recalling the inequality $q_1\leq\alpha_0^{-1}\rho q_1$, we now see that

$$\|\rho q_1\|_{L^2([0,T];L^1(\mathbb{R}^d))}\lesssim_d\alpha_0^{-\frac{1}{d+1}}\|\rho\|_{L^\infty([0,T];L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d))}\|\nabla q\|^{\frac{d}{d+1}}_{L^2(Q_T)}\|\rho q_1\|^{\frac{1}{d+1}}_{L^2([0,T];L^1(\mathbb{R}^d))},$$

which gives

$$\|\rho q_1\|_{L^2([0,T];L^1(\mathbb{R}^d))}\lesssim_d\alpha_0^{-\frac{1}{d}}\|\rho\|^{\frac{d+1}{d}}_{L^\infty([0,T];L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d))}\|\nabla q\|_{L^2(Q_T)}.$$

Combining this with (4.20), we obtain (4.16).

Next, let us estimate $\|\rho q\|_{L^2(Q_T)}$. Again,

$$\|\rho q\|_{L^2(Q_T)}\leq\beta\|\rho\|_{L^2(Q_T)}+\|\rho\|_{L^\infty(Q_T)}\|q_1\|_{L^2(Q_T)}.$$

Gagliardo-Nirenberg gives us

$$\|q_1\|^2_{L^2(Q_T)}\lesssim_d\int_0^T\|q_1(t,\cdot)\|^{\frac{4}{d+2}}_{L^1(\mathbb{R}^d)}\|\nabla q(t,\cdot)\|^{\frac{2d}{d+2}}_{L^2(\mathbb{R}^d)}\,dt\leq\|q_1\|^{\frac{4}{d+2}}_{L^2([0,T];L^1(\mathbb{R}^d))}\|\nabla q\|^{\frac{2d}{d+2}}_{L^2(Q_T)}.$$

Combining our work, we see that

$$\|\rho q\|_{L^2(Q_T)}\lesssim_d\beta\|\rho\|_{L^2(Q_T)}+\alpha_0^{-\frac{2}{d+2}}\|\rho\|_{L^\infty(Q_T)}\|\rho q\|^{\frac{2}{d+2}}_{L^2([0,T];L^1(\mathbb{R}^d))}\|\nabla q\|^{\frac{d}{d+2}}_{L^2(Q_T)},$$

where we have used $q_1\leq\alpha_0^{-1}\rho q$. We have now attained the bound in (4.17).

Next, we estimate $\|\nabla q\|_{L^2(Q_T)}$. From the dissipation relation (4.9), we have

$$\int_{Q_\infty} \omega|\nabla q|^2 - e(\rho)\partial_t\omega + \omega e^*(p)\nabla\cdot V - \omega\mu q \leq \int_{\mathbb{R}^d} \omega(0)e(\rho^0)\,dx$$

for any nonnegative $\omega \in W_c^{1,\infty}((0,\infty))$. Fix a time $T > 0$ that is a Lebesgue point for the mapping $T \mapsto \|\nabla q\|_{L^2(Q_T)}$. Assume that ω is a decreasing function supported on $[0,T]$ and $\omega \leq 1$ everywhere. We can then eliminate the term $-e(\rho)\partial_t\omega$. Thus, it follows from our previous work and the dissipation relation that

$$\int_{Q_\infty} \omega|\nabla q|^2 \leq \int_{\mathbb{R}^d} e(\rho^0)\,dx + B\|\rho q\|_{L^1(Q_T)}$$

Using our previous work, we see that $\int_{Q_\infty}\omega|\nabla q|^2$ is

$$\lesssim_d \int_{\mathbb{R}^d} e(\rho^0)\,dx + BT^{1/2}\big(\beta\|\rho\|_{L^2([0,T];L^1(\mathbb{R}^d))} + \alpha_0^{-\frac{1}{d}}\|\rho\|^{\frac{d+1}{d}}_{L^\infty([0,T];L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d))}\|\nabla q\|_{L^2(Q_T)}\big)$$

Letting ω approach the characteristic function of $[0,T]$, the above bound holds for $\|\nabla q\|^2_{L^2(Q_T)}$. Using the quadratic formula, it follows that $\|\nabla q\|^2_{L^2(Q_T)} \lesssim_d c_1^2 + c_0$ where

$$c_1 = BT^{1/2}\alpha_0^{-\frac{1}{d}}\|\rho\|^{\frac{d+1}{d}}_{L^\infty([0,T];L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d))}, \quad c_0 = \int_{\mathbb{R}^d} e(\rho^0)\,dx + BT^{1/2}\beta\|\rho\|_{L^2([0,T];L^1(\mathbb{R}^d))}.$$

The estimate on $\|q\|_{L^2(K)}$ follows from the inequality $q \leq \beta + q_1 \leq \beta + \alpha_0^{-1}\rho q$.

Finally, it remains to prove the estimate for (4.19). Let $\eta : [0,\infty) \to [0,1]$ be a smooth increasing function such that $\eta(r) = 0$ if $r \leq \frac{1}{2}$ and $\eta(r) = 1$ if $r \geq 1$. Given some nonnegative $\omega \in W_c^{1,\infty}([0,\infty))$ and any $\epsilon > 0$ we define $\psi_\epsilon(t,x) := \omega(t)\log(1+|x|)\eta(|x||)e^{-\epsilon|x|}$ which is a valid test function for (4.8). Since ψ_ϵ is smooth in space, we can integrate by parts in (4.8) to obtain

$$\int_{\mathbb{R}^d} \varphi_\epsilon(0,x)\rho^0(x)\,dx = -\int_{Q_\infty} \rho\partial_t\varphi_\epsilon + (q+\gamma\rho)\Delta\varphi_\epsilon + \rho V\cdot\nabla\varphi_\epsilon + \mu\varphi_\epsilon$$

Fix some $T > 0$ such that the support of ω is contained in $[0,T]$. $\nabla\varphi_\epsilon$ and $\Delta\varphi_\epsilon$ are both uniformly bounded in $L^\infty(Q_T)$. Hence, we have

$$\int_{Q_T} -\rho\partial_t\varphi_\epsilon \lesssim \|\rho\|_{L^2(Q_T)}\|V\|_{L^2(Q_T)} + \gamma\|\rho\|_{L^1(Q_T)} + \int_{Q_T} B\rho\varphi_\epsilon + q\Delta\varphi_\epsilon + \int_{\mathbb{R}^d}\varphi_\epsilon(0,x)\rho^0(x)\,dx.$$

Thus, the only potentially problematic term is $q\Delta\varphi_\epsilon$ since $L^1(Q_T)$ bounds on q might not hold.

To estimate $\int_{Q_T} q\Delta\varphi_\epsilon$ we choose some set $K \subset \mathbb{R}^d$ with finite measure such that K contains the unit ball. We then have

$$\int_{Q_T} q\Delta\varphi_\epsilon \leq \|q\|_{L^1([0,T]\times K)} + \int_{[0,T]\times(\mathbb{R}^d\setminus K)} q\Delta\varphi_\epsilon \lesssim \beta|K| + \alpha_0^{-1}\|\rho q\|_{L^2(Q_T)} + \int_{[0,T]\times(\mathbb{R}^d\setminus K)} q\Delta\varphi_\epsilon$$

On $|x| > 1$, $\eta = 1$, hence $\nabla\varphi_\epsilon(t,x) =$

$$\omega(t)e^{-\epsilon|x|}\Big(\frac{1)}{1+|x|} - \epsilon\varphi_\epsilon(t,x)\Big)\frac{x}{|x|},$$

$\Delta\varphi_\epsilon(t,x) =$

$$(\frac{(d-1)}{|x|} - 2\epsilon)\nabla\varphi_\epsilon(t,x)\cdot\frac{x}{|x|} - \omega(t)e^{-\epsilon|x|}\frac{1}{(1+|x|)^2},$$

and $\max(\Delta\varphi_\epsilon(t,x),0) \leq$

$$\omega(t)e^{-\epsilon|x|}\Big(\frac{(d-1)}{|x|(1+|x|)} + 2\epsilon^2\log(1+|x|)\Big)$$

If we set $f_\epsilon(t,x)=\omega(t)e^{-\epsilon|x|}\Big(\frac{(d-1)}{\max(|x|,1)(1+|x|)}+2\epsilon^2\log(1+|x|)\Big)$ for all $(t,x)\in Q_T$, then combining our work thus far, we have

$$\int_{Q_T} q\Delta\varphi_\epsilon \lesssim 1+\|\rho q\|_{L^2(Q_T)}+\int_{Q_T} qf.$$

We again decompose $q=q_0+q_1$ where $q_0=\min(q,\beta)$ and $q_1=\max(q-\beta,0)$. Since f is uniformly bounded and $q_1\leq \alpha_0^{-1}\rho q$ we get $\int_{Q_T} qf\lesssim \|\rho q\|_{L^1(Q_T)}+\int_{Q_T} q_0 f$. Fix some $v>\min(1-\frac{2}{d},0)$ and let $r=\min(v,1)$. If we define $C_r=\sup_{a\in[0,\alpha_0]} a^{-r}\sup\partial e(a)$, assumption (e3) implies that $C_r<\infty$. From the definition of C_r, we see that $q_0\lesssim C_r\rho^r$, therefore $\int_{Q_T} q_0 f\leq \|\rho\|^r_{L^1(Q_T)}\|f\|_{L^{\frac{1}{1-r}}(Q_T)}$. Since $\frac{1}{1-r}>d/2$ and for any fixed $\delta>0$

$$\int_{Q_T}\frac{e^{-\epsilon|x|}}{(1+|x|)^{d+\delta}}+\epsilon^{d+\delta}\log(1+|x|)^{d+\delta}e^{-\epsilon|x|}$$

are uniformly bounded from above with respect to ϵ, it now follows that

$$\int_{Q_T} q\Delta\varphi_\epsilon\lesssim 1+\|\rho q\|_{L^2(Q_T)}+\|\rho q\|_{L^1(Q_T)}+\|\rho\|^r_{L^1(Q_T)}$$

