(4) With your calculator set to use the chain calculation method, consider the following sequences of keystrokes. In each case, describe what they accomplish if the calculator has just been cleared.

(a) [1] [•] [0] [6] [y^x] [5] [=] [×] [5] [4] [2] [=] [STO] [7] [7] [6] [8] [y^x] [4] [1/x] [=] [STO] [+] [7]

(b) [2ND] [MEM] [2ND] [CLR WORK] [1] [ENTER] [↓] [2] [ENTER] [↓] [3] [ENTER] [↓] [4] [2ND] [QUIT]

(c) [6] [2] [5] [×] [5] [2] [=] [4] [0] [8] [9] [+] [2ND] [ANS] [=]

(d) [×] [2ND] [K] [1] [•] [0] [4] [=] [2] [=] [3] [=] [5] [8] [=] [STO] [4] [=] [+] [5] [4] [=] [STO] [3] [2] [=]

(e) [5] [+] [2ND] [K] [1] [0] [%] [=] [1] [0] [0] [=] [2] [0] [0] [=] [%] [%]

(f) [5] [4] [3] [→] [2] [÷] [0] [=] [CE/C] [5] [4] [2] [+/−] [LN] [CE/C] [5] [4] [2] [2ND] [+/−]

CHAPTER 1

The growth of money

1.1 INTRODUCTION

Throughout this book, the growth of money due to investment will be discussed. In our first section (Section (1.2)) we take a brief look at the rationale behind deposits increasing in value, and we end the chapter with Section (1.14) describing how inflation can erode that growth. In Section (1.3), we establish units of time and of money and the fundamental **amount functions**. Linear **simple interest** amount functions are discussed in Section (1.4), while in Section (1.5) we discuss the **usual compound interest accumulation function.** It is important to read that section carefully so that you understand why growth by compound interest is so prevalent. You are probably familiar with loans where interest is applied at the end of each time period, but the concept of loans **made at a discount** where interest is due at the beginning of the loan may be new to you. This is discussed in Section (1.6), and **discount rates** are introduced. The fundamental concept of **equivalence of rates** is also established. In Section (1.7) we introduce **discount functions,** which are reciprocals of amount functions. Discount functions are useful for determining the value of a promised future payment, and linear discount functions are discussed in Section (1.8). In

Section (1.9) we discover that what we might reasonably call "compound discount" is **equivalent** to compound interest. The interest rates and discount rates introduced prior to Section (1.9) are called **effective** rates. **Nominal** rates are explained in Section (1.10). These are rates that do not give rise to the expected growth for periods differing from the so-called compounding period. As we look at nominal rates compounded more and more frequently, we are led to consider the **force of interest**. The force of interest is a measure of how strongly interest is working to increase investments at a given instant.**force of interest** In Section (1.11) we find that compound interest gives rise to a constant force of interest. For those with a background including calculus, Section (1.12) gives a general discussion of the force of interest. If you do not know calculus, skip Section (1.12), but Section (1.13) was written for you.

1.2 WHAT IS INTEREST ?

People frequently participate in financial transactions. These are varied and at times quite complicated. However, it is fair to say that when lenders invest their money, they usually do so with the expectation (or at least the hope) of financial gain. If an investment amount $\$K$ grows to an amount $\$S$, then the difference $\$S - \$K$ is **interest**. The interest may be thought of as a rent paid by the borrower for the use of the $\$K$.

You probably take it for granted that you will be charged interest if you borrow money from a bank, and that you will receive interest if you lend money to a bank by opening a savings account. Nowadays, except for loans to family members or friends, interest-free loans are a rarity. In Western society the charging of interest is generally accepted business practice, but this was not always true. A powerful moral argument suggests you help your neighbor if he is in need. In the Middle Ages, the Catholic church viewed the charging of interest as sinful, and this view is still held in much of the Islamic world.

Part of the economic rationale for the charging of interest is based on the economic productivity of capital. This is sometimes referred to as the **investment opportunities theory**. If a farmer borrows seed and then harvests the crop he grew from it, he may return the quantity of seed he borrowed plus a bit more as interest. The loan allowed productivity, and the farmer shares his gain. Likewise, if you borrow money to run a successful business, the borrowed money allows you to produce more money and you should assign some of that gain to the lender. (A problem occurs for the borrower if his venture does not regenerate at least the original loan amount plus the interest due.)

Another justification for the charging of interest is the **time preference theory**. Generally, people prefer to have money now rather than the same amount of money at some later date. After all, if you have the money now, you have a choice as to whether you use it now or save it for the future. If you lend it, you no longer have the option of immediately using your money. Interest compensates a lender for this loss of choice.

One of the most widely accepted excuses for interest being charged is that a lender should be compensated for the possibility that the borrower defaults and capital is lost.

In the real world, investments have an element of risk and investors sometimes lose money. Some investments, such as deposits insured by the FDIC[1] or Treasury securities backed by the United States government, have low risk. Risk-free investments earn a positive amount of interest. In Section (1.14) we briefly discuss the risk due to inflation, but until Chapters 8 and 9, we rarely consider questions of risk. Therefore, *we assume that a nonnegative amount, and usually a positive amount, of interest is paid.* Of course in real life, issues of risk are of great importance!

1.3 ACCUMULATION AND AMOUNT FUNCTIONS

We begin by introducing units of money. Most commonly we will think of these as dollars, and hence we will use a dollar sign to indicate our units of money. However, what follows is equally valid for other units of money such as euros, yen, or gold pieces. Occasionally, it might be more useful to have our basic unit of money be thousands of dollars or millions of dollars.

A common financial transaction is for one party to lend another party a lump sum of money. The lender views this as an investment since interest is expected in addition to the return of the loan amount. The amount $\$K$ of money the investor loans is called the **principal**. If we turn this around to the borrower's perspective, the amount of money borrowed is the principal.

Let us introduce units of time. The units of time are most often years, but sometimes it is more convenient to use some other unit such as months or days or quarters of a year. We need to choose a time to call time 0 and it is natural to make that the time of our initial financial transaction. Thus, we suppose that $\$K$ is invested at time $t = 0$.

We can now define a real-valued function $A_K(t)$ with domain $\{t | t \geq 0\}$ by insisting that $\$A_K(t)$ equals the balance at time t. The function $A_K(t)$ is called the **amount function** for principal $\$K$. It is standard to write $a(t)$ for $A_1(t)$ where $A_1(t)$ is the amount function for principal $\$1$. The function $a(t)$ is called the **accumulation function**.

[1] The Federal Deposit Insurance Corporation (FDIC) is a federal government agency, that was created in 1933 in response to the many bank failures of the 1920s and 1930s. It oversees more than 5000 member banks and insures the deposits of individual investors at these banks. The amount insured per investor at each bank is limited to $100,000, but deposits held with different ownership (e.g., single with spouse as survivor, joint) may be insured individually, effectively allowing a depositor more insurance. The total amount of insured deposits surpasses three trillion dollars. FDIC does *not* insure investments other than deposits, even if they are offered by a member bank. The FDIC is funded by premiums that banks pay and by interest they earn.

FACT 1.3.1
$\$K$ invested at time 0 grows to $\$A_K(t)$ at time t, and \$1 invested at time 0 grows to $\$a(t)$ at time t.

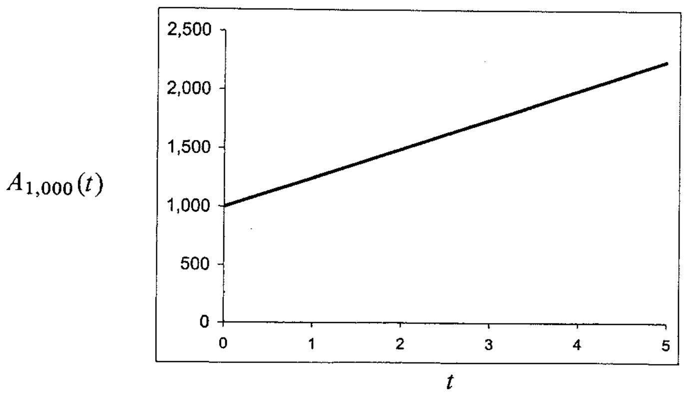
Typically $A_K(t) = Ka(t)$. This equality means that the amount your investment is worth at time t is proportional to the amount $\$K$ you deposited at time 0. However, this relationship need not hold. For instance, you may encounter tiered investment accounts in which the rate of interest you earn depends on the interval in which your balance lies. (See Example 1.5.5.) We will assume that $A_K(t) = Ka(t)$ unless a tiered growth structure is specifically indicated.

We note that $a(0) = 1$ and $A_K(0) = K$. The functions $a(t)$ and $A_K(t)$ are most often nondecreasing functions. However, if a fund loses money over an interval, the associated accumulation and amount functions will decrease. If we have continuous accrual of interest, the amount function is continuous. On the other hand, if interest is only paid at the end of each interest period, say at the end of each quarter of a year, the associated accumulation and amount functions are step functions.

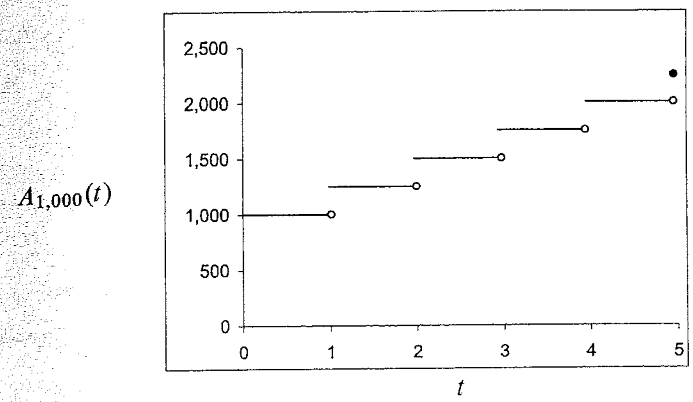
EXAMPLE 1.3.2

Problem: An investment of \$1,000 grows by a constant amount of \$250 each year for five years. What does the graph of the amount function $A_{1,000}(t)$ look like if interest is paid continuously using the linear relationship $A_{1,000}(t) = \$1{,}000 + \$250t$? How about if interest is only paid at the end of each year?

Solution If interest is earned continuously using the given linear relationship, the graph of $A_{1,000}(t)$ is a line segment with slope 250.



If interest is only paid at the end of each year, the graph of $A_{1,000}(t)$ is as follows:

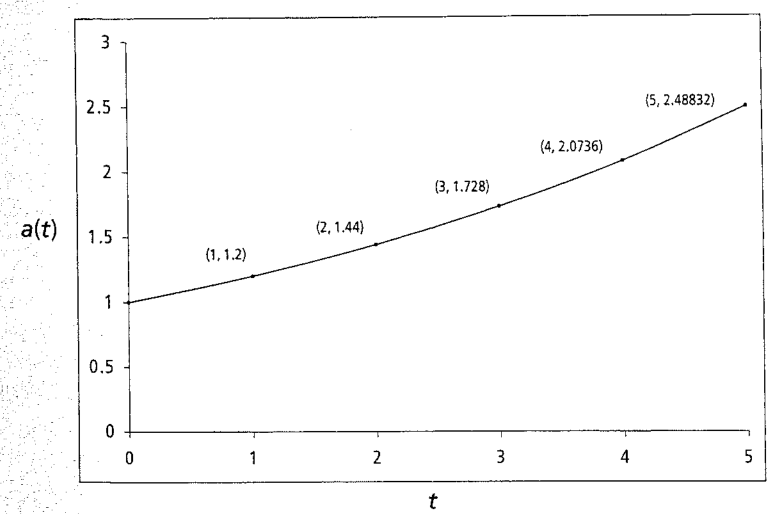


■

EXAMPLE 1.3.3

Problem: Suppose that time is measured in years and an investment fund grows according to $a(t) = (1.2)^t$ for $0 \le t \le 5$. Then the investment fund grows at a constant rate of 20% per year. (In Section 1.5, we will call $a(t)$ a compound interest accumulation function with annual effective interest rate $i = .2$.) Graph the accumulation function $a(t)$.

Solution



■

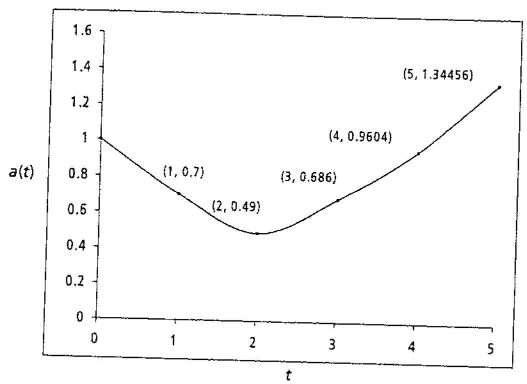
EXAMPLE 1.3.4

Problem: The Risky Investment Fund declines at a constant rate of 30% for two years, then grows at a constant rate of 40% for three years. That is to say,

$$a(t) = \begin{cases} (1-.3)^t & \text{for } 0 \le t \le 2 \\ (1-.3)^2(1+.4)^{t-2} & \text{for } 2 \le t \le 5. \end{cases}$$

Graph the associated accumulation function.

Solution



■

If $t_2 > t_1 \ge 0$, then $\$(A_K(t_2) - A_K(t_1))$ gives the **amount of interest** earned on an investment of $\$K$ (made at $t = 0$) between time t_1 and time t_2. We define the **effective interest rate for the interval** $[t_1, t_2]$ to be

(1.3.5) $$i_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_1)}.$$

Note that whenever $A_K(t) = Ka(t)$, we also have

(1.3.6) $$i_{[t_1,t_2]} = \frac{A_K(t_2) - A_K(t_1)}{A_K(t_1)}.$$

An investor of $\$K$ at time 0 is concerned with $A_K(t)$ and hence when $A_K(t) \neq Ka(t)$, he still regards the right-hand side of (1.3.6) as being the rate of interest for the interval $[t_1, t_2]$, even if it is not equal to the left-hand side of (1.3.6). [Of course this only happens if $A_K(t) \neq Ka(t)$.]

If n is a positive integer, the interval $[n-1, n]$ is called the **n-th time period** and we agree to write i_n for $i_{[n-1,n]}$. Thus,

(1.3.7) $$i_n = \frac{a(n) - a(n-1)}{a(n-1)} \quad \text{and} \quad a(n) = a(n-1)(1+i_n).$$

In particular,

(1.3.8) $$i_1 = \frac{a(1) - a(0)}{a(0)} = \frac{a(1) - 1}{1} = a(1) - 1.$$

Note that i_n represents the interest rate earned by an investor during the n-th period in which the investment is governed by the accumulation function $a(t)$.