Finally, we have obtained

$$\int_{Q_T} -\rho(\partial_t\varphi_\epsilon+B\varphi_\epsilon)\lesssim$$

$$\|\rho\|_{L^2(Q_T)}\|V\|_{L^2(Q_T)}+\gamma\|\rho\|_{L^1(Q_T)}+1+\|\rho q\|_{L^2(Q_T)\cap L^1(Q_T)}+\|\rho\|^r_{L^1(Q_T)}+\int_{\mathbb{R}^d}\varphi_\epsilon(0,x)\rho^0(x)\,dx.$$

ω was arbitrary, hence, Gronwall's inequality gives us

$$\|\rho\log(1+|x|)e^{-\epsilon|x|}\|_{L^\infty([0,T];L^1(\mathbb{R}^d))}\lesssim$$

$$e^{BT}\Big(\|\rho\|_{L^2(Q_T)}\|V\|_{L^2(Q_T)}+\gamma\|\rho\|_{L^1(Q_T)}+1+(\|\rho q\|_{L^2(Q_T)\cap L^1(Q_T)}+\|\rho\|^r_{L^1(Q_T)}+\int_{\mathbb{R}^d}\log(1+|x|)e^{-\epsilon|x|}\rho^0(x)\,dx\Big).$$

The unspecified constants are independent of ϵ and so sending $\epsilon\to 0$ we are done

□

5. Main results

At last, we are ready to combine our work to prove the main results of this paper. We will begin by constructing solutions to the system (1.3) and then we will show that these can be converted into solutions to the original system (1.1).

The construction of solutions to (1.3) is based on a vanishing viscosity approach. To that end, we consider a viscous analogue of system (1.3) where we add viscosity to both of the species ρ_1,ρ_2. Given a viscosity parameter $\gamma\geq 0$, we introduce the system:

$$\begin{cases}\partial_t\rho_1-\gamma\Delta\rho_1-\nabla\cdot(\frac{\rho_1}{\rho}\nabla q)+\nabla\cdot(\rho_1 V)=\rho_1F_{1,1}\big((z^*)^{-1}(q),n\big)+\rho_2F_{1,2}\big((z^*)^{-1}(q),n\big),\\ \partial_t\rho_2-\gamma\Delta\rho_2-\nabla\cdot(\frac{\rho_2}{\rho}\nabla q)+\nabla\cdot(\rho_2 V)=\rho_1F_{2,1}\big((z^*)^{-1}(q),n\big)+\rho_2F_{2,2}\big((z^*)^{-1}(q),n\big),\\ (\rho_1+\rho_2)q=e(\rho_1+\rho_2)+e^*(q),\\ \partial_t n-\alpha\Delta n=-n(c_1\rho_1+c_2\rho_2).\end{cases}\tag{5.1}$$

We define weak solutions to this system as follows.

Definition 5.1. Given a viscosity parameter $\gamma \geq 0$ and initial data $\rho_1^0, \rho_2^0 \in X(e)$ and $n^0 \in L^2(\mathbb{R}^d)$, we say that $(\rho_1, \rho_2, q, n) \in \mathcal{X}(e) \times \mathcal{X}(e) \times \mathcal{Y}(e^*) \times L^2_{\text{loc}}([0,\infty); H^1(\mathbb{R}^d))$ is a weak solution to the system (5.1) with initial data $(\rho_1^0, \rho_2^0, n^0)$, if $\rho q = e(\rho) + e^*(q)$ almost everywhere, $(\rho, q) \in \mathcal{D}(e, e^*)$, $\gamma\nabla\rho_1, \gamma\nabla\rho_2 \in L^2_{\text{loc}}([0,\infty); L^2(\mathbb{R}^d))$, and for every test function $\psi \in H^1_c([0,\infty); H^1(\mathbb{R}^d))$

(5.2)
$$\int_{\mathbb{R}^d} \psi(0,x)\rho_1^0 = \int_{Q_\infty} \nabla\psi\cdot\big(\frac{\rho_1}{\rho}\nabla q + \gamma\nabla\rho_1 - \rho_1 V\big) - \rho_1\partial_t\psi - \psi\big(\rho_1 F_{1,1}\big((z^*)^{-1}(q), n\big) + \rho_2 F_{1,2}\big((z^*)^{-1}(q), n\big)\big),$$

(5.3)
$$\int_{\mathbb{R}^d} \psi(0,x)\rho_2^0 = \int_{Q_\infty} \nabla\psi\cdot\big(\frac{\rho_2}{\rho}\nabla q + \gamma\nabla\rho_2 - \rho_2 V\big) - \rho_2\partial_t\psi - \psi\big(\rho_1 F_{2,1}\big((z^*)^{-1}(q), n\big) + \rho_2 F_{2,2}\big((z^*)^{-1}(q), n\big)\big),$$

$$\int_{\mathbb{R}^d} \psi(0,x)n^0 = \int_{Q_\infty} \alpha\nabla\psi\cdot\nabla n - n\partial_t\psi + n(c_1\rho_1 + c_2\rho_2)\psi \tag{5.4}$$

where $\rho = \rho_1 + \rho_2$.

When $\gamma > 0$, the existence of weak solutions to (5.1) is straightforward, as the individual densities will be bounded in $L^2_{\text{loc}}([0,\infty); H^1(\mathbb{R}^d)) \cap H^1_{\text{loc}}([0,\infty); H^{-1}(\mathbb{R}^d))$. Since this space is compact in $L^2_{\text{loc}}([0,\infty); L^2(\mathbb{R}^d))$, one can construct the solutions as limits of an even more regularized system (with enough regularity existence of solutions can be shown with a standard but tedious Picard iteration). Thus, we can assume the existence of a sequence $(\rho_{1,k}, \rho_{2,k}, q_k, n_k)$ such that for each k the variables are a weak solution to (5.1) with viscosity parameter $\gamma_k > 0$. We will then use our efforts from the past two sections to show that when $\gamma_k \to 0$ we can still pass to the limit in equations (5.2-5.4) to obtain a solution to (1.3). In fact, we will show that we can pass to the limit even when the underlying energy function e_k is changing along the sequence.

We begin with a result that establishes conditions under which the sequence of pressure variables converges strongly in $L^2_{\text{loc}}([0,\infty); H^1_{\text{loc}}(\mathbb{R}^d))$.

Proposition 5.2. *Let e_k be a sequence of energy functions satisfying (e1-e3) and suppose there exists an energy e satisfying (e1-e3) such that e_k converges pointwise everywhere to e. Let $(\rho_k, q_k) \in \mathcal{D}(e_k, e_k^*)$, and $\mu_k \in L^\infty(\frac{1}{\rho_k})$ be sequences of densities, pressure, and growth terms that converge weakly in $L^1_{\text{loc}}(Q_\infty)$ to limits $\rho \in \mathcal{X}(e), q \in \mathcal{Y}(e^*), \mu \in L^\infty(\frac{1}{\rho})$. Suppose that for all k the duality relation $\rho_k q_k = e_k(\rho_k) + e_k^*(q_k)$ holds almost everywhere and the product $\rho_k q_k$ converges weakly to ρq in $L^1_{\text{loc}}(Q_\infty)$. Furthermore, suppose that for every nonnegative $\omega \in W^{1,\infty}_c([0,\infty))$ the variables satisfy the energy dissipation properties*

$$\int_{Q_\infty} -e_k(\rho_k)\partial_t\omega + \omega|\nabla q_k|^2 + \omega e_k^*(q_k)\nabla\cdot V - \omega\mu_k q_k \leq \int_{\mathbb{R}^d} \omega(0)e_k(\rho^0(x))\,dx, \tag{5.5}$$

and

$$\int_{\mathbb{R}^d} \omega(0)e(\rho^0(x))\,dx \leq \int_{Q_\infty} -e(\rho)\partial_t\omega + \omega|\nabla q|^2 + \omega e^*(q)\nabla\cdot V - \omega\mu q. \tag{5.6}$$

If $\log(1+|x|)\rho_k(t,x)$ is uniformly bounded with respect to k in $L^1_{\text{loc}}([0,\infty); L^1(\mathbb{R}^d))$ and for every compact set $D \subset Q_\infty$

$$\limsup_{k\to\infty} \int_D \omega\mu_k q_k \leq \int_D \omega\mu q, \tag{5.7}$$

then q_k converges strongly in $L^2_{\text{loc}}([0,\infty); L^2_{\text{loc}}(\mathbb{R}^d) \cap \dot{H}^1(\mathbb{R}^d))$ to q.

Proof. Given some nonnegative $\omega \in W_c^{1,\infty}([0,\infty))$ we can combine (5.5) and (5.6) to obtain

$$\limsup_{k\to\infty} \int_{Q_\infty} -e_k(\rho_k)\partial_t\omega + \omega|\nabla q_k|^2 + \omega e_k^*(q_k)\nabla\cdot V - \omega\mu_k q_k$$
$$\leq \int_{Q_\infty} -e(\rho)\partial_t\omega + \omega|\nabla q|^2 + \omega e^*(q)\nabla\cdot V - \omega\mu q.$$

Applying Proposition 3.2, it follows that that $e_k(\rho_k)$, $e_k^*(q_k)$ and $\rho_k q_k$ converge weakly in $L^1_{\text{loc}}([0,\infty);L^1(\mathbb{R}^d))$ to $e(\rho)$ and $e^*(q)$ respectively. The uniform boundedness of $\log(1+|x|)\rho_k(t,x)$ implies that (5.7) holds when D is replaced by Q_∞. In addition, it implies that $\log(1+|x|)^{1/2}\rho_k q_k$ is uniformly bounded in $L^1_{\text{loc}}([0,\infty);L^1(\mathbb{R}^d))$. Thus, the weak convergence of $e_k(\rho_k)$, $e_k^*(q_k)$ against test functions in $L^1_{\text{loc}}(Q_T)$ can be extended to test functions in $W_c^{1,\infty}([0,\infty))$ that are independent of space. From these weak convergence properties we obtain

$$\limsup_{k\to\infty} \int_{Q_\infty} \omega|\nabla q_k|^2 \leq \int_{Q_\infty} \omega|\nabla q|^2.$$

Since $\omega \in W_c^{1,\infty}([0,\infty))$ was arbitrary, this automatically implies that ∇q_k converges strongly to ∇q in $L^2_{\text{loc}}([0,\infty);L^2(\mathbb{R}^d))$.

Now we turn to showing that q_k converges to q in $L^2_{\text{loc}}(Q_\infty)$. We recall from assumption (e3) that there exists some $\alpha_0 > 0$ such that e is differentiable at α_0. It immediately follows that e is finite on $[0,\alpha_0]$. Let $\alpha_n \in (0,\alpha_0]$ be a decreasing sequence of points such that e is differentiable at α_n for all n and $\lim_{n\to\infty}\alpha_n = 0$. Set $\beta_n = e'(\alpha_n)$ and note that (e3) implies that $\lim_{n\to\infty}\beta_n = 0$.

Split $q_k = q_{k,0}^n + q_{k,1}^n$, $q = q_1^n + q_0^n$ where $q_{k,1}^n = \max(q_k - \beta_n, 0)$, $q_{k,0}^n = \min(q_k, \beta_n)$ (and similarly for q_1^n, q_0^n). For n fixed, the duality relation and the convergence of e_k to e implies that $q_{k,1}^n \leq \frac{2}{\alpha_n}\rho_k q_k$, for all k sufficiently large (c.f. Lemma A.1). Therefore, for each n fixed $q_{k,1}^n$ is bounded with respect to k in $L^1(Q_T)$.

Let χ_n be the indicator function of the set $[0,\beta_n]$ and note that $\nabla q_{k,1}^n = \nabla q_k(1-\chi_n(q_k))$ and $\nabla q_1^n = \nabla q(1-\chi_n(q))$. Since $\partial e(\alpha_n) = \{e'(\alpha_n)\} = \{\beta_n\}$ is a singleton and χ_n is constant on $(0,\beta_n), (\beta_n,\infty)$, Lemma 3.3 implies that $\chi_n(q_k)$ converges locally in measure to $\chi_n(q)$.

Fix some compact set $D \subset Q_\infty$ and choose T sufficiently large such that $D \subset [0,T]\times\mathbb{R}^d$. For any fixed n we have the estimate

$$\|q_k - q\|_{L^2(D)} \leq \|q_{k,0}^n - q_0^n\|_{L^2(D)} + \|q_{k,1}^n - q_1^n\|_{L^2(D)} \leq \beta_n|D| + \|q_{k,1}^n - q_1^n\|_{L^2([0,T]\times\mathbb{R}^d)}$$
$$\lesssim_d \beta_n|D| + \|\nabla(q_{k,1}^n - q_1^n)\|_{L^2([0,T]\times\mathbb{R}^d)}^{\frac{d}{d+2}} \|q_{k,1}^n - q_1^n\|_{L^2([0,T];L^1(\mathbb{R}^d))}^{\frac{2}{d+2}}$$

where the second line follows from Gagliardo-Nirenberg. We can then bound

$$\|q_{k,1}^n - q_1^n\|_{L^2([0,T];L^1(\mathbb{R}^d))} \leq \frac{2}{\alpha_n}\|\rho_k q_k\|_{L^2([0,T];L^1(\mathbb{R}^d))} + \frac{1}{\alpha_n}\|\rho q\|_{L^2([0,T];L^1(\mathbb{R}^d))}$$

and

$$\|\nabla(q_{k,1}^n - q_1^n)\|_{L^2([0,T]\times\mathbb{R}^d)} \leq \|\nabla q - \nabla q_k\|_{L^2([0,T]\times\mathbb{R}^d)} + \|\big(\chi_n(q_k) - \chi_n(q)\big)\nabla q\|_{L^2([0,T]\times\mathbb{R}^d)}$$

Thus, there exists a constant independent of k such that

$$\limsup_{k\to\infty}\|q_k - q\|_{L^2(D)} \leq \beta_n|D| + C\limsup_{k\to\infty}\|\big(\chi_n(q_k) - \chi_n(q)\big)\nabla q\|_{L^2([0,T]\times\mathbb{R}^d)}$$

The local convergence in measure of $\chi_n(q_k)$ to $\chi_n(q)$ combined with the L^2 bound on ∇q is sufficient to conclude that $\limsup_{k\to\infty}\|\big(\chi_n(q_k) - \chi_n(q)\big)\nabla q\|_{L^2([0,T]\times\mathbb{R}^d)} = 0$. Thus, $\limsup_{k\to\infty}\|q_k - q\|_{L^2(D)} \leq \beta_n|D|$ for all $n \in \mathbb{Z}_+$. Sending $n\to\infty$, we conclude that $\limsup_{k\to\infty}\|q_k - q\|_{L^2(D)} = 0$ for all compact sets $D \subset Q_\infty$. $\square$

The next two Lemmas are technical results that will help us guarantee that we can pass to the limit in all of the terms in (5.2) and (5.3).

Lemma 5.3. *Let e_k be a sequence of energies satisfying (e1-e3) and suppose there exists an energy e satisfying (e1-e3) such that e_k converges pointwise everywhere to e. Let $(\rho_k, q_k) \in \mathcal{D}(e_k, e_k^*)$ be sequences of uniformly bounded density and pressure variables that satisfy the duality relation $\rho_k q_k = e_k(\rho_k) + e_k^*(q_k)$ almost everywhere. If q_k converges strongly in $L^2_{\text{loc}}([0,\infty); H^1_{\text{loc}}(\mathbb{R}^d))$ to a limit q and ρ_k converges weakly in $L^2_{\text{loc}}([0,\infty); L^2(\mathbb{R}^d))$ to a limit ρ, then*

$$\limsup_{k\to\infty} \int_D |\rho - \rho_k||\nabla q|^2 = 0$$

for any compact set $D \subset Q_\infty$

Proof. Clearly for any $\varphi \in C_c^\infty(Q_\infty)$ we have

$$\limsup_{k\to\infty} \int_{Q_\infty} \varphi \rho_k q_k = \int_{Q_\infty} \varphi \rho q.$$

Thus, by Proposition 3.2, the limiting variables satisfy the duality relation $\rho q = e(\rho) + e^*(q)$ almost everywhere.

Let $M = \sup_k \|\rho_k\|_{L^\infty(D)} < \infty$. Define $\bar{e}_k^*$ and $\bar{e}^*$ such that $\bar{e}_k^*(0) = 0, \bar{e}^*(0) = 0$, and

$$\partial \bar{e}_k^*(b) = \{\min(a, M) : a \in \partial e_k^*(b)\}, \quad \partial \bar{e}^*(b) = \{\min(a, M) : a \in \partial e^*(b)\}$$

Let $\bar{e}_k = (\bar{e}_k^*)^*$ and $\bar{e} = (\bar{e}^*)^*$. Clearly, we still have the duality relations $\rho_k q_k = \bar{e}(\rho_k) + \bar{e}^*(q_k)$ and $\rho q = \bar{e}(\rho) + \bar{e}^*(q)$ almost everywhere. It also follows that $\bar{e}_k^*, \bar{e}^*$ are uniformly Lipschitz on the entire real line and uniformly bounded on compact subsets of $\mathbb{R}$. As a result, $\bar{e}_k^*$ must converge uniformly on compact subsets of $\mathbb{R}$ to $\bar{e}^*$.

Fix some $\delta > 0$. Convexity and the duality relation imply that

$$\rho_k \le \frac{\bar{e}_k^*(q_k + \delta) - \bar{e}_k^*(q_k)}{\delta}, \quad \rho \le \frac{\bar{e}^*(q+\delta) - \bar{e}^*(q)}{\delta},$$

and

$$\rho_k \ge \frac{\bar{e}_k^*(q_k) - \bar{e}_k^*(q_k - \delta)}{\delta}, \quad \rho \ge \frac{\bar{e}_k^*(q) - \bar{e}^*(q-\delta)}{\delta}.$$

Therefore,

$$\int_D |\rho - \rho_k||\nabla q|^2$$

$$\le \int_D \Big(|\frac{\bar{e}_k^*(q_k+\delta) + \bar{e}^*(q-\delta) - \bar{e}_k^*(q_k) - \bar{e}^*(q)}{\delta}| + |\frac{\bar{e}^*(q+\delta) + \bar{e}_k^*(q_k-\delta) - \bar{e}_k^*(q_k) - \bar{e}^*(q)}{\delta}|\Big)|\nabla q|^2.$$

Thus, it follows that

$$\limsup_{k\to\infty} \int_D |\rho - \rho_k||\nabla q|^2 \le 2\int_D |\frac{\bar{e}^*(q+\delta) + \bar{e}^*(q-\delta) - 2\bar{e}^*(q)}{\delta}||\nabla q|^2$$

If $\bar{e}^*$ is continuously differentiable at a point $b \in \mathbb{R}$, then

$$\lim_{\delta\to 0} \frac{\bar{e}^*(b+\delta) + \bar{e}^*(b-\delta) - 2\bar{e}^*(b)}{\delta} = 0.$$

The singular set $S \subset \mathbb{R}$ of values where $\bar{e}^*$ is not continuously differentiable is at most countable. Therefore, $|\nabla q|$ is zero almost everywhere on the set $\{(t,x) \in D : q(t,x) \in S\}$. Hence, by dominated convergence,

$$\lim_{\delta\to 0} 2\int_D |\frac{\bar{e}^*(q+\delta) + \bar{e}^*(q-\delta) - 2\bar{e}^*(q)}{\delta}||\nabla q|^2 = 0.$$