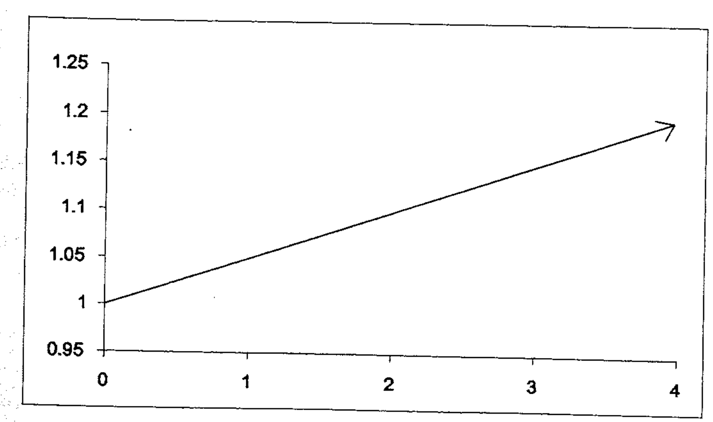
1.4 SIMPLE INTEREST / LINEAR ACCUMULATION FUNCTIONS

In the world of day-to-day financial transactions, you don't hear talk of accumulation and amount functions. However, investments and loans are usually made according to carefully spelled out rules, and many of these algorithms are easily translated into the language of amount functions. Among the simplest amount functions which occur in real life are linear functions, and we next consider how these occur.

Consider two parties negotiating a loan. They might agree that for each \$1 borrowed, at the time of repayment the borrower will pay $\$.05T$ interest where T is the number of years until repayment. (They will also presumably agree upon allowable values of T.) Then the amount owed at time t per dollar borrowed is $a(t) = 1 + (.05)t$, a linear accumulation function with y-intercept 1 and slope .05.

Focus now on an arbitrary linear accumulation function $a(t)$. If $a(t)$ were linear, since $a(0) = 1$, the accumulation function $a(t)$ would satisfy $a(t) = 1 + st$ for some constant s. Then $a(1) = s + 1$. But by (1.3.8), $s = i_1$ and $a(t) = 1 + i_1 t$.

$A_K(t) = K(1 + st)$ is called the **amount function for $\$K$ invested by simple interest at rate s.** The function $a(t) = 1 + st$ is the **simple interest accumulation function at rate s.**



Graph of $a(t) = 1 + st$ for $s = .05$

We next consider a couple of problems involving simple interest.

EXAMPLE 1.4.1

Problem: Tonya loans Renu \$1,600. Renu promises that in return, she will pay Tonya \$2,000 at the end of four years. To what rate of simple interest does this correspond?

Solution The equation $\$2{,}000 = \$A_{1{,}600}(4) = \$1{,}600(1+4s)$ is equivalent to $s = \frac{1}{4}\left(\frac{2{,}000}{1{,}600} - 1\right) = .0625$, so the loan corresponds to a simple interest rate of 6.25%. ■

EXAMPLE 1.4.2

Problem: Antonio loans his brother Bob \$2,400 for three years at 5% simple interest. The brothers agree that if Bob wishes to repay the loan early, he may do so, and the repayment amount will still be based on 5% simple interest. Find the amount Bob would be required to pay if he makes his repayment at the end of three years. What if the repayment is after two years or after one year? Calculate i_1, i_2, and i_3 if the loan lasts the full three years.

Solution If Bob repays the loan after three years, he must pay Antonio $\$2{,}400[1+3(.05)] = \$2{,}760$. If the repayment comes at the end of two years, the repayment amount is $\$2{,}400[1+2(.05)] = \$2{,}640$, while if it comes after one year, the amount due is $\$2{,}400[1+1(.05)] = \$2{,}520$. We therefore have the following annual effective interest rates.

$$i_1 = \frac{\$2{,}520 - \$2{,}400}{\$2{,}400} = 5\%,$$

$$i_2 = \frac{\$2{,}640 - \$2{,}520}{\$2{,}520} \approx 4.76\%,$$

and

$$i_3 = \frac{\$2{,}760 - \$2{,}640}{\$2{,}640} \approx 4.55\%.$$

Note that the annual effective interest rates are decreasing. Each year there is less incentive for Bob to repay the loan early since the rate of interest he is paying is lower. ■

When the growth of money is governed by simple interest at rate s,

$$i_n = \frac{a(n)-a(n-1)}{a(n-1)} = \frac{(1+sn)-[1+s(n-1)]}{1+s(n-1)} = \frac{s}{1+s(n-1)}. \tag{1.4.3}$$

Hence $\{i_n\}$ is a decreasing sequence (and, for those of you familiar with calculus, $\{i_n\}$ converges to 0). **In part because $\{i_n\}$ is a decreasing sequence, simple interest is rarely used for loans of long duration.**

In Examples (1.4.1) and (1.4.2), the loans were for an integral number of years and we counted the length of time in the simplest possible manner. However, when simple interest is used, there are a number of different methods used for measuring the time of the loan. We now briefly mention some of the more common rules.

With the **exact simple interest** method, the term D of the loan is first measured in days and then divided by the number of days in a year (usually 365) to yield the length of the loan in years. Exact simple interest is sometimes referred to as the **"actual/actual"** method, the first "actual" for the number of days, the second "actual" for dividing by 365. To calculate D, it is important to know the number of days in each month. January, March, May, July, August, October, and December each have 31 days while April, June, September, and November each have 30. February has 28 days unless it occurs in a leap year, in which case it has 29 days. You are likely familiar with the fact that leap years only occur in years divisible by 4. A more obscure fact is that if a year is divisible by 100, it must also be divisible by 400 in order to be a leap year. Hence 1988, 1956, and 2000 were leap years, while 1987, 1953, and 1900 were not leap years.

EXAMPLE 1.4.3 Exact simple interest actual/actual

Problem: Brad borrows \$5,000 from Julio on October 14, 1998 at 8% exact simple interest and agrees to repay the loan on May 7, 1999. What is the amount of Brad's required May repayment?

Solution The duration of the loan in days is

$$\underbrace{(31-14)}_{\text{Oct.}} + \underbrace{30}_{\text{Nov.}} + \underbrace{31}_{\text{Dec.}} + \underbrace{31}_{\text{Jan.}} + \underbrace{28}_{\text{Feb.}} + \underbrace{31}_{\text{Mar.}} + \underbrace{30}_{\text{Apr.}} + \underbrace{7}_{\text{May}} = 205$$

and therefore Brad must repay Julio $\$5{,}000(1+.08(\frac{205}{365})) \approx \$5{,}224.66$. ■

With the **ordinary simple interest** method, you pretend each month has 30 days and hence a year has 360 days.

If the loan is from day d_1 of month m_1 of year y_1 to day d_2 of month m_2 of year y_2, then you compute the number of days using the formula

$$\boxed{d = 360(y_2 - y_1) + 30(m_2 - m_1) + (d_2 - d_1).} \tag{1.4.5}$$

Ordinary simple interest is sometimes referred to as the **"30/360"** rule, the "30" for the number of days in a month, the "360" for the number of days in a year.

EXAMPLE 1.4.6 Ordinary simple interest 30/360

Problem: Suppose that the loan of Example (1.4.4) is made at 8% ordinary simple interest instead of 8% exact simple interest. What is the amount of Brad's required May repayment?

Solution According to the method of ordinary simple interest, the duration of the loan in days is calculated to be

$$360(1999-1998)+30(5-10)+(7-14)=203.$$

Therefore Brad must repay Julio $\$5{,}000[1+.08(\frac{203}{360})] \approx \$5{,}225.56$. ■

The **Banker's rule** is a hybrid of the above methods that is usually more advantageous to the lender. As in the exact simple interest method, count the actual number of days but imagine that a year has 360 days. The Banker's rule is sometimes referred to as the **"actual/360"** method.

EXAMPLE 1.4.7 Banker's rule actual/360

Problem: Suppose that the loan of Example (1.4.4) is made by 8% simple interest computed using the Banker's rule instead of 8% exact simple interest. What is the amount of Brad's required May repayment?

Solution The term of the loan is calculated as in Example (1.4.4) and we use 360 for the number of days in a year. Therefore, Brad must repay Julio

$$\$5{,}000\left[1+.08\left(\tfrac{205}{360}\right)\right] \approx \$5{,}227.78.$$ ■

The duration of the loan, be it an actual count of the number of days or an estimate based on the assumption that all months have thirty days, may be quickly determined using the BA II Plus calculator's **Date worksheet** so long as the loan takes place during the years 1950–2049. (In fact, in the event that you are interested in an interval other than during this one hundred year span, the worksheet is still quite helpful [see Problem (1.4.7)]. We next illustrate the use of the **Date worksheet** with the loan interval of Examples (1.4.4) and (1.4.6), namely the period from October 14, 1998 until May 7, 1999.

Press [2ND] [DATE] to open the date worksheet. Next push the keys [2ND] [CLR WORK] if you need to check whether the calculator is formatted to accept dates in U.S. order (Month–Day–Year) or European order (Day–Month–Year). If it is ready to accept U.S. order, the display will read "DT1 = 12 - 31 - 1990", and otherwise the display will show "DT1 = 31 - 12 - 1990", indicating European formatting. The default formatting is for the U.S. ordering, and assuming this is in place, you should push [1] [0] [•] [1] [4] [9] [8] [ENTER] to enter a starting date of October 14, 1998. (Should your display have read "DT1 = 31 - 12 -1990", indicating European formatting, you should enter October 14, 1998 by keying [1] [4] [•] [1] [0] [9] [8] [ENTER].) Push [↓] and the display will show "DT2", indicating that the worksheet is ready to accept the loan completion date of May 7, 1999. Enter this date by keying [0] [5] [•] [0] [7] [9] [9] [ENTER] or [0] [7] [•] [0] [5] [9] [9] [ENTER] depending on whether you have U.S. or European formatting. (The BA II Plus calculator accepts calendar dates from January 1, 1950 through December 31, 2049. Regardless of which formatting is in effect, the year is entered by keying the last two-digits after the month and day have been entered. The month and day are each recorded by entering a two digit number, and they are separated by entering a decimal point.)

Now that the loan commencement and termination dates have been entered, you should press [↓] [↓]. This will result in the display reading either "ACT" or "360." The first of these tells you that the calculator is prepared to make an exact calculation of the days the loan lasted, while the latter alerts you that it will estimate the loan duration using thirty-day months. Should you wish to have the other basis for the calculation of the loan duration, push [2ND] [SET] [↑] [CPT]. On the other hand, if you are satisfied with the indicated calculation method, just push [↑] [CPT]. Now read the display. If you used the "ACT" calculation, your screen should show "DBD = 205", while the "360" method will produce "DBD = 203".

1.5 COMPOUND INTEREST (THE USUAL CASE!)

Although simple interest is easy to compute, practical applications of this method are limited. We now explain why this is so. Suppose you invest at a bank where savings accounts earn simple interest at a rate i. As noted in Section (1.4), the effective interest rate in the n-th year is a decreasing function of n. Consequently, you would do well to go into the bank, close your account, and then instantly reopen it. But this would be inconvenient for you and for the bank. Therefore, it is sensible for the bank to design an account that grows in a manner where there is no advantage or disadvantage to closing an account and then instantly reopening it. In particular, we want the effective interest rate for the n-th period i_n to be independent of n.

Define

(1.5.1) $$\boxed{i = i_1 = a(1) - 1.}$$

CLAIM 1.5.2

If an accumulation function $a(t)$ has the associated periodic interest rates i_n all equal to the constant i, then the accumulation function must satisfy $a(k) = (1+i)^k$ for all nonnegative integers k.

Claim (1.5.2) is a statement about the nonnegative integers. The Principle of Mathematical Induction is a valuable technique for proving facts about the nonnegative integers (or about any infinite set of consecutive integers with a smallest element), and we shall use this method to establish Claim (1.5.2). Mathematical Induction requires you to establish that your claim is true for the smallest integer in your set of consecutive integers (in this case for $k = 0$) and that the claim being true for a given integer k forces its validity for that integer's successor $k + 1$.[2]

Proof: (An induction argument) For a given k, refer to the equation $a(k) = (1+i)^k$ as equation k. Our task is to establish equation k for all nonnegative integers k. Note that equation 0 is true since $(1+i)^0 = 1$ and for any accumulation function, $a(0) = 1$. Having established this first equation, our method is to show that if equation k is true, equation $(k+1)$ must also be true. For this, note that we now have two assumptions, namely (1) $i_n = i$ for all n — in particular $i_{k+1} = i$, and (2) $a(k) = (1+i)^k$. From these it follows that

$$i = i_{k+1} = \frac{a(k+1) - a(k)}{a(k)} = \frac{a(k+1) - (1+i)^k}{(1+i)^k}.$$

But this is equivalent to

$$a(k+1) = i(1+i)^k + (1+i)^k = (1+i)(1+i)^k = (1+i)^{k+1}.$$

So, we have verified equation $(k+1)$. ■

The situation for nonintegral investment periods is more subtle. In terms of accumulation functions, the condition that there should never be an advantage or disadvantage to closing and immediately reopening one's account means that

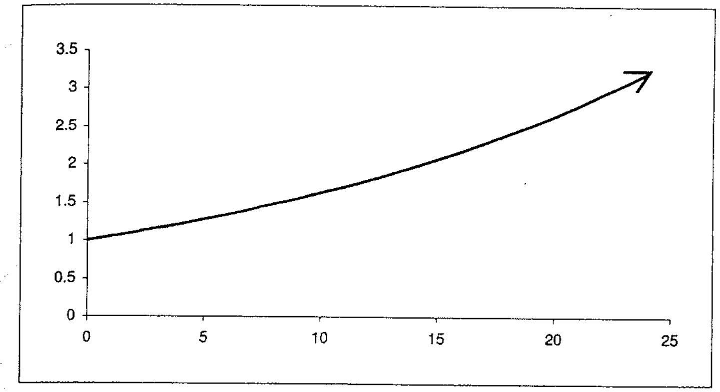
(1.5.3) $\quad a(s+t) = a(s)a(t) \quad$ for all positive real numbers t and s.

In calculus-based Problem (1.5.11), we help the student show that assuming $a(t)$ is differentiable at all $t > 0$ and is differentiable from the right at $t = 0$, then equation (1.5.3) forces $a(t) = (1+i)^t$ for all nonnegative t, not just for integral t. In practice however, banks do not always use this formula when t is nonintegral. Some banks pay compound interest for an integral number of years followed by simple interest for the final portion of a year. For instance, they may set $a(3.25) = a(3)(1+.25i) = (1+i)^3(1+.25i)$ rather than using $a(3.25) = (1+i)^{3.25}$.

Unless otherwise stated, we will use $a(t) = (1+i)^t$ for all $t \geq 0$ and will call this the **compound interest accumulation function** at interest rate i.

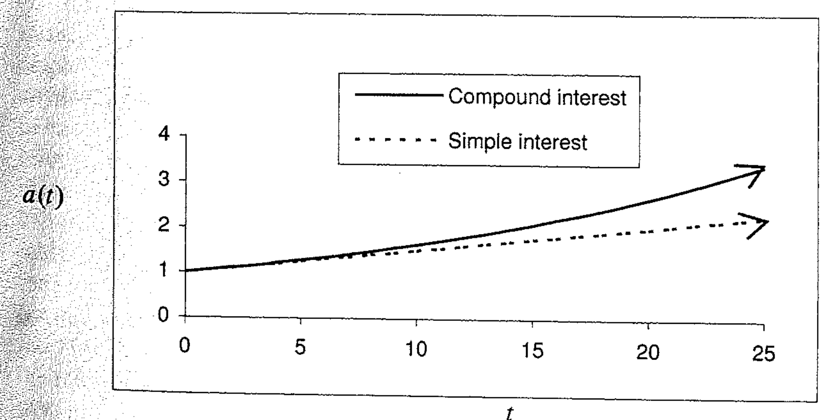
[2] The following commonly used analogy involves dominoes or other thin blocks. It may help you visualize why Mathematical Induction should be allowed. Suppose you place dominoes close together with their faces parallel to one another. Do not use glue! If the block at one end falls toward the next block, the blocks will all fall.

For accounts governed by the compound interest accumulation function at interest rate i, money earns interest at the constant interest rate i. As interest is paid, it is reinvested and also earns interest at rate i.



Graph of $a(t) = (1+i)^t$ **for** $i = .05$

You might wonder how simple interest and compound interest compare. Here is a graph showing both kinds of accumulation with annual rates of 5%.



You can easily note that if you go far enough to the right (large t), the compound interest accumulation function lies above the simple interest accumulation function. The scale on our graph is not large enough for you to see that the simple interest function is larger up until time 1, at which point the compound interest function begins dominating.

EXAMPLE 1.5.4 Compound interest from deposit to withdrawal

Problem: Fernando deposits \$12,000 in an account at Victory Bank where accounts grow according to the compound interest accumulation function $a(t) = (1.05)^t$ for all nonnegative t. He makes no further deposits or withdrawals until he closes the account six and a half years after he opened it. How much money does Fernando receive when he closes the account?

Solution Fernando receives the account balance which is

$$\$12{,}000(1.05)^{6.5} = \$16{,}478.27.$$

■

In the next example, and again later in the book, we use the **greatest integer function**. This function is defined by

$$\lfloor t \rfloor = \text{the largest integer which does not exceed } t.$$

You may say "the greatest integer in t" for $\lfloor t \rfloor$. Computer scientists refer to the function $\lfloor t \rfloor$ as the **floor function**.

EXAMPLE 1.5.5 Compound interest with simple interest for fractional parts

Problem: Sarafina also had \$12,000 to deposit. She invested her money at Simpler Bank where accounts grow according to the accumulation function

$$a(t) = (1.05)^{\lfloor t \rfloor}(1 + .05(t - \lfloor t \rfloor)).$$

She also closes her balance after six and a half years. Compare her closing balance to the closing balance Fernando received in Example (1.5.4).

Solution Sarafina's closing balance is

$$\begin{aligned}\$12{,}000a(6.5) &= \$12{,}000(1.05)^{\lfloor 6.5 \rfloor}\left[1 + .05(6.5 - \lfloor 6.5 \rfloor)\right] \\ &= \$12{,}000(1.05)^{6}\left[1 + .05(.5)\right] \approx \$16{,}483.18.\end{aligned}$$

Her balance is close to Fernando's but is slightly higher. ■

EXAMPLE 1.5.6 Tiered investment account

Problem: Patriot Bank offers a "bracketed account" with compound interest at an annual effective interest rate of 2% on balances of less than \$2,000, 3% on balances of at least \$2,000 but less than \$5,000, and 4% on balances of at least \$5,000. Moises opens an account with \$1,800. Determine the amount function $A(t) = A_{1,800}(t)$ and show that the *function* $A_{1,800}(t)$ is different from the *function* $1{,}800a(t)$, although their values agree for t in $[0, 5.32]$.

Solution Define t_1 to be the length of time it takes for Moises' balance to grow to \$2,000 and t_2 to be the time it takes to grow from \$2,000 to \$5,000. Then $\$1{,}800(1.02)^{t_1} = \$2{,}000$ and $\$2{,}000(1.03)^{t_2} = \$5{,}000$. It follows that

$$t_1 = \frac{\ln(2{,}000/1{,}800)}{\ln(1.02)} \approx 5.320532174,$$

and

$$t_2 = \frac{\ln(5{,}000/2{,}000)}{\ln(1.03)} \approx 30.99891276.$$

Then

$$A_{1,800}(t) = \begin{cases} 1{,}800(1.02)^t & \text{if } 0 \le t \le t_1 \approx 5.3205; \\ 2{,}000(1.03)^{t-t_1} & \text{if } 5.3205 \approx t_1 \le t \le t_1 + t_2 \approx 36.319; \\ 5{,}000(1.04)^{t-t_1-t_2} & \text{if } 36.319 \approx t_1 + t_2 \le t. \end{cases}$$

On the other hand, $a(t) = \$1(1.02)^t$ for $0 \le t \le t_3$ where $(1.02)^{t_3} = \$2{,}000$, that is for

$$t_3 = \frac{\ln(2{,}000)}{\ln(1.02)} \approx 383.8330311.$$

So, $A_{1,800} = 1,800a(t)$ only on the interval $[0, t_1]$. Beyond that interval, for any argument t, $A_{1,800}(t)$ exceeds $1{,}800a(t)$. For instance, $A_{1,800}(10) = \$2{,}000(1.03)^{10-t_1} \approx 2{,}340.62$, and $1,800a(10) = 1{,}800(1.02)^{10} \approx 2{,}194.19$. ■

Commonly, the interest rate applied will vary from bank statement to bank statement. It may be determined in a specified manner from a monetary index (for example, the Federal Funds rate or the prime rate), or the bank may be free to offer whatever rate it wants. In either case, supply and demand play a major role, and this in turn is influenced by the government's fiscal and monetary policies. **Fiscal policy** refers to the government's decisions concerning spending and taxation. If the government spends more, this introduces money into the economy and should eventually increase the amount of money available to consumers. This tends to drive interest rates down. On the other hand, taxation decreases the amount of money available for individuals and companies, so interest rates may rise due to increased demand for loans. **Monetary policy** refers to the regulation of the money supply and interest rates by a central bank. In the United States, the Federal Reserve has direct control of the rate it charges banks for "overnight" borrowing. The Federal Reserve may also strongly influence the **Federal Funds Rate** charged for interbank "overnight" loans by competing (or not competing) for bank's money with U.S. Treasury securities. The Federal Reserve has a less direct influence on the **prime rate**, the rate that a bank charges to its "best" customers. The level of these rates tends to impact the interest rates available to all borrowers.

Savings accounts are generally governed by compound interest, but the rate at which compound interest is paid is subject to change.

EXAMPLE 1.5.7 Varying interest rates

Problem: Arisha deposits \$8,000 in a savings account at Victory Bank. For the first three years the money is on deposit, the annual effective rate of compound interest paid is 5%, for the next two years it is 5.5%, and for the following four years it is 6%. If Arisha closes her account after nine years, what will her balance be?

Solution If Arisha closes her account after nine years, her balance will be $\$8{,}000(1.05)^3(1.055)^2(1.06)^4 \approx \$13{,}013.26$. ■

It is not uncommon for investors to wish to determine a rate of compound interest that would provide them with a certain amount of money at a given later date.

EXAMPLE 1.5.8 Unknown rate

Problem: Pedro wishes to have \$12,000 in three years, so he can buy his father's car. Pedro has \$9,800 to invest in a three year certificate of deposit. [3] What annual effective rate of interest must the CD earn so that it will have a redemption value of \$12,000? When we speak of a CD, we will refer to an investment with a fixed term and fixed rate.

Solution Denote the required effective interest rate by i. Then $12{,}000 = A_{9{,}800}(3) = 9{,}800(1+i)^3$. Solving for i, we find $i = \left(\frac{12{,}000}{9{,}800}\right)^{\frac{1}{3}} - 1 \approx .069838912$. A rate of 7% would produce $\$9{,}800(1.07)^3 \approx \$12{,}005.42$. ■

1.6 INTEREST IN ADVANCE / THE EFFECTIVE DISCOUNT RATE

When you rent an apartment, usually you are required to pay rent for each month at the beginning of the month. In other words, you pay the rent before you have the use of the apartment. We said [Section (1.1)] that interest may be thought of as a rent for the use of the investor's money. It is therefore not surprising that there are financial arrangements in which the interest must be paid by the borrower before the borrowed money becomes available.

[3] A **certificate of deposit** (CD) requires the investor to deposit money to the issuing bank or savings and loan for a fixed term. Should the investor decide to withdraw deposited funds before the end of the term, there is usually a substantial penalty — perhaps one quarter's interest payment — but withdrawals of interest are usually allowed. **Liquid CDs** may allow one or more partial withdrawals without penalty before the CD matures. With a traditional CD, the interest rate is fixed at the time the account is opened. However, the CD market has expanded to include market-linked CDs and CDs for which the investor may request one-or-more adjustments to the interest rate should interest rates go up. Most often, additional funds may not be added to a CD once the account has been opened; however, some CDs allow the customer to make a limited number of additional deposits.

> When money is borrowed with interest due before the money is released, we describe the relationship using discount rates. If an investor lends $\$K$ for one basic period at a **discount rate** D, then the borrower will have to pay $\$KD$ in order to receive the use of $\$K$. Therefore, instead of having the use of an extra $\$K$, the borrower only has the use of an extra $\$K - \$KD = (1-D)\$K$. The quantity $\$KD$ is called the **amount of discount** for the loan.

Note that in Section (1.3), we defined the amount of interest on an interval, and now we have defined the amount of discount on an interval. In any cashflow with a beginning and ending balance and no withdrawals or deposits, the amount of interest and the amount of discount are the same; they are both equal to the change in the balance.

EXAMPLE 1.6.1

Problem: Chan borrows \$1,000 at a discount rate of 7%. How much extra money does he have the use of?

Solution In order to get the \$1,000, he must first pay $(.07)\$1{,}000 = \$70$. Chan therefore has the use of an extra $\$1{,}000 - \$70 = \$930$. ■

In our example, the discount rate is

$$7\% = \frac{\$70}{\$1{,}000} = \frac{\$1{,}000 - \$930}{\$1{,}000}.$$

In other words, the discount rate for the loan period may be obtained by first calculating the difference between the stated amount of the loan and the amount of extra money actually available, then dividing this difference by the stated amount of the loan.

We wish to define an effective discount rate analogous to the effective interest rate [defined by (1.3.5)]. Suppose that the loan period is the interval $[t_1, t_2]$ from time t_1 to t_2. Also suppose that at time t_1 the borrower will have the use of an extra $\$a(t_1)$. At time t_2 this debt will have grown to $\$a(t_2)$. Therefore, the amount of discount for the interval $[t_1, t_2]$ is $\$a(t_2) - \$a(t_1)$, and the discount per dollar borrowed at a discount is $\$\left(\frac{a(t_2)-a(t_1)}{a(t_2)}\right)$. We define the effective discount rate for the interval $[t_1, t_2]$ to be

(1.6.2)
$$d_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_2)}.$$

Comparing definition (1.6.2) with definition (1.3.5), we see that the definitions for $i_{[t_1,t_2]}$ and $d_{[t_1,t_2]}$ have the same numerators but different denominators. To compute the interest rate $i_{[t_1,t_2]}$, your denominator is the accumulated amount $a(t_1)$ at the **beginning** of the interval $[t_1, t_2]$. To compute the discount rate $d_{[t_1,t_2]}$, you divide by the accumulated amount $a(t_2)$ at the **end** of the interval $[t_1, t_2]$.

If, as is commonly the case, $A_K(t) = Ka(t)$, then

(1.6.3) $$d_{[t_1,t_2]} = \frac{A_K(t_2) - A_K(t_1)}{A_K(t_2)}.$$

Recall that if n is a positive integer, the interval $[n-1, n]$ is called the ***n*-th time period**. We agree to write d_n for $d_{[n-1,n]}$. Thus

(1.6.4) $$d_n = \frac{a(n) - a(n-1)}{a(n)} \quad \text{and} \quad a(n-1) = a(n)(1-d_n).$$

Compare this definition with the definition of i_n given in Equation (1.3.7).

EXAMPLE 1.6.5 Computing interest and discount rates; Solution includes important calculator information on using stored intermediate results

Problem: Suppose that the growth of money is governed by the accumulation function $a(t) = (1.05)^{\frac{t}{2}}(1 + .025t)$. Find d_4 and i_4.

Solution Note that $a(4) = (1.05)^2(1.1) = 1.21275$ and $a(3) = (1.05)^{\frac{3}{2}}(1.075) \approx 1.156624568$. Therefore $d_4 = \frac{a(4)-a(3)}{a(4)} \approx .046279474$ and $i_4 = \frac{a(4)-a(3)}{a(3)} \approx .048525195$.

Note that in our computations, we will always use the stored values resulting from previous calculations and these may include more places of accuracy than we have reported; we usually only report the displayed value resulting from 9-formatting. For example, the calculator stores 1.156624567708 for $a(3)$ rather than the announced 1.156624568. However, if you use the number 1.156624568 (displayed for $a(3)$ if you have 9-formatting) instead of the stored value 1.156624567708, your calculated value for i_4 will be approximately .048525194. ■

As was the case in Example (1.6.5), usually $i_{[t_1,t_2]}$ and $d_{[t_1,t_2]}$ are not equal. However, they are clearly related. With an eye to pursuing this relationship, we define what it means for a rate of interest and a rate of discount to be **equivalent**.

IMPORTANT DEFINITION 1.6.6
A rate of interest and a rate of discount are said to be **equivalent** for an interval $[t_1, t_2]$ if for each \$1 invested at time t_1, the two rates produce the same accumulated value at time t_2. More generally, two different methods of specifying an investment's growth (over a given time period) are called **equivalent** if they correspond to the same accumulation function.