□

Lemma 5.4. *Let z_k be a sequence of energies satisfying (z1-z3) and suppose there exists an energy z satisfying (z1-z3) such that z_k converges pointwise everywhere to z. Define e_k, e by formula (2.1). Suppose that $(\rho_{1,k}, \rho_{2,k}, q_k, n_k) \in \mathcal{X}(e_k) \times \mathcal{X}(e_k) \times \mathcal{Y}(e_k^*) \times L^2_{\mathrm{loc}}([0,\infty); H^1(\mathbb{R}^d))$ is a sequence such that $(\rho_{1,k}+\rho_{2,k})q_k = e_k(\rho_{1,k}+\rho_{2,k}) + e_k^*(q_k)$ almost everywhere. Suppose that $\rho_{1,k}, \rho_{2,k}$ converge weakly in $L^r_{\mathrm{loc}}([0,\infty); L^r(\mathbb{R}^d)$ to limits $\rho_1, \rho_2 \in \mathcal{X}(e)$, q_k converges strongly in $L^2_{\mathrm{loc}}([0,\infty); H^1_{\mathrm{loc}}(\mathbb{R}^d))$ to a limit q, and n_k converges strongly in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$ to a limit n. If the growth terms $F_{i,j}$ satisfy assumptions (F1-F2), then $\rho_{j,k}F_{i,j}\big(z_k^{-1}(q_k), n_k\big)$ converges weakly in $L^r_{\mathrm{loc}}(Q_\infty)$ to $\rho_j F_{i,j}\big(z^{-1}(q), n\big)$ for all $i, j \in \{1,2\}$ and any $r < \infty$.*

Proof. It suffices to prove the convergence of $\rho_{1,k}F_{1,1}\big(z_k^{-1}(q_k), n_k\big)$ to $\rho_1 F_{1,1}\big(z^{-1}(q), n\big)$, the argument for the other terms is identical. Let $\varphi \in C_c^\infty(Q_\infty)$ and let $D \subset Q_\infty$ be a compact set containing the support of φ. For $N \in \mathbb{R}$ define $S_{k,N} := \{(t,x) \in D : q_k(t,x) + n_k(t,x) > N\}$. From the uniform bounds on the norms of q_k, n_k it follows that $\lim_{N\to\infty} \sup_k |S_{k,N}| = 0$. Thus, we can assume without loss of generality that q_k, n_k are uniformly bounded by some $M > 0$ (and of course this same logic applies to q, n as well).

Let $b_\infty = \sup\{b \in \mathbb{R} : z^*(b) < \infty\}$. Fix $\epsilon \in (0, z^*(b_\infty)/2)$ and let $q_{k,\epsilon} = \min(\max(\epsilon, q_k), z^*(b_\infty) - \epsilon)$, $q_\epsilon = \min(\max(\epsilon, q), z^*(b_\infty) - \epsilon)$. It now follows that $(z_k^*)^{-1}(q_{k,\epsilon}), (z^*)^{-1}(q_\epsilon)$ are uniformly bounded in $L^\infty(D)$. Thanks to Lemma A.1, we know that $(z_k^*)^{-1}$ converges uniformly to $(z^*)^{-1}$ on $(\epsilon, z^*(b_\infty) - \epsilon)$. Combining this with properties (F1-F2), and the various convergence properties of $q_k, n_k, \rho_{1,k}$ it follows that

$$\limsup_{k\to\infty} \Big| \int_{Q_\infty} \varphi\Big(\rho_{1,k}F_{1,1}\big((z_k^*)^{-1}(q_{k,\epsilon}), n_k\big) - \rho_1 F_{1,1}\big((z^*)^{-1}(q_\epsilon), n\big)\Big) \Big| = 0.$$

Thus, it remains to show that

$$\lim_{\epsilon\to 0^+} \Big| \int_{Q_\infty} \varphi\rho_1\Big(F_{1,1}\big((z^*)^{-1}(q_\epsilon), n\big) - F_{1,1}\big((z^*)^{-1}(q), n\big)\Big) \Big| = 0 \tag{5.8}$$

and

$$\lim_{\epsilon\to 0^+} \limsup_{k\to\infty} \Big| \int_{Q_\infty} \varphi\rho_{1,k}\Big(F_{1,1}\big((z_k^*)^{-1}(q_{k,\epsilon}), n_k\big) - F_{1,1}\big((z_k^*)^{-1}(q_k), n_k\big)\Big) \Big| = 0. \tag{5.9}$$

To do this we will exploit the density pressure duality relationship. Thanks to the relationship between e and z, we can express the duality relation as $(\rho_{1,k}+\rho_{2,k})(z_k^*)^{-1}(q_k) = z_k(\rho_{1,k}+\rho_{2,k}) + q_k$. Fix some $\delta > 0$ and split the support of $\rho_{1,k}$ into the sets $\rho_{1,k} < \delta$ and $\rho_{1,k} \geq \delta$. Again using duality, we have

$$0 \leq \rho_{1,k} \leq \rho_{1,k} + \rho_{2,k} \in \partial z_k^* \circ (z_k^*)^{-1} \circ q_k$$

Thus, for almost every (t,x) where $\rho_{1,k}(t,x) \geq \delta$, it follows that $(z_k^*)^{-1}$ is at worst δ^{-1} Lipschitz at the value $q_k(t,x)$ and $(z_k^*)^{-1}(q_k(t,x))$ is uniformly bounded with respect to k. Thus,

$$\Big| \int_{Q_\infty} \varphi\rho_{1,k}\Big(F_{1,1}\big((z_k^*)^{-1}(q_{k,\epsilon}), n_k\big) - F_{1,1}\big((z_k^*)^{-1}(q_k), n_k\big)\Big) \Big|$$

$$\leq B\delta \|\varphi\|_{L^1(D)} + \omega_\delta(2\epsilon\delta^{-1}) \|\rho_{1,k}\|_{L^1(D)} \|\varphi\|_{L^\infty(D)} + \|\rho_{1,k}\varphi\|_{L^\infty(D)} |D_{k,\epsilon}|$$

where B is a bound on $F_{1,1}$ and ω_δ is the modulus of continuity of $F_{1,1}$ on the bounded set $\Big(\bigcup_k \{(z_k^*)^{-1}(q_k(t,x)) : \rho_{1,k}(t,x) \geq \delta\}\Big) \times [0, M]$ and $D_{k,\epsilon} = \{(t,x) \in D : q_k(t,x) > z^*(b_\infty) + \epsilon\}$. The convergence of z_k to z implies that $\limsup_{k\to\infty} |D_{k,\epsilon}| = 0$ for all fixed $\epsilon > 0$. Thus, sending $k \to \infty$, then $\epsilon \to 0^+$, and then $\delta \to 0^+$, we get (5.9). The strong convergence of q_k implies that the duality relation $(\rho_1+\rho_2)(z^*)^{-1}(q) = z(\rho_1+\rho_2) + q$ holds, thus we can use a similar argument to obtain (5.8).

□

At last, we are ready to prove our main result, which will let us pass to the limit when we consider sequences of weak solutions to (5.1). Note that the following theorem applies in the case where the viscosity is decreasing to zero along the sequence, as well as when the viscosity is zero along the entire sequence.

Theorem 5.5. *Let z_k be a sequence of energies satisfying (z1-z3). Suppose there exists an energy z satisfying (z1-z3) such that z_k converges pointwise everywhere to z. Define e_k, e by formula (2.1). Let $\rho_1^0, \rho_2^0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d), n^0 \in L^2(\mathbb{R}^d)$ be initial data such that $e(\rho_1^0+\rho_2^0) \in L^1(\mathbb{R}^d)$. Let $V \in L^2_{\text{loc}}([0,\infty); L^2(\mathbb{R}^d))$ be a vector field such that $\nabla \cdot V \in L^\infty(Q_\infty)$ and let $F_{i,j}$ be source terms satisfying (F1-F2). Let $\rho_{1,k}, \rho_{2,k} \in \mathcal{X}(e_k)$, $q_k \in \mathcal{Y}(e_k^*)$, $n_k \in L^2_{\text{loc}}([0,\infty); H^1(\mathbb{R}^d))$ be sequences of density pressure and nutrient variables such that for each k, the variables are weak solutions to the system (5.1) with energy e_k, viscosity constant $\gamma_k \geq 0$, and initial data $(\rho_1^0, \rho_2^0, n^0)$. Furthermore, suppose that $\gamma_k \nabla \rho_{1,k}, \gamma_k \nabla \rho_{2,k} \in L^2_{\text{loc}}([0,\infty); L^2(\mathbb{R}^d))$. If γ_k converges to 0 and at least one of the following two conditions hold:*

(a) $\partial z(a)$ is a singleton for all $a \in (0,\infty)$,
(b) the source terms satisfy the additional condition (F3),

then any limit point (ρ_1, ρ_2, q, n) of the sequence is a solution of (1.3).