Focus on a loan lasting from time t_1 to time t_2. If the loan is for an amount $\$L$ and the interest rate is $i_{[t_1,t_2]}$, then we must repay $\$L(1 + i_{[t_1,t_2]})$. On the other hand, if the loan is made at a discount with discount rate $d_{[t_1,t_2]}$ and the repayment amount is $\$L(1 + i_{[t_1,t_2]})$, then the borrower walked away at the beginning of the loan period with $\$L(1 + i_{[t_1,t_2]})(1 - d_{[t_1,t_2]})$ of the lender's money. Consequently, on the interval $[t_1, t_2]$, an interest rate of $i_{[t_1,t_2]}$ is **equivalent to** a discount rate of $d_{[t_1,t_2]}$ precisely when $\$L = \$L(1 + i_{[t_1,t_2]})(1 - d_{[t_1,t_2]})$ for all loan amounts $\$L$. It follows that the rates are **equivalent** if and only if

(1.6.7) $$1 = (1 + i_{[t_1,t_2]})(1 - d_{[t_1,t_2]}).$$

Equation (1.6.7) may also be algebraically derived using Equations (1.3.5) and (1.6.2), but this demonstration is less instructive from an interest theory point of view.

Equation (1.6.7) is equivalent to

$$0 = i_{[t_1,t_2]} - d_{[t_1,t_2]} - i_{[t_1,t_2]}d_{[t_1,t_2]}$$

which in turn gives rise to

(1.6.8) $$i_{[t_1,t_2]} = \frac{d_{[t_1,t_2]}}{1 - d_{[t_1,t_2]}}$$

and

(1.6.9) $$d_{[t_1,t_2]} = \frac{i_{[t_1,t_2]}}{1 + i_{[t_1,t_2]}}.$$

If we have a positive interest rate $i_{[t_1,t_2]}$, it follows from equation (1.6.9) that the discount rate $d_{[t_1,t_2]}$ is less than $i_{[t_1,t_2]}$.

Recalling (1.6.4) and that we write d_n for $d_{[n-1,n]}$, (1.6.7), (1.6.8), and (1.6.9) respectively, give us the equations

(1.6.10) $$(1 + i_n)(1 - d_n) = 1,$$

(1.6.11) $$i_n = \frac{d_n}{1 - d_n},$$

and

(1.6.12) $$\boxed{d_n = \frac{i_n}{1+i_n}.}$$

EXAMPLE 1.6.13 Discount rates and compound interest

Problem: Suppose that an account is governed by compound interest at an annual effective interest rate of 8%. Find an expression for d_n, the discount rate for the n-th year.

Solution Since the account is governed by compound interest at an annual effective rate of 8%, $i_n = .08$ for all positive integers n. Therefore (1.6.12) yields $d_n = \frac{.08}{1+.08} = \frac{.08}{1.08} = \frac{2}{27}$, a constant. We will return to constant d_n in Section (1.9). ■

1.7 DISCOUNT FUNCTIONS / THE TIME VALUE OF MONEY

In a world where interest may be earned, an amount $\$K$ present on hand is worth more than a payment of $\$K$ t years in the future. This is because you could invest the $\$K$ today and after t years it would have grown to a larger amount $\$A_K(t)$. In particular, \$1 invested now will grow to $\$a(t)$ in t years and, assuming the growth of money is proportional to the amount invested, $\frac{\$1}{a(t)}$ will grow to \$1 in t years. This leads us to define the **discount function** $v(t)$ by

(1.7.1) $$\boxed{v(t) = \frac{1}{a(t)}.}$$

The value $\$v(t_0)$ is the amount of money you must invest at time 0 to have \$1 after t_0 years.

EXAMPLE 1.7.2

Problem: Suppose that the growth of money for the next five years is governed by simple interest at 5%. How much money should you invest now in order that you have a balance of \$23,000 three years from now?

Solution Note that the discount function is $v(t) = \frac{1}{a(t)} = \frac{1}{1+(.05)t}$. Therefore $v(3) = \frac{1}{1+(.05)3} = 1/1.15$. If we wish to have \$23,000 three years from now, we should invest $\$23{,}000v(3) = \$23{,}000(1/1.15) = \$20{,}000$. ■

The next example considers a more subtle question since you wish to invest at a time which is not 0. The reason that this is potentially trickier is that the accumulation function is defined as giving the new value of 1, deposited *at time zero*, t years after it was deposited.

EXAMPLE 1.7.3

Problem: Again, suppose that the growth of money for the next five years is governed by the linear accumulation function $a(t) = 1 + .05t$ so that

$$v(t) = \frac{1}{a(t)} = \frac{1}{1+(.05)t}.$$

If you wish to invest money two years from now so as to have \$23,000 five years from now, how much money should you invest?

Solution Let's begin by focusing on the desired \$23,000 five years from now. Were you to invest money *now* so that it would grow to this, you would need to invest $\$23{,}000v(5) = \$23{,}000(1/1.25) = \$18{,}400$. But \$18,400 would grow after two years to $\$18{,}400a(2) = \$18{,}400(1+2(.05)) = \$20{,}240$. Hence you should invest \$20,240 at time two to achieve \$23,000 at time five.

It may be puzzling why the relevant accumulation function is not one that begins at the time ($t = 2$) when you invest. This mystery is resolved by the realization that the accumulation function was specified without regard to your particular investment behavior. Your involvement began after the accumulation function governing the growth of investments was set. ■

The answer to the problem posed in Example (1.7.2) is \$20,000 while the answer to the problem asked in Example (1.7.3) is \$20,240. The lesson is that in general, the accumulation of money over time depends not only on the length of the time interval but also on where in time the interval lies.

If we review the solution to Example (1.7.3), we see that the answer was obtained by computing $\$23{,}000v(5)a(2) = \$23{,}000\frac{a(2)}{a(5)} = \$23{,}000\frac{v(5)}{v(2)}$. This method of solution applies more generally and gives us the following result.

> **IMPORTANT FACT 1.7.4**
> If we wish to invest money t_1 years from now in order to have $\$S$ t_2 years from now, we should invest $\$Sv(t_2)a(t_1) = \$S\frac{a(t_1)}{a(t_2)} = \$S\frac{v(t_2)}{v(t_1)}$.

Next concentrate on the compound interest accumulation function $a(t) = (1+i)^t$. We define the **discount factor**

(1.7.5) $$\boxed{v = \frac{1}{1+i}.}$$

Then

$$v(t) = \frac{1}{a(t)} = \frac{1}{(1+i)^t} = v^t,$$

and

$$S\frac{v(t_2)}{v(t_1)} = S\frac{v^{t_2}}{v^{t_1}} = Sv^{t_2-t_1}.$$

So, for compound interest, the amount you need to invest in order to have $\$S$ at a later time depends only on the amount of time until you need the $\$S$.

We next introduce **present value**, a concept that is very useful if you wish to make a comparison of investment alternatives. If the growth of money is proportional to the amount of money invested, and if we have an accumulation function $a(t)$ with associated discount function $v(t) = \frac{1}{a(t)}$, $\$Lv(t_0)$ invested at time 0 will grow to $\$L$ at time t_0. (This is true because $(\$Lv(t_0))a(t_0) = \$L(1/a(t_0))a(t_0) = \$L$.) We therefore define the **present value** *with respect to* $a(t)$ **of** $\$L$ **to be received at time** t_0 to be $\$Lv(t_0)$. We denote this present value by $PV_{a(t)}$ ($\$L$ at t_0).

(1.7.6) $$\boxed{PV_{a(t)}(\$L \text{ at } t_0) = \$Lv(t_0).}$$

When there is a clear choice of accumulation function, we drop the subscript $a(t)$ in $PV_{a(t)}(\$L$ at $t_0)$.

EXAMPLE 1.7.7

Problem: What is the present value of $2,000 to be paid in four years assuming money grows by compound interest at an annual effective interest rate of 6%? What if money grows by compound interest at an annual effective interest rate of 3%?

Solution In the two scenarios we have compound accumulation functions $a(t) = (1.06)^t$ and $a(t) = (1.03)^t$ respectively. We calculate that

$$PV_{(1.06)^t}(\$2{,}000 \text{ at } 4) = \$2{,}000(1.06)^{-4} \approx \$1{,}584.19$$

and

$$PV_{(1.03)^t}(\$2{,}000 \text{ at } 4) = \$2{,}000(1.03)^{-4} \approx \$1{,}776.97.$$

When we have the lower interest rate 3%, it takes more money invested now to produce $2,000 four years from now. This is why $PV_{(1.03)^t}(\$2{,}000 \text{ at } 4) > PV_{(1.06)^t}(\$2{,}000 \text{ at } 4)$. ■

Suppose that the growth of money is viewed as being governed by an accumulation function $a(t)$. Then the **net present value** or **NPV** of a sequence of investment returns $R_0, R_1, R_2, \ldots, R_n$ received at times $0, t_1, t_2, \ldots, t_n$ is defined to be the sum

$$\sum_{k=0}^{n} R_k\, v(t_k).$$

Note that a return R_k is positive if the investor receives money and negative if the investor pays out money. If you have a savings account, you are the investor and from your perspective, withdrawals are positive cashflows and deposits are negative cashflows.

EXAMPLE 1.7.8

Problem: Compare the net present values of two certificates of deposit available to Helga, one at Bank Alpha with purchase price $1,000 and redemption value $1,250 at the end of two years, the other at Bank Beta with purchase price $1,000 and redemption value $1,300 at the end of three years, first using the compound interest accumulation function $a(t) = (1.05)^t$ and next using the compound interest accumulation function $a(t) = (1.03)^t$.

Solution Under compound interest at 5%, the Bank Alpha certificate of deposit has NPV equal to $-\$1{,}000 + \$1{,}250(1.05)^{-2} \approx \$133.79$ and the Bank Beta certificate has NPV equal to $-\$1{,}000 + \$1{,}300(1.05)^{-3} \approx \$122.99$. So, using the compound interest accumulation function $a(t) = (1.05)^t$, the certificate of deposit at Bank Alpha has a higher NPV. On the other hand, if we repeat our calculation using the compound interest accumulation function $a(t) = (1.03)^t$, Bank Alpha's certificate has NPV $-\$1{,}000 + \$1{,}250(1.03)^{-2} \approx \$178.24$ which is lower than the NPV $-\$1{,}000 + \$1{,}300(1.03)^{-3} \approx \$189.68$ of Bank Beta's certificate.

Note: Using the accumulation function $a(t) = (1.05)^t$, at Bank Alpha we have a present value

$$PV_{(1.05)^t}(\$1{,}250 \text{ at } 2) = \$1{,}250(1.05)^{-2} \approx \$1{,}133.79,$$

while at Bank Beta we have a present value

$$PV_{(1.05)^t}(\$1{,}300 \text{ at } 3) = \$1{,}300(1.05)^{-3} \approx \$1{,}122.99.$$

We might look at the values $1,133.79 and $1,122.99 as follows. If Helga had a 5% savings account, $1,133.79 deposited at time 0 would grow to $(\$1{,}133.79)(1.05)^2 = \$1{,}250$ at time 2. Similarly, $1,122.99 deposited at time 0 would grow to $1,300 at time 3. Thus having the opportunity to invest in the two-year Bank Alpha CD is comparable to being given an extra $133.79 at time 0 to invest in a 5% savings account. The opportunity to invest in the three year Bank Beta CD is comparable to being given an extra $122.99 at time 0 to invest in a 5% savings account for three years. Of course $133.79 and $122.99 are the net present values we found using the accumulation function $a(t) = (1.05)^t$. ■

Example (1.7.8) involved a comparison of investments using net present values. Another way of comparing investments is by seeing what annual effective rate of interest each of the investments corresponds to.

EXAMPLE 1.7.9

Problem: As in Example (1.7.8), Helga has \$1,000 to invest. She has a choice of two investments. Bank Alpha offers a two-year certificate of deposit in which her \$1,000 would grow to \$1,250. Bank Beta's three-year certificate of deposit would allow her \$1,000 to grow to \$1,300. Find the annual effective rate of compound interest to which each of these investments correspond.

Solution We first consider the certificate of deposit offered by Bank Alpha. If Helga opens this account, the \$1,000 she deposits at time $t = 0$ grows to \$1,250 at $t = 2$. Accordingly, the applicable annual effective rate of compound interest is i_α where

$$\$1{,}250 = \$1{,}000(1 + i_\alpha)^2.$$

Therefore,

$$i_\alpha = \left(\frac{\$1{,}250}{\$1{,}000}\right)^{\frac{1}{2}} - 1 \approx .118033989 \approx 11.8\%.$$

In contrast, the \$1,000 she deposits at time $t = 0$ in a Bank Beta certificate of deposit grows to \$1,300 at $t = 3$. So, the applicable annual effective rate of compound interest is i_β where

$$\$1{,}300 = \$1{,}000(1 + i_\beta)^3.$$

Therefore,

$$i_\beta = \left(\frac{\$1{,}300}{\$1{,}000}\right)^{\frac{1}{3}} - 1 \approx .091392883 \approx 9.1\%.$$

■

In Example (1.7.9), the higher interest rate at Bank Alpha suggests that Helga might prefer banking there. However, she would have to reinvest her money two years from now, and expectations of what rates will be available at that point must be considered. We will look at problems with reinvestment rates later in the book.

We end this section with a more difficult net present value problem, followed by a discussion of special BA II Plus calculator capabilities for solving certain net present value problems.

EXAMPLE 1.7.10

Problem: Project 1 requires an investment of \$10,000 at time 0 and an additional investment of \$5,000 at $t = 1$. It returns \$2,000 at $t = 3$ and \$6,000 at times $t = 4$, $t = 5$, and $t = 6$. Project 2 requires an investment of \$6,000 at $t = 0$ and returns \$3,500 at $t = 1$ and \$5,000 at an unknown time. The net present values of the two projects are equal when calculated using compound interest at 4%. Find the unknown time for the second return of Project 2.

Solution The net present value of Project 1 is

$$-\$10{,}000 - \$5{,}000(1.04)^{-1} + \$2{,}000(1.04)^{-3} + \$6{,}000\big[(1.04)^{-4} + (1.04)^{-5} + (1.04)^{-6}\big] \approx \$1{,}772.575351.$$

Project 2 has net present value

$$-\$6{,}000 + \$3{,}500(1.04)^{-1} + \$5{,}000(1.04)^{-T}$$

where T is the time of the unknown \$5,000 return. Therefore,

$$-\$6{,}000 + \$3{,}500(1.04)^{-1} + \$5{,}000(1.04)^{-T} \approx \$1{,}772.575351.$$

This is equivalent to

$$\$5{,}000(1.04)^{-T} \approx \$1{,}772.575351 + \$6{,}000 - \$3{,}500(1.04)^{-1} \approx \$4{,}407.190735,$$

so

$$T \approx \frac{\ln\left(\frac{5{,}000}{4{,}407.190735}\right)}{\ln 1.04} \approx 3.217709596.$$

(To obtain this value, we have followed through the calculations with the stored values, thereby using more significant figures than our equations record. If you just worked with the equations, you would get 3.217709594.) Project 2 has its \$5,000 return after approximately 3.217698947 years or, based on a 365-day year, 1,174.46 days. ■

Net present values may also be calculated using the **Cash Flow worksheet** and the associated **NPV subworksheet** of the BA II Plus calculator. These frequently used worksheets are the only two, except for the **TVM worksheet**, that are accessed without using the [2ND] button. Observe that the second row of the calculator looks like

[2ND] [CF] [NPV] [IRR] [→].