Proof. Step 1: Uniform bounds, basic convergence properties, and parabolic structure.
Summing the first two equations of (5.1) together, we see that for any test function $\psi \in W_c^{1,1}([0,\infty); H^1(\mathbb{R}^d))$ ρ_k, q_k are weak solutions to the parabolic equation

$$\int_{\mathbb{R}^d} \psi(0,x)\rho^0 = \int_{Q_\infty} -\rho_k \partial_t \psi + \nabla\psi \cdot (\nabla q_k + \gamma_k \nabla \rho_k) - \rho_k \nabla \psi \cdot V - \psi \mu_k \tag{5.10}$$

where $\rho_k = \rho_{1,k}+\rho_{2,k}$, $\mu_k = \mu_{1,k}+\mu_{2,k}$ and $\mu_{i,k} = \rho_{1,k}F_{i,1}\big((z_k^*)^{-1}(q_k,n_k)\big)+\rho_{2,k}F_{i,2}\big((z_k^*)^{-1}(q_k,n_k)\big)$.
Thanks to Proposition **??**, ρ_k, q_k, μ_k must satisfy the energy dissipation inequality

$$\int_{Q_\infty} -e(\rho_k)\partial_t \omega + \omega |\nabla q_k|^2 + \omega e^*(q_k)\nabla \cdot V - \omega \mu_k q_k \leq \int_{\mathbb{R}^d} \omega(0) e(\rho^0(x))\, dx,$$

for every nonnegative $\omega \in W^{1,\infty}([0,\infty))$ and the estimates (4.11)-(4.18). After plugging estimate (4.11) into estimate (4.13), it follows that all of the estimates (4.12)-(4.18) are independent of k and only depend on ρ^0, V the bounds on $F_{i,j}$, and the constants $\beta_{1,k} := \min(1, \frac{1}{2}\liminf_{b\to\infty} \frac{e_k^*(b)}{b})$, $\beta_{2,k} := \inf\{b \geq 0 : \alpha_0 \leq \inf \partial e^*(b)\}$. Thanks to property (e3) and the convergence of e_k^* to e^* it follows that $\beta_{1,k}^{-1}$ and $\beta_{2,k}$ are uniformly bounded in k (c.f. Lemma A.1). Thus, ρ_k, q_k are uniformly bounded in the norms estimated in (4.12)-(4.18). As a result, there must exist $\rho \in \mathcal{X}(e)$, $q \in \mathcal{Y}(e^*)$ and $\mu \in L^\infty_{\text{loc}}([0,\infty); L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$ such that ρ_k, q_k, μ_k converge weakly in $L^2_{\text{loc}}(Q_\infty)$ (along a subsequence that we do not relabel) to ρ, q, μ respectively. Note that for ρ_k, μ_k the weak convergence in fact holds in $L^r_{\text{loc}}(Q_\infty)$ for any $r < \infty$. Furthermore, the convexity of the mapping $(\alpha,\beta) \mapsto \frac{|\alpha|}{\beta}$ over $\mathbb{R} \times (0,\infty)$ implies that $\mu \in L^\infty(\frac{1}{\rho})$.

Property (F2) implies that $0 \leq \rho_{1,k}, \rho_{2,k} \leq \rho_k$. Hence, $\rho_{1,k}, \rho_{2,k}$ are uniformly bounded in $L^\infty_{\text{loc}}([0,\infty); L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$ and there exist limit points ρ_1, ρ_2 (and a subsequence that we do not relabel) such that $\rho_{1,k}, \rho_{2,k}$ converge weakly in $L^r_{\text{loc}}([0,\infty); L^1(\mathbb{R}^d)\cap L^r(\mathbb{R}^d))$ to ρ_1, ρ_2 respectively for any $r < \infty$. Furthermore, the bounds on $\rho_{1,k}, \rho_{2,k}$ combined with standard results for the heat equation imply that n_k is uniformly bounded in $L^2_{\text{loc}}([0,\infty); H^1(\mathbb{R}^d))\cap H^1_{\text{loc}}([0,\infty); H^{-1}(\mathbb{R}^d))$. Hence, the Aubin-Lions Lemma implies that there exists a limit point $n \in L^2_{\text{loc}}([0,\infty); H^1(\mathbb{R}^d))$ and a subsequence (that we do not relabel) such that n_k converges to n in $L^2_{\text{loc}}([0,\infty); L^2(\mathbb{R}^d))$.

Thanks to the linear structure of equation (5.10), the convergence properties we have established are strong enough to send $k \to \infty$. Thus, ρ, q, μ satisfy the weak equation

$$\int_{\mathbb{R}^d} \psi(0,x)\rho^0(x)\,dx = \int_{Q_\infty} \nabla q \cdot \nabla \psi - \rho \partial_t \psi - \rho V \cdot \nabla \psi - \mu \psi \tag{5.11}$$

for any $\psi \in W_c^{1,1}([0,\infty); H^1(\mathbb{R}^d))$. After taking the limit, the bounds on ρ, q, μ inherited from the estimates (4.12-4.18) allow us to conclude that (5.11) holds for any $\psi \in W_c^{1,1}([0,\infty); L^1(\rho) \cap \dot{H}^1(\mathbb{R}^d))$.

Step 2: Weak convergence of the products $\rho_{1,k}q_k, \rho_{2,k}q_k$.

We want to use Lemma 3.5 to prove that $\rho_{i,k}q_k$ converges weakly to $\rho_i q$ for $i = 1, 2$. This will imply that $\rho_k q_k$ converges weakly to ρq. Fix some $\epsilon > 0$ and let η_ϵ be a spatial mollifier. Define $\rho_{i,k,\epsilon} = \eta_\epsilon * \rho_{i,k}$ and $\rho_{i,\epsilon} = \eta_\epsilon * \rho_i$. Thanks to estimates (4.12-4.14), it follows that

$$\sup_k \|\partial_t \rho_{i,k,\epsilon}\|_{L^2(Q_T)} + \|\nabla \rho_{i,k,\epsilon}\|_{L^2(Q_T)} \lesssim_\epsilon \sup_k \|\rho_{i,k}\|_{L^2(Q_T)} + \|\rho_{i,k}\|_{H^1([0,T];H^{-1}(\mathbb{R}^d))} < \infty.$$

Thus, for $\epsilon > 0$ fixed, $\rho_{i,k,\epsilon}$ is uniformly equicontinuous in $L^2(Q_T)$. The uniform bounds (4.12) and (4.14) automatically upgrade this to uniform equicontinuity in $L^r(Q_T) \cap L^1(Q_T)$ for any $r < \infty$. In addition, the estimates (4.18) and (4.15) imply that q_k is spatially equicontinuous in $L^2_{\mathrm{loc}}(Q_\infty)$. Thus, we can apply Lemma 3.5 to conclude that $\rho_{i,k}q_k$ converges weakly in $(C_c(Q_\infty))^*$ to $\rho_i q$ for $i = 1, 2$. The uniform boundedness of $\rho_{i,k}q_k$ in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$ gives us the automatic upgrade to weak convergence in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$. Now Proposition 3.2 implies that $\rho q = e(\rho) + e^*(q)$ almost everywhere and $\rho_k q_k, e(\rho_k)$ and $e^*(q_k)$ converge weakly in $L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))$ to $\rho q, e(\rho)$ and $e^*(q)$ respectively. Now we can use Proposition 4.2 to conclude that for every $\varphi \in W_c^{1,\infty}([0,\infty); L^1(\rho) \cap \dot{H}^1(\mathbb{R}^d))$ the limit variables ρ, μ, q satisfy the energy dissipation relation (4.2).

Step 3: Strong convergence of ∇q_k to ∇q in $L^2_{\mathrm{loc}}(Q_\infty)$

We now want to use our work in Proposition 5.2 to prove the strong convergence of the pressure gradient. From Proposition 4.3, we know that (5.5) holds. The pointwise everywhere convergence of z_k to z implies the pointwise everywhere convergence of e_k to e. From step 2, we know that $\rho_k q_k$ converges weakly to ρq and that (4.2) holds for the limit variables. Thus, to apply Proposition 5.2 it remains to show that $\limsup_{k\to\infty} \int_D \omega \mu_k q_k \leq \int_D \omega \mu q$ for every $\omega \in W_c^{1,\infty}([0,\infty))$ and compact set $D \subset Q_\infty$. We will split our work into two cases.

Step 3a: Scenario (a) holds. When $\partial z(a)$ is a singleton for all $a \in (0,\infty)$, it follows that $\partial e(a)$ is a singleton for all $a \in (0,\infty)$. Thus, Lemma 3.3 implies that q_k converges in measure to q. Since q_k is uniformly bounded in $L^2_{\mathrm{loc}}(Q_\infty) \cap L^2_{\mathrm{loc}}([0,\infty); \dot{H}^1(\mathbb{R}^d))$, we can upgrade the convergence in measure to strong convergence in $L^r_{\mathrm{loc}}(Q_\infty)$ for any $r < 2$. Thus, Proposition 5.2 implies that ∇q_k converges strongly to ∇q in $L^2_{\mathrm{loc}}(Q_\infty)$.

Step 3b: Scenario (b) holds

Without strict convexity of the dual energy, the weak convergence of $e_k^*(q_k)$ does not give us strong convergence of q_k. Thus, we will instead need to establish the nonlinear limit (**??**) even though we do not have access to any strong convergence properties. To succeed in this endeavor, we will employ a delicate argument that exploits the structure of the product $q_k\mu_k$

We begin by fixing some $\delta > 0$ and letting J_δ be a space time mollifier. Set $q_{k,\delta} := J_\delta * q_k$ and $q_\delta := q * J_\delta$. It is clear that $q_{k,\delta}$ converges strongly to q_δ in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$ and q_δ converges strongly to q in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$. Thus, it will be enough to show that

$$\liminf_{\delta\to 0} \limsup_{k\to\infty} \int_D \omega(q_k - q_{k,\delta})\mu_{i,k} \leq 0,$$

for $i = 1, 2$.

We focus on the case $i=1$ (the argument for $i=2$ is identical). Assumption (F3) and the monotonicity of $(z_k^*)^{-1}$ guarantees that $q \mapsto F_{1,1}\big((z_k^*)^{-1}(q), n\big) + F_{1,2}\big((z_k^*)^{-1}(q), n\big)$ is decreasing for each fixed value of n. As a result, there must exist a function $f_k : [0,\infty)\times[0,\infty) \to \mathbb{R}$ such that for each fixed value of n, we have $f_k(0,n)=0$, $q \mapsto f_k(q,n)$ is convex, and $-\partial_q f_k(q,n) = F_{1,1}\big((z_k^*)^{-1}(q), n\big) + F_{1,2}\big((z_k^*)^{-1}(q), n\big)$. The structure of $\mu_{1,k}$ combined with the convexity of f_k implies that

$$\int_D \omega(q_k - q_{k,\delta})\mu_{i,k} \leq \int_D \omega\rho_{1,k}\big(f_k(q_{k,\delta}, n_k) - f_k(q_k, n_k)\big).$$

Since $F_{1,1}+F_{1,2}$ is uniformly bounded over $\mathbb{R}\times[0,\infty)$, it follows that f_k is uniformly Lipschitz in the first argument. Uniform equicontinuity in the second argument is clear when $q=0$. For $q>0$, fix some $\epsilon \in (0,q)$ and consider $n_1, n_2 \geq 0$. We see that

$$|f_k(q,n_1) - f_k(q,n_2)| \leq \sum_{i=1}^{2} \int_0^q |F_{1,i}\big((z_k^*)^{-1}(a), n_1\big) - F_{1,i}\big((z_k^*)^{-1}(a), n_2\big)| da.$$

$$\leq 2B\epsilon + q \sup_{b \in [(z_k^*)^{-1}(\epsilon), (z_k^*)^{-1}(q)]} \sum_{i=1}^{2} |F_{1,i}\big(b, n_1\big) - F_{1,i}\big(b, n_2\big)|,$$

where B is a bound on $F_{1,1}+F_{1,2}$. Assumption (z3) and the pointwise everywhere convergence of z_k to z implies that $(z_k^*)^{-1}(\epsilon), (z_k^*)^{-1}(q)$ are uniformly bounded with respect to k. Thus, it now follows that f_k is uniformly equicontinuous in the second argument on compact subsets of $[0,\infty)^2$. As a result, f_k must converge uniformly on compact subsets of $[0,\infty)^2$ (possibly along some subsequence that we do not relabel) to a limit function f that is convex in the first variable and continuous in the second.

For all k we have $|f_k(q,n)| \leq Bq$. Thus, it is now clear that

$$\liminf_{\delta\to 0} \limsup_{k\to\infty} \int_D \omega\rho_{1,k}\Big(|f_k(q_{k,\delta}, n_k) - f(q,n)| + |f_k(q_k,n_k) - f(q_k,n_k)| + |f(q_k,n) - f(q_k,n_k)|\Big) = 0.$$

It remains to prove that

$$\limsup_{k\to\infty} \int_D \omega\rho_{1,k}\big(f(q,n) - f(q_k,n)\big) \leq 0.$$

Let $f^*(a,n) = \sup_{q\in[0,\infty)} aq - f(q,n)$. Given any smooth function $\psi \in C_c^\infty(Q_\infty)$, we have

$$\int_D \omega\rho_{1,k}\big(f(q,n) - f(q_k,n)\big) \leq \int_D \omega\rho_{1,k}\big(f(q,n) - q_k\psi\big) + \omega\rho_{1,k} f^*(\psi, n).$$

Using the weak convergence of the product $\rho_{1,k}q_k$ to $\rho_1 q$ we see that

$$\limsup_{k\to\infty} \int_D \omega\rho_{1,k}\big(f(q,n) - q_k\psi\big) + \rho_{1,k} f^*(\psi,n) = \int_D \omega\rho_1\big(f(q,n) - q\psi\big) + \omega\rho_1 f^*(\psi,n).$$

Taking an infimum over ψ, we get

$$\limsup_{k\to\infty} \int_D \omega\rho_{1,k}\big(f(q,n) - f(q_k,n)\big) \leq 0.$$

as desired.

Step 4: Passing to the limit in the weak equations

Now that we have obtained the strong convergence of the pressure gradient, we are ready to pass to the limit in the weak equations. In Lemma 5.4, we showed that the source terms converge weakly to the desired limit under the convergence properties that we have established. The weak convergence of the remaining terms is clear except for the weak convergence of the product $\frac{\rho_{i,k}}{\rho_k}\nabla q_k$ to $\frac{\rho_i}{\rho}\nabla q$. Given some $\delta>0$ and a compact set $D\subset Q_\infty$ it follows from Lemma 5.3 that $\frac{1}{\rho_k+\delta}\nabla q_k$ converges strongly in $L^2_{\mathrm{loc}}(D)$ to $\frac{1}{\rho+\delta}\nabla q$. Thus, if we can show that

$$\liminf_{\delta\to 0}\Big(\int_D \frac{\delta\rho_i}{\rho(\rho+\delta)}|\nabla q|^2+\limsup_{k\to\infty}\int_D\frac{\delta\rho_{i,k}}{\rho_k(\rho_k+\delta)}|\nabla q_k|^2\Big)=0, \tag{5.12}$$

then it will follow that $\frac{\rho_{i,k}}{\rho_k}\nabla q_k$ converges weakly in $L^2_{\mathrm{loc}}(D)$ to $\frac{\rho_i}{\rho}\nabla q$.

Since $\rho_{i,k}\le\rho_k$ and $\rho_i\le\rho$, the left hand side of (5.12) is bounded above by

$$\liminf_{\delta\to 0}\Big(\int_D \frac{\delta}{\rho+\delta}|\nabla q|^2+\limsup_{k\to\infty}\int_D\frac{\delta}{\rho_k+\delta}|\nabla q_k|^2\Big)$$
$$=\liminf_{\delta\to 0}\int_D\frac{2\delta}{\rho+\delta}|\nabla q|^2,$$

where we have used Lemma 5.3 to go from the first line to the second. The property $\limsup_{a\to 0^+}\frac{e(a)}{a}=0$ combined with the duality relation implies that $q=0$ whenever $\rho=0$. As a result, $|\nabla q|$ gives no mass to the set of points where $\rho=0$. By dominated convergence

$$\liminf_{\delta\to 0}\int_D\frac{2\delta}{\rho+\delta}|\nabla q|^2=0.$$

□

Corollary 5.6. *Let e be an energy satisfying (e1-e3) such that $\partial e(a)$ is a singleton for all $a\in(0,\infty)$. Let $F_{i,j}$ be source terms satisfying (F1-F2). Given initial data $\rho_1^0,\rho_2^0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d), n^0\in L^2(\mathbb{R}^d)$ such that $e(\rho_1^0+\rho_2^0)\in L^1(\mathbb{R}^d)$, there exists a weak solution $(\rho_1,\rho_2,q,n)\in\mathcal{X}(e)\times\mathcal{X}(e)\times\mathcal{Y}(e^*)\times L^2_{\mathrm{loc}}([0,\infty);H^1(\mathbb{R}^d))$ to the system (1.3).*

Proof. For $\gamma_k=\frac{1}{k}$, the existence of a solution to the system (5.1) for the fixed energy e is straightforward. Using these solutions, we can pass to the limit as $k\to\infty$ using Theorem 5.5. □

Corollary 5.7. *Let e be an energy satisfying (e1-e3) and let $F_{i,j}$ be source terms satisfying (F1-F3). Given initial data $\rho_1^0,\rho_2^0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d), n^0\in L^2(\mathbb{R}^d)$ such that $e(\rho_1^0+\rho_2^0)\in L^1(\mathbb{R}^d)$, there exists a weak solution $(\rho_1,\rho_2,q,n)\in\mathcal{X}(e)\times\mathcal{X}(e)\times\mathcal{Y}(e^*)\times L^2_{\mathrm{loc}}([0,\infty);H^1(\mathbb{R}^d))$ to the system (1.1).*

Proof. See Corollary 5.6. □

Corollary 5.8. *Let $F_{i,j}$ be source terms satisfying (F1-F3). Given initial data $\rho_1^0,\rho_2^0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d), n^0\in L^2(\mathbb{R}^d)$ such that $\rho_1^0+\rho_2^0\le 1$ almost everywhere, let $(\rho_{1,m},\rho_{2,m},q_m,n_m)\in\mathcal{X}(e)\times\mathcal{X}(e)\times\mathcal{Y}(e^*)\times L^2_{\mathrm{loc}}([0,\infty);H^1(\mathbb{R}^d))$ be weak solutions of the system (1.3) with the energy $e_m(a)=\frac{1}{m}a^m$. As $m\to\infty$, any weak limit point of the sequence $(\rho_{1,m},\rho_{2,m},q_m,n_m)$ is a solution to the system (1.3) with the incompressible energy*

$$e_\infty(a)=\begin{cases}0 & \text{if } a\in[0,1],\\ +\infty & \text{otherwise.}\end{cases}$$

Furthermore, for every $\varphi\in L^\infty_c([0,\infty);W^{1,\infty}(\mathbb{R}^d)\cap\dot{H}^1(\mathbb{R}^d))$ the pressure satisfies the complementarity condition

$$\int_{Q_\infty}(\nabla q-\rho V)\cdot\nabla(q\varphi)-\varphi q\Big(\rho_1\big(F_{1,1}(q,n)+F_{1,2}(q,n)\big)+\rho_2\big(F_{2,1}(q,n)+F_{2,2}(q,n)\big)\Big)=0. \tag{5.13}$$

Proof. It is clear that e_m converges pointwise everywhere to e_∞. We can use Corollary 5.6 to construct weak solutions of (1.3) for each $m > 0$. We can then use Theorem 5.5 to pass to the limit $m \to \infty$. Note that the complementarity condition is precisely the energy dissipation relation (4.2) applied to the sum of the first two equations of (1.3). Indeed, since $\rho_1^0 + \rho_2^0 \le 1$, it follows that $e(\rho^0) = 0$ and $e(\rho) = 0$ almost everywhere. Thus, (4.2) simplifies to (5.13). $\square$

At last, we will show that weak solutions to (1.3) can be converted into weak solutions to (1.1).