Here, CF stands for cashflow, NPV stands for net present value, and IRR stands for internal rate of return. Internal rate of return will be introduced in Section (2.3) and the [IRR] key will only be used in conjunction with the **Cash Flow worksheet**.

To open the **Cash Flow worksheet** of the BA II Plus calculator, push [CF]. At this point it is advisable to clear the worksheet by pushing

[2ND] [CLR WORK].

Your display should now include "CFo = 0". The register CFo is designed to hold any cashflow made at time 0. The remaining registers in the cashflow worksheet of the BA II Plus calculator are

C01=0, F01=0, C02=0, F02=0, C03=0, . . . , C24=0, F24=0,

while the BA II Plus Professional contains registers numbered through C32=0 and F32=0. The letters C in the above sequence stand for "contribution" while the letters F signifies "frequency."

If the **Cash Flow worksheet** is open, repeatedly pushing [↓] will cause your calculator to cycle through the filled registers of the worksheet; so, if all the registers are filled, on a standard BA II Plus calculator, you will need to press [↓] forty-nine times to cycle through all the registers.

With a goal of filling these registers in a useful manner, first choose a time to denote as time $t = 0$. This will often be the time a financial relationship is originated. Enter any cashflow that occurs at time $t = 0$ in the CFo register by having the display show CFo and then keying the desired cashflow amount at time 0, followed by [ENTER]. Remember, you will need to enter all your cashflows with a consistent viewpoint relative to their signs.

Next choose an increment of time between successive cashflows. Here it is important to note that you may wish to enter 0 for some of your cashflows. The increment should in general be chosen to be the longest time interval that allows you to include all of your non-zero cashflows. So, if you have cashflows at three months, nine months, and twelve months after your $t = 0$ payment, go ahead and choose three months as your basic interval. (In this case your second contribution will be 0 since there is no cashflow at six months.) On the other hand, if contributions occur at three months and five months, the best you can do is to select one month as your time interval. Had you chosen three months, you would have skipped over the payment at the end of five months. Furthermore, neither two months nor one-and-a-half months would have been a suitable selection.[4]

The cashflows are now entered successively by showing the next available contribution register, keying a numerical amount, and then remembering to depress [ENTER] [↓]. The just depressed [↓] keystroke moves you from a contribution register to the frequency register indexed by the same number. It will be showing a frequency of 1, but if there are consecutive payments that are for an identical amount (perhaps 0), save yourself time and registers by keying in an appropriate frequency greater than 1, then depressing [ENTER] [↓].

EXAMPLE 1.7.11 Detailed instructions on the use of the Cash Flow worksheet and NPV subworksheet

Problem: Suppose that Ivy receives payments of $300 three months from now and again at the end of seven months. Furthermore, she receives $1,500 at the end of each month starting nine months from now and continuing through two years from now. That is to say, she receives a payment of $1,500 ten months from now and these payments continue monthly, the last payment occurring twenty-four months from now. Find the net present value of these payments if the monthly interest rate is $(1.04)^{\frac{1}{12}} - 1$, a rate that we will learn in Section (1.10) is equivalent to the annual effective interest rate 4%.

[4] If you know a little number theory, you might realize that the fact that the desired time increment is one month results from 3 and 5 having 1 as their greatest common divisor. In contrast, in our example where we used a three-month increment, there were non-zero cashflows at three, nine, and twelve months and the greatest common divisor of 3, 6, and 9 is 3.

Solution Take the basic time interval to be a month and let the present be $t = 0$. With these choices, consider the appropriate settings for the cashflow registers.

- CFo = 0 since there is no payment at $t = 0$.
- C01 = 0 and F01 = 2 since there are no payments at the ends of the first two months, that is to say at time $t = 1$ and $t = 2$.
- C02 = 300 and F02 = 1 since there is a payment of $300 at the end of the next month ($t = 3$), and it is not repeated the following month.
- C03 = 0 and F03 = 3 since there is no payment at the end of the next three months, that is to say at time $t = 4$, $t = 5$, and $t = 6$.
- C04 = 300 and F04 = 1 since there is a payment of $300 at the end of the next month ($t = 7$), and it is not repeated the following month.
- C05 = 0 and F05=2 since following the payment at $t = 7$, there are two successive months with no payment, namely $t = 8$ and $t = 9$.
- C06 = 1,500 and F06 = 15 since there are 15 successive months ($t = 10, 11, \ldots, 24$) at the end of each of which there is a payment of $1,500.

Graphically, we may show this as

0	1 2	3	4 5 6	7	8 9	10 11 12 . . . 24.
CFo=0	C01=0 F01=2	C02=300 F02=1	C03=0 F03=3	C04=300 F04=1	C05=0 F05=2	C06=1,500 F06=15

Note that in this schematic, the contributions C01–C06 give you the amount of the cashflow at each of the times indicated above them, while F01–F06 each indicate the number of different times shown above them.

The sequence of keystrokes that accomplishes the entry of the desired entries for CFo through F06 is as follows. Push [CF] [2ND] [CLR WORK] to open the **Cash Flow worksheet** and set the entries of all the registers equal to 0. Key [↓] [↓] to move past CF0 and C01 to F01. Next push [2] [ENTER] [↓]. This enters 2 as the value of the F01 register and moves you on to the C02 register. You may take care of the C02 and F02 entries by pushing [3] [0] [0] [ENTER] [↓] [↓]. Here you don't have to enter the F02 entry because when you enter a nonzero entry into a C-register, it automatically changes the contents of the corresponding F-register to 1. Next push [↓] [3] [ENTER] [↓] to fill the C03 and F03 registers with 0 and 3, respectively, and move to the registers indexed by 4. Keying [3] [0] [0] [ENTER] [↓] [↓] takes care of these and moves you to index 5.

(Once again you took advantage of the default frequency of 1 being entered for a nonzero contribution.) The values for the registers indexed by 5 are correctly assigned by depressing [↓] [2] [ENTER] and you push [↓] to move to the registers indexed by 6. Key [1] [5] [0] [0] [ENTER] [↓] [1] [5] [ENTER] to fill these as indicated above.

You now are ready to compute the desired net present value. This requires you to open the **NPV subworksheet** by depressing [NPV]. The subworksheet will show "I =" and you need to enter the interest rate (as a percent) per month. You were given a monthly interest rate of $(1.04)^{\frac{1}{12}} - 1$. Enter this by pushing

[1] [•] [0] [4] [y^x] [1] [2] [1/x] [=] [−] [1] [=] [×] [1] [0] [0] [=] [ENTER].

Now, push [↓] [CPT] to move to the NPV register and compute (and enter) the net present value $21,876.34572 corresponding to the above payments. ■

In our next example, we practice the skills introduced in Example (1.7.11) by recalculating the net present value of Example (1.7.10). The only new feature of this example is that we have a cashflow at $t = 0$.

EXAMPLE 1.7.12 Cash Flow worksheet and NPV subworksheet

Problem: As in Example (1.7.10), Project 1 requires an investment of $10,000 at time 0 and an additional investment of $5,000 at $t = 1$. It returns $2,000 at $t = 3$ and $6,000 at times $t = 4$, $t = 5$, and $t = 6$. Use the **Cash Flow worksheet** and **NPV subworksheet** to find the net present value of this project if the effective interest rate per basic time period is 4%.

Solution The desired entries are given schematically [as in Example (1.7.11)] by

$$\underbrace{0}_{\text{CFo}=-10{,}000}\quad \underbrace{1}_{\substack{\text{C01}=-5{,}000\\ \text{F01}=1}}\quad \underbrace{2}_{\substack{\text{C02}=0\\ \text{F02}=1}}\quad \underbrace{3}_{\substack{\text{C03}=2{,}000\\ \text{F03}=1}}\quad \underbrace{4\ 5\ 6}_{\substack{\text{C04}=6{,}000\\ \text{F04}=3}}\ .$$

These may be achieved by keying

[CF] [2ND] [CLR WORK] [1] [0] [0] [0] [0] [+/−] [ENTER] [↓]
[5] [0] [0] [0] [+/−] [ENTER] [↓] [↓] [↓] [1] [ENTER] [↓] [2] [0] [0] [0]
[ENTER] [↓] [↓] [6] [0] [0] [0] [ENTER] [↓] [3] [ENTER].

Next push [NPV]. The calculator should now display "I = ". Enter the interest rate of 4% by pushing [4] [ENTER], and subsequently depress [↓] [CPT]. Then 1,772.575351 is shown, and this is the net present value (in dollars). ■

On occasion you wish to revise the holdings of the cashflow registers without having to clear the worksheet and start anew. We next explain how this may be done efficiently. The procedure, which is described in (1) below, replaces a previously-entered register value with a new one; it works for any BA II Plus calculator register that allows entered (as opposed to computed) values. You may have already discovered this with the **Date worksheet**.

(1) To change the amount entered as the k-th cashflow, display the current value of that cashflow, and then key the desired value and push [ENTER]. Similarly, to change the frequency of the k-th cashflow, view the current value stored as the frequency, then key the desired frequency and depress [ENTER].

(2) To delete the k-th cashflow and its accompanying frequency, display the current holding of that register and key [2ND] [DEL]. This will result in the previous k-th cashflow being removed. Any subsequent cashflows will be moved, along with their accompanying frequencies, to the registers with labels one less than those they previously occupied. For instance, if you delete the 5th cashflow, the old 6th cashflow (if any) becomes the 5th cashflow, the 7th cashflow (if any) becomes the 6th cashflow, etc.

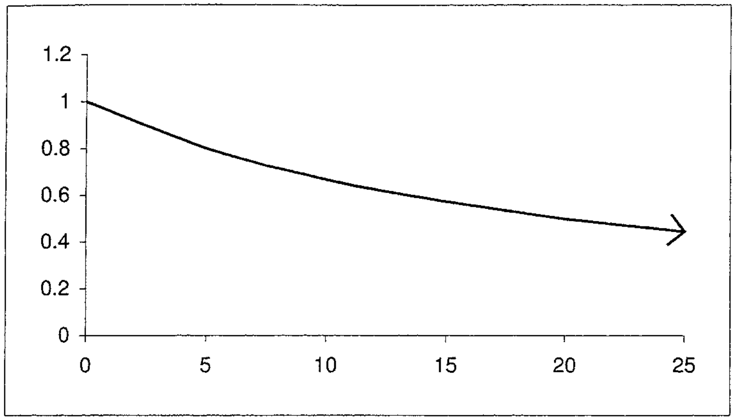
(3) To insert a forgotten cashflow as the k-th cashflow and move any subsequent cashflows to registers indexed by one more than those they previously occupied, first display the label for the k-th register along with its current entry. Then push [2ND] [INS] followed by the desired new numerical value of the k-th entry and [ENTER]. The frequency of this new entry will be entered as 1. If this is not as desired, change it according to the instructions given in (1). Naturally you may only insert a new cashflow if all your cashflow registers have not already been filled. (The display will include "↑↓ DEL INS" when inserting is possible. If all the registers are full, the "INS" will be omitted.)

1.8 SIMPLE DISCOUNT

In Section (1.4) we considered two parties negotiating a loan with a fixed amount of interest per basic time period for each $1 borrowed. Suppose that we again consider two parties negotiating a loan, but this time they agree on a fixed amount of **discount** D per basic time period for each $1 borrowed. Then, if the loan period is $[0, t]$ and K is the loan amount, the borrower receives $\$K - \$KtD = K(1 - tD)$. In particular, the borrower receives $1 if $K = (1 - tD)^{-1}$. It follows that $a(t) = (1 - tD)^{-1} = \frac{1}{1-tD}$. Then the discount function $v(t) = \frac{1}{a(t)} = 1 - tD$ is linear.

Momentarily, we will further examine the situation where the discount function $v(t)$ is linear. We caution you that this is different from the accumulation function $a(t)$ being linear. When $a(t)$ is linear, say $a(t) = 1 + st$, the discount function

$v(t) = (1 + st)^{-1}$ is a decreasing function that is asymptotic to the t-axis, hence is **not** linear.



Graph of $v(t) = (1 + st)^{-1}$ for $s = .05$

If $v(t)$ is linear, $v(t) = mt + b$, we say that we have **simple discount**. In fact, since $v(0) = \frac{1}{a(0)} = \frac{1}{1} = 1$, the discount function must then satisfy $v(t) = mt + 1$. To find m, we combine two observations. First note that

$$m + 1 = v(1) = \frac{1}{a(1)} = \frac{1}{i_1 + 1}.$$

Secondly, according to Equation (1.6.10), $\frac{1}{i_1+1} = 1 - d_1$. Therefore, $m + 1 = 1 - d_1$, $m = -d_1$, and

$$v(t) = -d_1 t + 1.$$

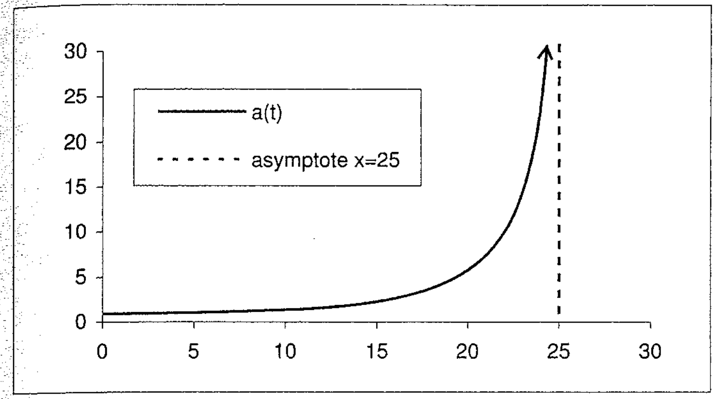
So the **simple discount accumulation function** is

$$a(t) = \frac{1}{1 - d_1 t}.$$

$A_K(t) = \frac{K}{(1-dt)}$ is called the **amount function for $\$K$ invested by simple discount at rate d.**

$a(t) = \frac{1}{1-dt}$ is called the **simple discount accumulation function at rate d.**

Then $a(t)$ is an increasing function that is asymptotic to the line $t = \frac{1}{d_1}$. Therefore, it only makes sense to talk about simple discount on the interval $[0, \frac{1}{d_1})$.