Proposition 5.9. *Let z be an energy satisfying (z1-z3) and define e by formula (2.1). Suppose that $\rho_1^0, \rho_2^0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), n^0 \in L^2(\mathbb{R}^d)$ is initial data such that $e(\rho_1^0+\rho_2^0), z(\rho_1^0+\rho_2^0) \in L^1(\mathbb{R}^d)$. If $(\rho_1, \rho_2, q, n) \in \mathcal{X}(e) \times \mathcal{X}(e) \times \mathcal{Y}(e^*) \times L^2_{\mathrm{loc}}([0,\infty); H^1(\mathbb{R}^d))$ is a weak solution to the system (1.3) and we set $p = (z^*)^{-1}(q)$ then $(\rho_1, \rho_2, p, n) \in \mathcal{X}(e) \times \mathcal{X}(e) \times L^2_{\mathrm{loc}}([0,\infty); L^1_{\mathrm{loc}}(\rho)) \cap \dot{H}^1(\rho)) \times L^2_{\mathrm{loc}}([0,\infty); H^1(\mathbb{R}^d))$ is a weak solution of (1.1).*

Proof. The duality relation $\rho q = e(\rho) + e^*(q)$ is equivalent to $p\rho = z(\rho) + z^*(p) = z(\rho) + q$. Since $\rho \in L^\infty_{\mathrm{loc}}([0,\infty); L^\infty(\mathbb{R}^d))$ it follows that $z(\rho)$ is bounded from below on any compact subset of Q_∞. If there exists $a_1 > 0$ such that $\lim_{a\to a_1} z(a) = +\infty$, then there must be some $0 \le a_2 < a_1$ such that z is increasing on $[a_2, a_1)$ and decreasing on $[0, a_2]$. Let $a_3 = \max(a_2, \frac{3}{4}a_1)$. We can then compute

$$\lim_{a\to a_1} \frac{z(a)}{e(a)} \le \lim_{a\to a_1} \frac{z(a)}{az(a) - 2\int_{a_2}^a z(s)\,ds} \le \lim_{a\to a_1} \frac{z(a)}{\frac{a_1}{2}z(a) - 2(a_3 - a_2)z(a_3)} = \frac{2}{a_1} < \infty.$$

Therefore, $e(\rho) \in L^\infty_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d))$ implies that $z(\rho) \in L^\infty_{\mathrm{loc}}([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^d))$. Since $q \in L^2_{\mathrm{loc}}(Q_\infty)$ it now follows that $p \in L^2_{\mathrm{loc}}([0,\infty); L^1_{\mathrm{loc}}(\rho))$.

If $(z^*)^{-1}$ is uniformly Lipschitz on $[0,\infty)$, then the chain rule for Sobolev functions implies that $\nabla p = \frac{1}{\rho}\nabla q$ and $\nabla p \in L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$. In this case, it is now clear that (ρ_1, ρ_2, p, n) is a solution to the system (1.1). The regularity of p can then be improved by arguing as in Propositions 4.2.

Now we suppose that $(z^*)^{-1}$ is not uniformly Lipschitz on $[0,\infty)$. Fix some $\delta > 0$ and define $q_\delta := \max(q, \delta)$, and $p_\delta := (z^*)^{-1}(q_\delta)$. The monotonicity of z^* implies that p_δ is decreasing with respect to δ. Therefore,

$$\int_D \rho|p_\delta - p| = \int_D \rho(p_\delta - p) \le \int_D z^*(p_\delta) - z^*(p) = \int_D q_\delta - q \le \delta|D|.$$

Hence, p_δ converges to p in $L^1_{\mathrm{loc}}(\rho)$. Since $(z^*)^{-1}$ is Lipschitz on $[\delta,\infty)$, the chain rule for Sobolev functions allows us to compute $\nabla p_\delta = \frac{\chi_\delta(q)}{\rho}\nabla q$ where χ_δ is the characteristic function of the interval $[\delta,\infty)$. A direct computation reveals that

$$\int_D \rho|\nabla p_{\delta_1} - \nabla p_{\delta_0}| = \int_D |\nabla q||\chi_{\delta_1}(q) - \chi_{\delta_0}(q)|.$$

Since $|\nabla q|$ vanishes almost everywhere on the set $\{(t,x) \in Q_\infty : q(t,x) = 0\}$, it follows that ∇p_δ is a Cauchy sequence in $L^1_{\mathrm{loc}}(\rho)$ that converges to $\frac{1}{\rho}\nabla q$. In particular, for any smooth vector field v with compact support we can conclude that

$$\lim_{\delta\to 0} |\int_{Q_\infty} (\rho_i \nabla p_\delta - \frac{\rho_i}{\rho}\nabla q) \cdot v| = 0.$$

for $i = 1, 2$.

It remains to show that $\lim_{\delta\to 0} \nabla p_\delta = \nabla p$ in the sense of distributions. Fix some $\epsilon > 0$ and let $\eta_\epsilon : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that $\eta_\epsilon(a) = 0$ if $a \le \epsilon/2$ and $\eta_\epsilon(a) = 1$ if $a \ge \epsilon$. Since $\limsup_{a\to 0^+} \frac{e(a)}{a} = 0$, it follows that $\frac{1}{\rho}$ is bounded on the set $q \ge \epsilon$. Hence, $\eta_\epsilon(q)$ and $\eta_\epsilon'(q)$

are absolutely continuous with respect to ρ. Given a test function $\varphi \in L^\infty_c([0,\infty); W^{1,\infty}_c(\mathbb{R}^d))$, we can compute

$$\int_{Q_\infty} p\nabla\cdot(\varphi\eta_\epsilon(q)) = \lim_{\delta\to 0}\int_{Q_\infty} p_\delta\nabla\cdot(\varphi\eta_\epsilon(q)) = -\lim_{\delta\to 0}\int_{Q_\infty}\nabla p_\delta\cdot(\varphi\eta_\epsilon(q))$$
$$= -\lim_{\delta\to 0}\int_{Q_\infty}\frac{\chi\delta(q)}{\rho}\eta_\epsilon(q)\nabla q\cdot\varphi = -\int_{Q_\infty}\frac{\eta_\epsilon(q)}{\rho}\nabla q\cdot\varphi$$

Thus, $\nabla p = \frac{1}{\rho}\nabla q = \lim_{\delta\to 0}\nabla p_\delta$ in the sense of distributions when tested against functions of the form $\varphi\eta_\epsilon(q)$. When z^* fails to be Lipschitz on $[0,\infty)$ it follows that ρ approaches zero wherever q approaches zero. Thus by sending $\epsilon\to 0$ we can conclude that $\nabla p = \frac{1}{\rho}\nabla p$ on $\rho > 0$. It now follows that (ρ_1, ρ_2, p, n) is a solution to the system (1.1). Again, the regularity of p can then be improved by arguing as in Propositions 4.2.

□

The proofs of Theorems 1.1 1.2, and 1.3 are now just corollaries of the previous proposition, Theorem 5.5, and Corollaries 5.6 and 5.7.

Appendix A. Some properties of convex functions

Lemma A.1. *Let $f:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ be a proper, lower semicontinuous, convex function such that $f^{-1}(\mathbb{R})$ is not a singleton. If $f_k:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is a sequence of proper, lower semicontinuous convex functions such that f_k converges pointwise everywhere to f then the following properties hold:*

(1) If f is differentiable at a point $a\in\mathbb{R}$, then
$$\limsup_{k\to\infty}\max\Big(|\sup\partial f_k(a) - f'(a)|\ ,\ |\inf\partial f_k(a) - f'(a)|\Big) = 0.$$
(2) The convergence of f_k to f is uniform on compact subsets of the interior of $f^{-1}(\mathbb{R})$.
(3) f_k^* converges pointwise everywhere to f^* except possibly at the two exceptional values $b^+_\infty = \sup\{b\in\mathbb{R} : f^*(b) < \infty\}, b^-_\infty = \inf\{b\in\mathbb{R} : f^*(b) < \infty\}$.
(4) If f^* is differentiable at a point $b\in\mathbb{R}$, then
$$\limsup_{k\to\infty}\max\Big(|\sup\partial f_k^*(b) - f^{*\prime}(b)|\ ,\ |\inf\partial f_k^*(b) - f^{*\prime}(b)|\Big) = 0,$$
and the convergence of f_k^* to f^* is uniform on compact subsets of the interior of $(f^*)^{-1}(\mathbb{R})$.

Proof. Let a be a point of differentiability for f. Since $f'(a)$ exists and is finite, there exists $\delta_0 > 0$ such that f is finite on $[a-\delta_0, a+\delta_0]$. Fix some $\delta\in(0,\delta_0)$. The convergence of f_k to f implies that there must exist some N, B sufficiently large such that $|f_k(a)|, |f_k(a-\delta)|, |f_k(a+\delta)| < B$ for all $k > N$. Now we can use convexity to bound
$$\frac{f_k(a) - f_k(a-\delta)}{\delta} \le \inf\partial f_k(a) \le \sup\partial f_k(a) \le \frac{f_k(a+\delta) - f_k(a)}{\delta}.$$
Thus,
$$\limsup_{k\to\infty}\max\Big(|\sup\partial f_k(a) - f'(a)|\ ,\ |\inf\partial f_k(a) - f'(a)|\Big) \le \Big|\frac{f(a) - f(a-\delta)}{\delta} - f'(a)\Big| + \Big|\frac{f(a+\delta) - f(a)}{\delta} - f'(a)\Big|.$$
Sending $\delta\to 0$ and using the fact that f is differentiable at a, we get the desired result.

Now suppose that $[a_0, a_1]$ is an interval in the interior of $f^{-1}(\mathbb{R})$ and choose some $\delta > 0$ such that $[a_0-\delta, a_1+\delta]$ is still in the interior of $f^{-1}(\mathbb{R})$ and f is differentiable at $a_0-\delta, a_1+\delta$. Given any $a\in[a_0, a_1]$, we have
$$f'(a_0-\delta) \le \inf\partial f(a) \le \sup\partial f(a) \le f'(a_1+\delta).$$

It then follows from our above work that $\partial f_k(a)$ is uniformly bounded on $[a_0, a_1]$ for all k sufficiently large. Hence, f_k is uniformly equicontinuous on $[a_0, a_1]$ and thus converges uniformly to f.

Now we consider f^*. Given any $b \in \mathbb{R}$, if we choose some $a \in f^{-1}(\mathbb{R})$, then

$$\liminf_{k\to\infty} f_k^*(b) \geq \liminf_{k\to\infty} ab - f_k(a) = ab - f(a).$$

Taking a supremum over a, it follows that $\liminf_{k\to\infty} f_k^*(b) \geq f^*(b)$. Hence we only need to worry about $b \in (f^*)^{-1}(\mathbb{R})$.

Let $b \in (f^*)^{-1}(\mathbb{R}) \setminus \{b_\infty^-, b_\infty^+\}$. It then follows that $\partial f^*(b) \neq \varnothing$ and so we can define $a_0 := \inf \partial f^*(b)$ and $a_1 := \sup \partial f^*(b)$. Again since $b \notin \{b_\infty^-, b_\infty^+\}$ a_0, a_1 must be finite. If we fix some $\delta > 0$ then $\frac{f(a_1+\delta)-f(a_1)}{\delta} > b$ and similarly $\frac{f(a_0)-f(a_0-\delta)}{\delta} < b$. Thus, the pointwise convergence of f_k to f implies that $\frac{f_k(a_1+\delta)-f_k(a)}{\delta} > b$ and $\frac{f(a_0)-f(a_0-\delta)}{\delta} < b$ for all k sufficiently large. Thus,

$$f_k^*(b) = \sup_{a\in[a_0-\delta, a_1+\delta]} ab - f_k(a),$$

for all k sufficiently large.

If $a_0 < a_1$, then there exists a point $a' \in (a_0, a_1)$ such that f is differentiable at a'. Choose $b_k' \in \partial f_k(a')$ and note that our earlier work shows that b_k' converges to $f'(a') = b$. Hence,

$$f_k^*(b) \leq \sup_{a\in[a_0-\delta, a_1+\delta]} ab - f_k(a') - b_k'(a - a') \leq \max(|a_0 - \delta||b - b_k'|, |a_1 + \delta||b - b_k'|) + b_k'a' - f_k(a').$$

So we obtain $\lim_{k\to\infty} f_k^*(b) \leq f'(a')a' - f(a') = f^*(b)$ where the final equality follows from the fact that $f'(a') = b$.

Now suppose that $a_0 = a_1$. Since $f^{-1}(\mathbb{R})$ is not a singleton, there exists a point $a_\delta \in [a_0 - \delta, a_0 + \delta]$ such that f is differentiable at a_δ (note that we can always choose our a_δ such that a_δ is either increasing or decreasing with respect to δ). If we choose $b_{k,\delta} \in \partial f_k(a_\delta)$, then we have

$$f_k^*(b) \leq \sup_{a\in[a_0-\delta, a_1+\delta]} ab - f_k(a_\delta) - b_{k,\delta}(a - a_\delta) \leq a_0 b - f_k(a_\delta) + \delta|b| + \delta|b_{k,\delta}|.$$

Since $b_{k,\delta}$ must converge to $f'(a_\delta)$, it follows that

$$\lim_{k\to\infty} f_k^*(b) \leq a_0 b - f(a_\delta) + \delta|b| + \delta|f'(a_\delta)|.$$

If a_δ is increasing with respect to δ then $\lim_{\delta\to 0} f'(a_\delta) = \inf \partial f(a_0)$ while $\lim_{\delta\to 0} f'(a_\delta) = \sup \partial f(a_0)$ if it is decreasing. Since either $\inf \partial f(a_0)$ or $\sup \partial f(a_0)$ is finite, we can assume that we chose a_δ such that $\limsup_{\delta\to 0} |f'(a_\delta)| < \infty$. Hence, sending $\delta \to 0$, we can conclude that $\lim_{k\to\infty} f_k^*(b) \leq a_0 b - f(a_0) = f^*(b)$. This completes the argument that $\lim_{k\to\infty} f_k^*(b) = f^*(b)$ if $b \notin \{b_\infty^-, b_\infty^+\}$.

Now that we have proven that $\lim_{k\to\infty} f_k^*(b) = f^*(b)$ for all $b \in \mathbb{R} \setminus \{b_\infty^-, b_\infty^+\}$ we can use the arguments we applied to f_k to conclude property (4).

□

Lemma A.2. *Let $z : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be an energy satisfying (z1-z3) and let $z_k : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a sequence of energies satisfying (z1-z3) such that z_k converges pointwise everywhere to z. If we set $b_\infty = \inf\{b \in \mathbb{R} : z^*(b) = +\infty\}$ then $(z_k^*)^{-1}$ converges uniformly to $(z^*)^{-1}$ on compact subsets of $\big(0, z^*(b_\infty)\big)$.*

Proof. If $z^*(b_\infty) = 0$, then there is nothing to prove. Otherwise, given $\epsilon \in (0, z^*(b_\infty))$ there must exist $b_{\epsilon/2} < b_\epsilon \in \mathbb{R}$ such that $z^*(b_{\epsilon/2}) = \epsilon/2$ and $z^*(b_\epsilon) = \epsilon$. It then follows that for all $b \geq b_\epsilon$ and k sufficiently large

$$\frac{\epsilon}{4(b_\epsilon - b_{\epsilon/2})} \leq \inf \partial z_k^*(b).$$

As a result, $(z_k^*)^{-1}$ is uniformly Lipschitz on $[\epsilon, z^*(b_\infty))$. Choose some value $a \in [\epsilon, z^*(b_\infty))$ and let $\bar{b} = (z^*)^{-1}(a)$. Let $a_k = z_k^*(\bar{b})$ and note that once k is sufficiently large we must have $a \in z_k^*(\mathbb{R})$. Thus,

$$|(z^*)^{-1}(a) - (z_k^*)^{-1}(a)| = |\bar{b} - (z_k^*)^{-1}(a_k + a - a_k)| \le L_\epsilon |a - a_k| = L_\epsilon |z^*(\bar{b}) - z_k^*(\bar{b})|$$

Now the uniform convergence of z_k^* to z^* on compact subsets of $(-\infty, b_\infty)$ combined with the Lipschitz bound implies the uniform convergence of $(z_k^*)^{-1}$ to $(z^*)^{-1}$ on compact subsets of $(0, z^*(b_\infty))$.

□

Lemma A.3. *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function and let $f^* : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be its convex conjugate. Suppose that $a \in f^{-1}(\mathbb{R})$ and there exists some $b \in \partial f(a)$. Given any $\epsilon > 0$, there exists $\lambda_f(a, b, \epsilon) > 0$ such that*

$$f(\bar{\alpha}) - f(a) - b(\bar{\alpha} - a) \ge \epsilon \lambda_f(a, b, \epsilon)$$

for any $\bar{\alpha} \in \{\alpha \in \mathbb{R} : [\alpha - \epsilon, \alpha + \epsilon] \cap \partial f^*(b) = \varnothing\}$.

Proof. Define

$$\alpha_f^+(a, b, \epsilon) := \inf\{\alpha > a : [\alpha - \epsilon, \alpha] \cap \partial f^*(b) = \varnothing\},$$

$$\alpha_f^-(a, b, \epsilon) := \sup\{\alpha < a : [\alpha, \alpha + \epsilon] \cap \partial f^*(b) = \varnothing\},$$

$$\lambda_f^+(a, b, \epsilon) := \begin{cases} \sup_{\bar{b} \in \partial f\left(\alpha_f^+(a,\epsilon)\right)} \bar{b} - b & \text{if } \partial f\big(\alpha_f^+(a, \epsilon)\big) \neq \varnothing, \\ +\infty & \text{else,} \end{cases}$$

$$\lambda_f^-(a, b, \epsilon) := \begin{cases} \inf_{\bar{b} \in \partial f\left(\alpha_f^+(a,\epsilon)\right)} \bar{b} - b & \text{if } \partial f\big(\alpha_f^-(a, \epsilon)\big) \neq \varnothing, \\ +\infty & \text{else,} \end{cases}$$

Since subdifferentials are closed sets, it follows that $\lambda_f^+(a, b, \epsilon), |\lambda_f^-(a, b, \epsilon)| > 0$ for all $\epsilon > 0$. With these definitions, we now see that for any $\alpha_0 \in \{\alpha < a : [\alpha, \alpha + \epsilon] \cap \partial f^*(b) = \varnothing\}$ we have

$$\begin{aligned} &f(\alpha_0) - f(a) - b(\alpha_0 - a) \\ &\ge f\big(\alpha_f^-(a, b, \epsilon)\big) - f(a) - b\big(\alpha_f^-(a, b, \epsilon) - a\big) + \lambda_f^-(a, b, \epsilon)\big(\alpha_0 - \alpha_f^-(a, b, \epsilon)\big) \\ &\ge \lambda_f^-(a, b, \epsilon)\big(\alpha_0 - \alpha_f^-(a, b, \epsilon)\big) \\ &\ge \epsilon|\lambda_f^-(a, b, \epsilon)|. \end{aligned}$$

Similarly, for any $\alpha_1 \in \{\alpha > a : [\alpha, \alpha + \epsilon] \cap \partial f^*(b) = \varnothing\}$ we have

$$\begin{aligned} &f(\alpha_1) - f(a) - b(\alpha_0 - a) \\ &\ge f\big(\alpha_f^+(a, b, \epsilon)\big) - f(a) - b\big(\alpha_f^+(a, b, \epsilon) - a\big) + \lambda_f^+(a, b, \epsilon)\big(\alpha_1 - \alpha_f^+(a, b, \epsilon)\big) \\ &\ge \lambda_f^+(a, b, \epsilon)\big(\alpha_1 - \alpha_f^+(a, b, \epsilon)\big) \\ &\ge \epsilon\lambda_f^+(a, b, \epsilon). \end{aligned}$$

Finally, if we define $\lambda_f(a, b, \epsilon) := \min(\lambda_f^+(a, b, \epsilon), |\lambda_f^-(a, b, \epsilon)|)$, then it follows that for any $\bar{\alpha} \in \{\alpha \in \mathbb{R} : [\alpha - \epsilon, \alpha + \epsilon] \cap \partial f^*(b) = \varnothing\}$, we have

$$f(\bar{\alpha}) - f(a) - b(\bar{\alpha} - a) \ge \epsilon\lambda_f(a, b, \epsilon) > 0.$$

□

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