Graph of $a(t) = (1 - d_1 t)^{-1}$ for $d_1 = .04$

We next want to see how simple discount might arise in real life. Let us suppose that we have a borrowing relationship with interest paid in advance and that this loan is to extend for more than one period. If all the interest is required to be paid at the beginning, the interest per period is d, and the loan lasts for a length of time t, then the interest on \$1 is dt. We therefore get $1 - dt$ and are expected to repay 1 at time t. Hence, $v(t) = 1 - dt$, and we have simple discount.

The scenario of the previous paragraph (which resulted in accumulation being governed by the simple discount accumulation function) is analogous to the situation in which you rent an apartment for several months (or years) and are expected to pay for the whole rental in advance. This is an unlikely rental arrangement except for short periods.

1.9 COMPOUND DISCOUNT

In Section (1.5) we considered what happens when the interest rate $i_n = i_{[n-1,n]}$ is independent of n. We found that this forces us to have the compound interest accumulation function $a(t) = (1 + i)^t$. Now let us look at the consequences of stipulating that the discount function $d_n = d_{[n-1,n]}$ is a constant d.

If d_n is a constant d, then recalling (1.6.11), we see that $i_n = \frac{d_n}{1-d_n} = \frac{d}{1-d}$ is also constant. As usual we call this constant i. We then have

(1.9.1) $$i = \frac{d}{1-d},$$

and therefore

(1.9.2) $$i = \frac{1}{1-d} - 1.$$

The constant d is called the effective discount rate for the basic time period. It is the effective discount rate for the interval $[t-1, t]$ for any positive integer t [see Equation (1.6.2)]. When the basic time period is a year, it is referred to as the **annual effective discount rate.**

As demonstrated in example (1.9.3), if one is given d and wishes to compute i, it may be more efficient to use Equation (1.9.2) rather than Equation (1.9.1).

> On the BA II Plus calculator, to convert from an effective discount rate d to an effective interest rate i, enter d (NOT as a percent), then push
>
> [+/−] [+] [1] [=] [1/x] [−] [1] [=]
>
> to obtain i (again NOT as a percent).

EXAMPLE 1.9.3 Finding an effective interest rate equivalent to a given discount rate

Problem: Given that the annual effective discount rate is 4.386286%, compute the annual effective interest rate i as a percent.

Solution 1 Use formula (1.9.2) to obtain i. On the BA II Plus calculator, push

[•] [0] [4] [3] [8] [6] [2] [8] [6] [+/−] [+] [1] [=] [1/x] [−] [1] [=]

to display .045875072. Then $i = 4.5875072\%$.

Solution 2 Use formula (1.9.1) to obtain i. On the BA II Plus calculator, push

[•] [0] [4] [3] [8] [6] [2] [8] [6] [STO] [α] [+/−] [+] [1] [=] [1/x] [×] [RCL] [α] [=],

thereby displaying .045875072. Then $i = 4.5875072\%$. ■

Once it is known that i and d are constant, equation (1.6.12) gives

(1.9.4) $$d = \frac{i}{1+i}.$$

That is, the amount of discount is the ratio of the interest i for the period divided by the amount 1 will have grown to at the end of the period. Two consequences of Equation (1.9.4) are

(1.9.5) $$d = iv,$$

and

(1.9.6) $$d = 1 - \frac{1}{1+i}.$$

Note that by the definition of v as the reciprocal of $1+i$ [Equation (1.7.5)], it is immediate from (1.9.6) that

(1.9.7) $$d = 1 - v \quad \text{and} \quad v = 1 - d.$$

A verbal interpretation of Equation (1.9.5) is that d, the amount of discount on 1, is equal to the amount of interest on 1, discounted for one year by multiplying by the discount factor v.

Equations (1.9.4) and (1.9.6) give formulas for calculating the effective discount rate from the effective interest rate. Just as Equation (1.9.2) is often more efficient than (1.9.1) for changing rates, many times Equation (1.9.6) leads more quickly to the interest rate than Equation (1.9.4). The reader might observe this by converting from $i = .03263529$ to $d \approx .031603888$.

> On the BA II Plus calculator, enter i (NOT as a percent), then push
>
> [+] [1] [=] [1/x] [+/−] [+] [1] [=]
>
> to obtain d (again NOT as a percent).

EXAMPLE 1.9.8 Comparing interest and discount rates

Problem: Cassandra needs to borrow money to pay her tuition. She has a choice of borrowing at an annual effective interest rate of 5.1% or at an annual effective discount rate of 4.9%. Which rate should she choose?

Solution 1 Using Equation (1.9.1), an annual effective discount rate of 4.9% is equivalent to an annual effective interest rate of $\frac{.049}{1-.049} \approx .0515$. Since $.0515 > .051 = 5.1\%$, and it benefits the borrower to have a lower effective interest rate, Cassandra should borrow at the 5.1% interest rate.

Solution 2 According to Equation (1.9.4), the annual effective interest rate $i = 5.1\% = .051$ corresponds to an annual effective discount rate of $\frac{.051}{1.051} \approx .0485$. Since $.0485 < .049 = 4.9\%$, Cassandra should borrow at the 5.1% interest rate. This is because it assists the borrower of a loan to have the loan governed by a lower effective discount rate. ■

EXAMPLE 1.9.9 Equation (1.9.7) is useful here.

Problem: Radhika is guaranteed a payment of \$5,000 in exactly four years. She needs \$4,500 now in order to pay her tuition bill. The best loan Radhika qualifies for has a discount rate of 4.9% and requires repayment in exactly four years. Radhika can borrow the full \$4,500, in which case she will owe $\$4{,}500(1-d)^{-4} = \$4{,}500(.951)^{-4} \approx \$5{,}501.62$ at the end of four years. She can repay this with her guaranteed \$5,000 and an additional \$501.62 that she will have to raise at the end of four years. Alternatively, Radhika may be able to sell her guaranteed \$5,000 payment and use the proceeds from the sale to cover all or part of her tuition payment. How much should she be willing to sell her \$5,000 payment for?

Solution The present value of \$5,000 four years from now with respect to $a(t) = (1+i)^t = (1-d)^{-t} = (1-.049)^{-t}$ is $\$5{,}000v(4) = \$5{,}000v^4 = \$5{,}000(1-d)^4 = \$5{,}000(1-.049)^4 \approx \$4{,}089.705844$. Radhika should be willing to sell her \$5,000 payment if she is offered more than \$4,089.70. For example, if she can sell it for \$4,200, then Radhika will only have to borrow \$4,500 − \$4,200 = \$300 now. In four years she would need to repay $\$300(1+i)^4 = \$300(1-d)^{-4} = \$300(.951)^{-4} \approx \$366.77$. Thus the amount she needs to raise after four years would be reduced from \$501.62 to \$366.77. ■

According to Equation (1.9.1), $(1-d)^{-1} = 1+i$, and since a constant i_n forces $a(t) = (1+i)^t$,

$$a(t) = (1+i)^t = (1-d)^{-t}. \tag{1.9.10}$$

Equation (1.9.10) tells us that compound discount at a discount rate d is equivalent to compound interest at an interest rate $i = \frac{d}{1-d}$.

> The accumulation function $a(t) = (1-d)^{-t}$ is called the **compound discount accumulation function** at discount rate d. It is equal to the compound interest accumulation function $a(t) = (1+i)^t$ if i is the effective interest rate equivalent to d.

When the accumulation function is $a(t) = (1-d)^{-t}$, if at time 0 you borrow $\$K$ at a discount and agree to repay it at time n, you walk away with $\$K(1-d)^n$ of the lender's money. The amount of discount for the loan is $K - K(1-d)^n$.

Another way of looking at the situation where you have constant d_n is the following. At time 0 you ask the lender to allow you to borrow $\$K$. Since the lender charges a discount rate of d per basic period, he initially views this inquiry as a request that you use $\$K(1-d)$ of his money from time 0 to time 1 and then you repay $\$K$. However, you inform him that you wish to have a loan of longer duration. Consider first the case where this longer loan period is two basic time periods. The lender then thinks of your loan as a sequence of two loans, each for one basic time period, and you have promised to pay him $\$K$ at time $t = 2$. Since you will be paying the lender $\$K$ at time $t = 2$, the lender understands that you are owed $\$K(1-d)$ at the beginning of the second period, that is to say at time $t = 1$. Rather than actually paying you this $\$K(1-d)$, he will forgive you $\$K(1-d)$ that you would otherwise be required to pay him. This latter $\$K(1-d)$ is the amount you would owe at $t = 1$ if the lender loaned you $\$K(1-d)^2$ at $t = 0$. Thus, if you borrow $\$K$ from $t = 0$

to $t = 2$ at a discount rate d, you will receive $\$K(1-d)^2$ at $t = 0$. The situation for a two-period loan is represented in Figure (1.9.11).

get $\$K(1-d)^2$	pay $\$K(1-d)$	
	get $\$K(1-d)$	pay $\$K$
0	1	2

FIGURE (1.9.11)

More generally, if the loan lasts k basic time periods, the lender thinks of your loan as a sequence of k loans, each lasting one basic time period. As is suggested by Figure (1.9.12), the amount you receive at $t = 0$ is $\$K(1-d)^k$.

get $\$K(1-d)^k$	pay $\$K(1-d)^{k-1}$				
	get $\$K(1-d)^{k-1}$				
		…			
			pay $\$K(1-d)^2$		
			get $\$K(1-d)^2$	pay $\$K(1-d)$	
				get $\$K(1-d)$	pay $\$K$
0	1	…	$k-2$	$k-1$	k

FIGURE (1.9.12)

EXAMPLE 1.9.13

Problem: Ezra has the opportunity to borrow money according to compound discount at an annual effective discount rate of 8%. He would like to borrow money in order to completely pay for a \$3,000 used piano. Ezra wishes to repay the loan with a payment in exactly five years. How much money must he ask to borrow at 8% discount?

Solution Let $\$K$ denote the amount Ezra requests to borrow for five years at 8% discount. Then the amount he receives is $\$K(1-.08)^5 = \$K(.92)^5$. In order for this amount to be \$3,000, it is necessary that $K = 3{,}000(.92)^{-5} \approx \$4{,}551.79$. Since $\$4{,}551.79(.92)^5 \approx \$3{,}000.000686$, if Ezra borrows \$4,551.79 for five years at 8% discount, he receives exactly \$3,000. ■

1.10 NOMINAL RATES OF INTEREST AND DISCOUNT

Suppose that we have an investment governed by compound interest. This means that if i is the applicable effective interest rate for the investment, then the growth of the money is governed by the compound accumulation function $a(t) = (1+i)^t = 1 + [(1+i)^t - 1]$. Therefore, 1 invested for a period of length T earns $(1+i)^T - 1$ interest. The quantity $[(1+i)^T - 1]$ may be thought of as the effective interest rate for a period of length T. In particular, the effective interest rate for a period of length $T = \frac{1}{m}$ is $[(1+i)^{\frac{1}{m}} - 1]$.

Banks commonly credit interest more than once per year, say m times per year. They advertise a **nominal** (annual) **interest rate** of $i^{(m)}$ **convertible** or **compounded** or **payable *m* times per year.**[5] The word "nominal" means "in name only," and this is indeed the case. The bank pays interest at a rate of $\frac{i^{(m)}}{m}$ per m-th of a year. But we just observed that the rate for such an interval is $[(1+i)^{\frac{1}{m}} - 1]$. We therefore have the following important statement.

IMPORTANT FACT 1.10.1
If an account is governed by a nominal interest rate of $i^{(m)}$ payable m times per year, the bank pays interest at a rate of $\frac{i^{(m)}}{m} = [(1+i)^{\frac{1}{m}} - 1]$ per m-th of a year.

Observe that it follows from the equation of Fact (1.10.1) that

(1.10.2) $$i^{(m)} = m[(1+i)^{\frac{1}{m}} - 1],$$

and

(1.10.3) $$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1.$$

There is a nice way to visualize why Equation (1.10.3) must hold. Money grows by a factor of $1 + \frac{i^{(m)}}{m}$ each m-th of a year, and it therefore changes by a factor of $\left(1 + \frac{i^{(m)}}{m}\right)^m$ each year. This implies that $1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$, and this equality is essentially Equation (1.10.3).

Banks are required to report the rate i as the "annual percentage yield" or **APY**. It follows from the binomial theorem and equation (1.10.3), that if m is an integer greater than 1 and $i^{(m)} > 0$, then $i > i^{(m)}$. This is as it should be since if compounding takes place more frequently, interest earns interest, and the stated rate $i^{(m)}$ need not be as large as i.

EXAMPLE 1.10.4

Problem: National Bank advertises a savings account paying 4% nominal interest compounded quarterly. What is the annual effective yield **APY** for this account?

[5] You may see **APR**, which is an acronym for "annual percentage rate," used to indicate a nominal rate.

Solution To say that the account pays 4% interest compounded quarterly means that the interest rate per quarter is $\frac{4\%}{4} = 1\%$. Therefore the annual effective yield is $(1.01)^4 - 1 = 4.060401\%$. ■

EXAMPLE 1.10.5 Varying rates

Problem: Sandra inherits \$10,000. She deposits it in a five-year certificate of deposit paying 6% nominal interest compounded monthly and the interest remains on deposit. At the end of the five years, Sandra decides to renew her CD for another five years at the then current nominal interest rate of 7.5% compounded quarterly. Again, interest is left to accrue. At the time her second CD matures, what is her investment worth?

Solution The interest rate on the first five-year CD was 6% convertible monthly and there are $5 \times 12 = 60$ months in five years. Therefore, at the time her first CD matured, it was worth $\$10{,}000(1 + \frac{.06}{12})^{60}$. [Each month the interest rate is $\frac{.06}{12}$, so each month the balance grows by a factor of $(1 + \frac{.06}{12})$. The original balance was \$10,000, so after sixty months it is $\$10{,}000(1 + \frac{.06}{12})^{60}$.] So, Sandra's initial deposit to open her second five-year CD was $\$10{,}000(1+\frac{.06}{12})^{60}$. The interest rate for the five-year reinvestment was 7.5% nominal convertible quarterly, and there are $5 \times 4 = 20$ quarters in five years. It follows that when Sandra's second CD matures, it has a value of $\$10{,}000(1+\frac{.06}{12})^{60}(1+\frac{.075}{4})^{20} \approx \$19{,}557.63$.

Note: At the time that the money was transferred to the second CD, the balance was $\$10{,}000(1 + \frac{.06}{12})^{60} \approx \$13{,}488.50153$. At that point, the bank might have rounded it to the nearest cent. In this case her final balance would be

$$\$13{,}488.50(1 + \frac{.075}{4})^{20} \approx \$19{,}557.62394 \approx \$19{,}557.62,$$

a penny less than it would have been without the rounding. ■

EXAMPLE 1.10.6 Unknown rate

Problem: Vladimir deposits \$50,000 in a three-year certificate of deposit for which interest is compounded quarterly and is left to accrue. At the end of the three years, the balance in the CD is \$63,786.11. What is the nominal annual interest rate $i^{(4)}$ convertible quarterly?

Solution The three years that the money is on deposit consists of $3 \times 4 = 12$ quarters, and the quarterly interest rate is $\frac{i^{(4)}}{4}$. Therefore $\$50{,}000(1 + \frac{i^{(4)}}{4})^{12} = \$63{,}786.11$ and $i^{(4)} = 4[(\frac{\$63{,}786.11}{\$50{,}000})^{\frac{1}{12}} - 1] \approx .082000004 \approx .082 = 8.2\%$. ■

EXAMPLE 1.10.7 $i^{(m)}$ for m not an integer

Problem: Jolene invests money in a fund for which interest is paid once every two years. The effective rate per two-year period is 14%. Find the nominal interest rate convertible biennially[6] and the annual effective interest rate governing the fund.

Solution We are asked to find $i^{(\frac{1}{2})}$ and also i. $14\% = \frac{i^{(\frac{1}{2})}}{\frac{1}{2}}$ so $i^{(\frac{1}{2})} = 7\%$ and $1.14 = (1+i)^2$ so $i = (1.14)^{\frac{1}{2}} - 1 \approx .067707825 \approx 6.77\%$. ■

Just as banks might advertise nominal interest rates when the interest period is other than a year, you might be presented with nominal discount rates when the discount period is other than yearly. More specifically, you might be offered a **nominal discount rate $d^{(m)}$ convertible or compounded or payable m times for year**. Then the year is divided into m subintervals of equal length, and the discount rate per each subinterval is $\frac{d^{(m)}}{m}$. It follows that

$$1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m \quad \text{and} \quad d = 1 - \left(1 - \frac{d^{(m)}}{m}\right)^m . \tag{1.10.8}$$

Therefore,

$$d^{(m)} = m[1 - (1-d)^{\frac{1}{m}}]. \tag{1.10.9}$$

Just as we derived formula (1.6.7), we can show that

$$\left(1 - \frac{d^{(m)}}{m}\right)\left(1 + \frac{i^{(m)}}{m}\right) = 1. \tag{1.10.10}$$

This is equivalent to

$$0 = \frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} - \frac{i^{(m)}}{m}\frac{d^{(m)}}{m}, \tag{1.10.11}$$

from which one easily obtains

$$\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{1 - \frac{d^{(m)}}{m}} \quad \text{and} \quad i^{(m)} = \frac{d^{(m)}}{1 - \frac{d^{(m)}}{m}}. \tag{1.10.12}$$

[6]**Biennially** means once every two years. **Semiannually** means twice a year. The word biannually is sometimes used as a synonym for biennially and other times as a synonym for semiannually. We will *not* use biannual.

From (1.10.11) one also derives

$$\frac{d^{(m)}}{m} = \frac{\frac{i^{(m)}}{m}}{1 + \frac{i^{(m)}}{m}} \quad \text{and} \quad d^{(m)} = \frac{i^{(m)}}{1 + \frac{i^{(m)}}{m}}. \tag{1.10.13}$$

It is also worth noting that if n and p are integers, since Equations (1.10.3) and (1.10.8) hold for any integer m,

$$\left(1 + \frac{i^{(n)}}{n}\right)^n = 1 + i = (1-d)^{-1} = \left(1 - \frac{d^{(p)}}{p}\right)^{-p}. \tag{1.10.14}$$

This equation is useful for converting between a nominal interest rate and a nominal discount rate, perhaps having different compounding frequencies. We shall illustrate this in Example (1.10.17).

EXAMPLE 1.10.15

Problem: Trust Bank offers a savings account with a nominal discount rate of 4.8% payable monthly. Find the annual effective rate of interest i and the annual effective rate of discount d.

Solution Note that $\frac{d^{(12)}}{12} = \frac{4.8\%}{12} = .4\% = .004$. It follows that $1 - d = (1 - \frac{d^{(12)}}{12})^{12} = (1 - .004)^{12} = (.996)^{12}$ and $d = 1 - (.996)^{12} \approx .046957954 \approx 4.7\%$. Therefore, $1 + i = (1-d)^{-1} = (.996)^{-12}$ and $i = (.996)^{-12} - 1 \approx 4.927165\%$. ■

We observe that, in Example (1.10.15), $i > d^{(12)} > d$. In general, $d^{(m)} > d$ for $m > 1$ since with more frequent discounting, discount is discounted and the stated rate $d^{(m)}$ must be larger than d. Moreover, recalling (1.10.13), we see that

$$d^{(m)} = \frac{i^{(m)}}{1 + \frac{i^{(m)}}{m}} < i^{(m)}$$

provided $i^{(m)} > 0$. Also, as noted following Equation (1.10.3), $i > i^{(m)}$ when $i^{(m)} > 0$. Finally, it follows from (1.10.3) that $i^{(m)} > 0$ if and only if $i > 0$. We therefore have

IMPORTANT FACT 1.10.16
If $i > 0$ and $m > 1$, then

$$i > i^{(m)} > d^{(m)} > d.$$

Sometimes one is presented with a problem whose solution might be accomplished most easily by converting between an interest rate (possibly nominal) and a discount rate (again possibly nominal) or vice versa.

EXAMPLE 1.10.17

Problem: Given that $d^{(4)} = 4.6\%$, find $i^{(12)}$.

Solution Equation (1.10.14) tells us that $\left(1+\frac{i^{(12)}}{12}\right)^{12} = \left(1-\frac{d^{(4)}}{4}\right)^{-4}$. Consequently, $1+\frac{i^{(12)}}{12} = \left(1-\frac{d^{(4)}}{4}\right)^{-\frac{1}{3}}$ and $i^{(12)} = 12\left[\left(1-\frac{d^{(4)}}{4}\right)^{-\frac{1}{3}}-1\right]$. If $d^{(4)} = 4.6\%$, then this gives us $i^{(12)} = 4.6355852\%$. ■

The BA II Plus calculator has an **Interest Conversion worksheet** that can be used to change from an effective interest rate to an equivalent nominal interest rate or from a nominal interest rate to an equivalent effective interest rate. To open this worksheet, push [2ND] [ICONV]. At this point the display will show "NOM = ".

> Suppose that the Interest Conversion worksheet is open and displays "NOM = ". To find an effective interest rate equivalent to a given nominal interest rate $i^{(m)}$, push calculator buttons to display the numerical value of $i^{(m)}$ as a percent and then push [ENTER] [↑], at which time the display will show "C/Y = ". Push calculator buttons to display the numerical value of m and then push [ENTER] [↑], at which time the display will show "EFF = ". Push [CPT]. The equivalent effective interest rate is then displayed as a percent.

EXAMPLE 1.10.18

Problem: Use the BA II Plus calculator **Interest Conversion worksheet** to find an annual effective interest rate equivalent to a nominal rate of interest of 6% convertible quarterly.

Solution Push

[2ND] [ICONV] [6] [ENTER] [↑] [4] [ENTER] [↑] [CPT].

Displayed is now the desired effective rate as a percent, namely 6.136355063%. ■

If you have an effective interest rate and desire an equivalent nominal rate, proceed by pushing [2ND] [ICONV] to open the worksheet, then push [↓] so that "EFF = " is displayed.

> Suppose that the **Interest Conversion worksheet** is open and displays "EFF = ". To find a nominal interest rate convertible m times per period that is equivalent to an effective interest rate i for the period, push calculator buttons to display the numerical value of i as a percent and then push [ENTER] [↓], at which time the display will show "C/Y = ". Push calculator buttons to display the numerical value of m and then push [ENTER] [↓], at which time the display will show "NOM = ". Push [CPT]. The equivalent nominal interest rate convertible m times per year is then displayed as a percent.

EXAMPLE 1.10.19

Problem: Use the BA II Plus calculator **Interest Conversion worksheet** to find a nominal interest rate convertible bimonthly[7] that is equivalent to an annual effective interest rate of 3%.

Solution Push [2ND] [ICONV] [↓] [3] [ENTER] [↓] [6] [ENTER] [↓] [CPT]. Displayed now is the desired nominal rate as a percent, namely 2.963173219%. ■

The **Interest Conversion worksheet** may also be used for discount rates, more specifically to change from an effective discount rate to an equivalent nominal discount rate or from a nominal discount rate to an equivalent effective discount rate. To figure out how this should be accomplished, note that Equation (1.10.8) can be rewritten as

$$d = 1-\left(1+\frac{(-d^{(m)})}{m}\right)^{m}, \tag{1.10.20}$$

and equation (1.10.9) may be rewritten as

$$d^{(m)} = -m\left[((1+(-d))^{\frac{1}{m}}-1)\right]. \tag{1.10.21}$$

Comparing Equation (1.10.20) with Equation (1.10.3) and Equation (1.10.21) with Equation (1.10.2), one sees that an equivalent rate may be found for discount rates just as for interest rates except for the need for minus signs just after a discount rate is entered and just before the equivalent rate is found.

[7] We will always use **bimonthly** to mean once every two months. When we wish to indicate twice a month, we use **semimonthly**.

> Suppose that **the Interest Conversion worksheet** is open and displays "NOM = ". To find an effective discount rate equivalent to a given nominal discount rate $d^{(m)}$, push calculator buttons to display the numerical value of $d^{(m)}$ as a percent and then push [+/−] [ENTER] [↑], at which time the display will show "C/Y = ". Push calculator buttons to display the numerical value of m and then push [ENTER] [↑], at which time the display will show "EFF = ". Push [CPT] [+/−]. The equivalent effective discount rate is then displayed as a percent.

EXAMPLE 1.10.22

Problem: Use the BA II Plus calculator **Interest Conversion worksheet** to find an annual effective discount rate equivalent to a nominal rate of discount of 4% convertible semiannually.

Solution Push

[2ND] [ICONV] [4] [+/−] [ENTER] [↑] [2]
[ENTER] [↑] [CPT] [+/−].

Displayed is now the desired effective rate as a percent, namely 3.96%. ∎

> Suppose that the **Interest Conversion worksheet** is open and displays "EFF = ". To find a nominal discount rate convertible m times per period that is equivalent to an effective discount rate d for the period, push calculator buttons to display the numerical value of d as a percent and then push [+/−] [ENTER] [↓], at which time the display will show "C/Y = ". Push calculator buttons to display the numerical value of m and then push [ENTER] [↓], at which time the display will show "NOM = ". Push [CPT] [+/−]. The equivalent nominal discount rate convertible m times per year is then displayed as a percent.

EXAMPLE 1.10.23

Problem: Use the BA II Plus calculator **Interest Conversion worksheet** to find a nominal discount rate convertible monthly that is equivalent to an annual effective discount rate of 7%.

Solution Push

[2ND] [ICONV] [↓] [7] [+/−] [ENTER] [↓] [1] [2]
[ENTER] [↓] [CPT] [+/−].

Displayed is now the desired nominal rate as a percent, namely approximately 7.235169679%. ∎

Sometimes one wishes to convert between two nominal rates. If one of these is a discount rate and the other an interest rate, it is easiest to convert without using the special **Interest Conversion worksheet** of the BA II Plus calculator. If both the rates are interest rates or both are discount rates, the Interest Conversion worksheet may be used as may the equations of this section.

EXAMPLE 1.10.24

Problem: Given that $i^{(5)} = 3.2\%$, find $i^{(7)}$.

Solution 1 **(Interest Conversion worksheet)** Key [2ND] [ICONV] to open the Interest Conversion worksheet. Follow this by pressing the sequence of keys [3] [•] [2] [ENTER] [↑] [5] [ENTER] [↑] [CPT] to obtain the intermediate result "EFF= 3.241222984". Next push [↓] [7] [ENTER] [↓] [CPT]. The display should show "NOM = 3.197082281".

Solution 2 According to Equation (1.10.3), $\left(1+\frac{i^{(5)}}{5}\right)^5 = \left(1+\frac{i^{(7)}}{7}\right)^7$. So, $i^{(7)} = 7\left[\left(1+\frac{i^{(5)}}{5}\right)^{\frac{5}{7}} - 1\right]$. When $i^{(5)} = .032$, this gives $i^{(7)} \approx 3.197082281\%$. (You may obtain this by the twenty-three key sequence [•] [0] [3] [2] [÷] [5] [=] [+] [1] [=] [y^x] [5] [=] [y^x] [7] [1/x] [=] [−] [1] [=] [×] [7] [=], where some of the [=] keys are not necessary.) ∎

1.11 A FRIENDLY COMPETITION (CONSTANT FORCE OF INTEREST)

(calculus needed here)

Let us assume that we have compound interest at an annual effective rate of $i > 0$. Further suppose that m and p are positive integers, with $p > m$ and that the nominal rates $i^{(m)}$ and $i^{(p)}$ are each equivalent to the effective rate i. Then $i^{(p)} < i^{(m)}$ because when we compute interest p times per year, there is more compounding of interest than when we compute it m times per year. This additional compounding means that in order to produce the same effective interest rate i, we need a lower nominal rate $i^{(p)}$. The inequality $i^{(p)} < i^{(m)}$ can also be derived using calculus [see problem (1.11.4)].

As m increases, $i^{(m)}$ decreases. The following argument uses calculus to show that $i^{(m)}$ gets closer and closer to the number $\delta = \ln(1+i)$ as m grows without bound. Recalling equality (1.10.2)

$$i^{(m)} = m((1+i)^{\frac{1}{m}} - 1),$$

and l'Hospital's rule, we may find the limit of the sequence $\{i^{(m)}\}$.

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} m((1+i)^{\frac{1}{m}} - 1)$$
$$= \left(\lim_{m\to\infty} \frac{((1+i)^{\frac{1}{m}}-1)}{\frac{1}{m}}\right) = \lim_{m\to\infty} \frac{(1+i)^{\frac{1}{m}}\ln(1+i)(-m^{-2})}{-m^{-2}}$$
$$= \lim_{m\to\infty} (1+i)^{\frac{1}{m}}\ln(1+i) = \lim_{m\to\infty} \ln(1+i) = \ln(1+i).$$

We use the letter δ to denote this limit, and refer to it as the **force of interest**. We have shown that

(1.11.1) $$\boxed{\delta = \lim_{m\to\infty} i^{(m)} = \ln(1+i).}$$

Equivalently,

(1.11.2) $$\boxed{i = e^{\delta} - 1 \quad \text{and} \quad e^{\delta} = 1+i.}$$

Since we have compound interest at an effective rate i, $a(t) = (1+i)^t$ and (1.11.2) yields

(1.11.3) $$\boxed{a(t) = e^{\delta t}.}$$

This limiting process is rather abstract. We wish to visualize how it may occur in real life. Imagine that there are two banks competing hard for depositors. Bank A advertises that it pays 5% annual effective interest. Bank B decides to offer a better deal, namely 5% nominal interest convertible quarterly. This is equivalent to an annual effective interest rate of $(1+\frac{.05}{4})^4 - 1 \approx 5.0945\%$. Not to be bested, Bank A changes its accounts to pay 5% nominal interest convertible monthly. This is equivalent to an annual effective interest rate of $(1+\frac{.05}{12})^{12} - 1 \approx 5.1162\%$. Bank B continues the competition by offering 5% nominal interest convertible daily. In a nonleap year, this is equivalent to an annual effective interest rate of $(1+\frac{.05}{365})^{365} - 1 \approx 5.1267\%$. Finally, in order not to be surpassed in this competition, at least at a 5% rate of interest, Bank A offers 5% nominal interest convertible continuously. Recalling (1.11.2), the equivalent annual effective interest rate is $i = e^{.05} - 1 \approx 5.1271\%$.

We note that the effective interest rates corresponding to nominal rates of 5% convertible daily and 5% convertible continuously are very close. Continuous compounding is sometimes used to approximate daily compounding.

The discussion in this section has involved nominal interest rates. However, δ may also be realized as a limit of nominal discount rates. To see this, note that (1.11.1) implies that $\lim_{m\to\infty}\left(1+\frac{i^{(m)}}{m}\right) = 1$. Consequently, again recalling (1.11.1) and (1.10.13),

(1.11.4) $$\lim_{m\to\infty} d^{(m)} = \lim_{m\to\infty}\left(\frac{i^{(m)}}{1+\frac{i^{(m)}}{m}}\right) = \lim_{m\to\infty} i^{(m)} = \delta.$$

Since $\{i^{(m)}\}$ is decreasing and $\{d^{(m)}\}$ is increasing, we deduce from the limits $\lim_{m\to\infty} i^{(m)} = \delta$ and $\lim_{m\to\infty} d^{(m)} = \delta$ that

$$i^{(m)} > \delta > d^{(m)}.$$

We can thus extend the ordering (1.10.16) to

> **IMPORTANT FACT 1.11.5**
> If $i > 0$ and $m > 1$, then
> $$i > i^{(m)} > \delta > d^{(m)} > d.$$

In this section we have assumed that we had a compound interest accumulation function. Before moving on to consider the force of interest when you have a general accumulation function, we consider two examples.

EXAMPLE 1.11.6

Problem: Estelle deposits $12,500 in a four-year certificate of deposit at Community Trust Bank. If the annual percentage yield (APY) is 4.235%, find the nominal interest rate δ convertible continuously.

Solution We are given that $i = 4.235\%$. So, Equation (1.11.1) gives us $\delta = \ln(1.04235) \approx .041477779$. As expected [see Fact (1.11.5)], $\delta < i$. ■

> The expression "money grows at $r\%$ compounded continuously" means that the force of interest δ is numerically equal to $r\%$.

EXAMPLE 1.11.7

Problem: Money at Swift National Bank grows at 3.8% compounded continuously. Rafael Ortiz closed his savings account at Swift exactly three years after he opened it, and he received his balance which was \$24,812. Mr. Ortiz made a \$6,500 deposit exactly one year after he opened the account, but he made no other deposits or withdrawals except for an unspecified opening deposit X. Find X and his balance exactly two years after he opened the account.

Solution We are given that $\delta = 3.8\%$ so $a(t) = e^{.038t}$. At the time the account was closed, the \$6,500 deposit has grown to $\$6{,}500\frac{a(3)}{a(1)} = \$6{,}500e^{.076}$, and the initial deposit X has accumulated to $Xe^{.038\times 3} = Xe^{.114}$. So, $\$24{,}812 \approx \$6{,}500e^{.076} + Xe^{.114}$, and $X = e^{-.114}(\$24{,}812 - \$6{,}500e^{.076}) \approx \$15{,}881.07029$. X must be an integral number of cents so $X = \$15{,}881.07$. His balance at the end of two years was $\$15{,}881.07a(2) + 6{,}500\frac{a(2)}{a(1)} = \$15{,}881.07e^{.076} + 6{,}500e^{.038} \approx \$23{,}886.83$. This balance may also be found by computing $\$24{,}812\frac{a(2)}{a(3)} = \$24{,}812e^{-.038} \approx$ \$23,886.83. ■

1.12 FORCE OF INTEREST

(calculus needed here)

Suppose that you want to measure how well an investment is doing near a particular point in time, say time t. The interest rate for the unit interval $[t, t+1]$ is $\frac{a(t+1)-a(t)}{a(t)}$. However, if the interest rate varies a lot, this interest rate may not give a good idea of what is happening at the instant t: that is, it may not be helpful for determining how money grows on very short intervals containing t. Suppose that instead of looking at $[t, t+1]$, you let m denote a positive integer and look at the interval $[t, t+\frac{1}{m}]$. Then $\frac{a(t+\frac{1}{m})-a(t)}{a(t)}$ is the interest rate for your interval of length $\frac{1}{m}$, hence corresponds to the nominal interest rate $\left(\frac{a(t+\frac{1}{m})-a(t)}{a(t)}\right)/\frac{1}{m} = m\left(\frac{a(t+\frac{1}{m})-a(t)}{a(t)}\right)$. As the integer m under consideration increases, this nominal interest rate convertible m times per basic period represents a better and better approximation to the interest rate at time t. As m tends to infinity, $\frac{1}{m}$ tends to 0 and

$$m\left(\frac{a(t+\frac{1}{m})-a(t)}{a(t)}\right) = \frac{\left(\frac{a(t+\frac{1}{m})-a(t)}{\frac{1}{m}}\right)}{a(t)}$$

tend to $\frac{a'(t)}{a(t)}$. Thus $\frac{a'(t)}{a(t)}$ can be thought of as a nominal interest rate convertible continuously that describes the performance of the investment at the instant t. We call the ratio $\frac{a'(t)}{a(t)}$ the **force of interest** δ_t at time t.

(1.12.1)
$$\boxed{\delta_t = \frac{a'(t)}{a(t)}.}$$

Note that if money grows by compound interest, you have the force of interest δ from Section (1.11) as well as the newly defined force of interest δ_t. This is acceptable since, as we will verify in Example (1.12.4), when you have a compound interest accumulation function, $\delta_t = \delta$ for all t.

EXAMPLE 1.12.2 δ_t for simple interest

Problem: Suppose that you have simple interest at a rate r. Find the force of interest δ_t.

Solution The accumulation function is $a(t) = 1 + rt$. Therefore, $\delta_t = \frac{a'(t)}{a(t)} = \frac{r}{1+rt}$. ■

EXAMPLE 1.12.3 δ_t for simple discount

Problem: Suppose that you have simple discount at a rate d. Find the force of interest δ_t.

Solution The accumulation function is $a(t) = (1-dt)^{-1}$. So, $\delta_t = \frac{a'(t)}{a(t)} = \frac{-(1-dt)^{-2}(-d)}{(1-dt)^{-1}} = \frac{d}{1-dt}$. ■

While the force of interest may vary with t (as it does in our examples with simple interest or simple discount), the next example shows that under compound interest, δ_t is equal to the constant δ of Section (1.11).

EXAMPLE 1.12.4 δ_t for compound interest

Problem: Suppose that you have compound interest at a rate i. Find the force of interest δ_t as a function of i.

Solution The accumulation function is $a(t) = (1+i)^t$. Therefore,

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{(1+i)^t \ln(1+i)}{(1+i)^t} = \ln(1+i).$$

Recalling (1.11.1),

$$\delta_t = \ln(1+i) = \lim_{m\to\infty} i^{(m)} = \delta.$$

■

The definition (1.12.1) of the force of interest δ_t tells you how to obtain δ_t from the accumulation function $a(t)$. Observe that the ratio $\frac{a'(t)}{a(t)}$ is equal to the derivative $\frac{d}{dt}(\ln a(t))$. Therefore,

(1.12.5)
$$\boxed{\delta_t = \frac{d}{dt}(\ln a(t)).}$$

This gives us a second formula with which to calculate the force of interest δ_t from a given accumulation function $a(t)$.

EXAMPLE 1.12.6 Finding δ_t from $a(t)$ using (1.12.5)

Problem: Suppose $a(t) = (1.05)^{\frac{t}{2}}(1.04)^{\frac{t^2}{3}}(1.03)^{\frac{t^3}{6}}$. Find δ_t.

Solution According to (1.12.5),

$$\begin{aligned}\delta_t &= \frac{d}{dt}\left(\ln a(t)\right) = \frac{d}{dt}\left(\ln\left((1.05)^{\frac{t}{2}}(1.04)^{\frac{t^2}{3}}(1.03)^{\frac{t^3}{6}}\right)\right)\\ &= \frac{d}{dt}\left(\frac{t}{2}\ln(1.05) + \frac{t^2}{3}\ln(1.04) + \frac{t^3}{6}\ln(1.03)\right)\\ &= \frac{1}{2}\ln(1.05) + \frac{2t}{3}\ln(1.04) + \frac{t^2}{2}\ln(1.03)\\ &= \ln\left((1.05)^{\frac{1}{2}}(1.04)^{\frac{2t}{3}}(1.03)^{\frac{t^2}{2}}\right).\end{aligned}$$

■

We next learn how the accumulation function $a(t)$ may be found if the force of interest function δ_t is known.

It follows immediately from the definition of the force of interest at time r that if $r \in [0, t]$, then $\delta_r = \frac{a'(r)}{a(r)} = \frac{d}{dr}\ln a(r)$. (Note that $r \in [0, t]$ is a standard notation denoting that r is in the set $[0, t]$, and more generally $s \in S$ means that s is an element of the set S. We will henceforth use the symbol $\in$ without comment.) Integrating over the interval $[0, t]$ you obtain

$$\int_0^t \delta_r dr = \int_0^t \frac{d}{dr}\ln\left(a(r)\right)dr.$$

Now by the Fundamental Theorem of Calculus, $\int_0^t \frac{d}{dr}\ln a(r)dr = \ln a(t) - \ln a(0)$. Since $a(0) = 1$ and $\ln 1 = 0$, this gives us

$$\int_0^t \delta_r dr = \ln a(t).$$

Consequently,

(1.12.7) $$\boxed{a(t) = e^{\int_0^t \delta_r dr}.}$$

EXAMPLE 1.12.8 Finding $a(t)$ from δ_t

Problem: Suppose $\delta_t = \frac{3}{1-3t}$. Find the corresponding accumulation function $a(t)$.

Solution Note that

$$\begin{aligned}\int_0^t \delta_r dr &= \int_0^t \frac{3}{1-3r}dr = -\ln(1-3r)\Big|_0^t\\ &= -\ln(1-3t) + \ln 1 = -\ln(1-3t) = \ln\left((1-3t)^{-1}\right).\end{aligned}$$

Therefore

$$a(t) = e^{\int_0^t \delta_r dr} = e^{\ln((1-3t)^{-1})} = (1-3t)^{-1}.$$

This is the accumulation function for simple discountat a rate of 300% per basic period. ■

1.13 NOTE FOR THOSE WHO SKIPPED SECTIONS (1.11) AND (1.12)

Imagine that there are two banks competing hard for depositors. Bank A advertises that it pays 5% annual effective interest. Bank B decides to offer a better deal, namely 5% nominal interest convertible quarterly. The reason that this is a better deal is that there is more frequent compounding, and therefore more earning of interest by previous interest. The nominal interest rate of 5% convertible quarterly is equivalent to an annual effective interest rate of $(1 + \frac{.05}{4})^4 - 1 \approx 5.0945\%$. Not to be bested, Bank A changes its accounts to pay 5% nominal interest convertible monthly. This is equivalent to an annual effective interest rate of $(1 + \frac{.05}{12})^{12} - 1 \approx 5.1162\%$. Bank B continues the competition by offering 5% nominal interest convertible daily. In a nonleap year, this is equivalent to an annual effective interest rate of $(1 + \frac{.05}{365})^{365} - 1 \approx 5.1267\%$. Finally, in order not to be surpassed in this competition, at least at a 5% rate of interest, Bank A declares that it offers 5% nominal interest **convertible continuously**. That is, they say they will compound interest constantly, and this is the limiting case of compounding more and more frequently. Calculus is the mathematics of limits, so it takes calculus to analyze to what annual effective rate of interest this continuous compounding is equivalent. In fact, it is equivalent to an annual effective interest rate of $i = e^{.05} - 1 \approx 5.1271\%$.

More generally,

> A nominal interest rate of δ convertible continuously is equivalent to an annual effective rate of $i = e^{\delta} - 1$. When these equivalent rates govern the growth of money, $a(t) = (1+i)^t = e^{\delta t}$ and $v = \frac{1}{1+i} = e^{-\delta}$.

The constant δ is called the **force of interest**.

Alternatively, we may start with an annual effective interest rate and look for an equivalent force of interest (nominal rate convertible continuously).

> An annual effective interest rate of i is equivalent to a nominal rate of interest δ convertible continuously where $\delta = \ln(1+i)$.

EXAMPLE 1.13.1

Problem: Swift Bank promises 3.5% interest compounded continuously. If Ken deposits $3,000 in Swift Bank, what will his balance be four years later?

Solution At Swift Bank, $\delta = .035$, $a(t) = e^{\delta t} = e^{.035t}$, and Ken's balance after four years is $\$3{,}000e^{(.035)4} \approx \$3{,}450.82$. ■

1.14 INFLATION

In Section (1.2) we introduced units of money, saying that we would usually take these to be dollars. But what exactly does one mean by a dollar? A dollar at the beginning of the year 2000 was worth — that is, had the same purchasing power as — only about 89 cents in January 1, 1995 dollars and about seventy five cents in January 1, 1990 dollars. So, if a loaf of bread cost 75 cents in 1990, the price of a comparable loaf in the year 2000 might be about one dollar. Put another way, it would take approximately $\frac{100}{75} \approx 1.33$ dollars in 2000 to buy what you could buy for one dollar ten years earlier. This loss of purchasing power per dollar (or other monetary unit) is called **inflation**.

Inflation is formidable to measure because it gives the increase over time of a vague economic function, the price level function $p(t)$. Focus on an interval of time $[t_1, t_2]$. Analogously to how the effective interest rate $i_{[t_1,t_2]}$ was defined [see (1.3.5)], the inflation rate should be defined by

(1.14.1) $$r_{[t_1,t_2]} = \frac{p(t_2) - p(t_1)}{p(t_1)}.$$

However, it is unclear how best to define $p(t)$. Usually the price of some well-defined "basket of goods" is taken to define $p(t)$, say the "basket of goods" used to calculate a national **Consumer Price Index**[8]. But, for a particular investor, a price index of a specialized segment of the economy might be more relevant.

To analyze the impact of inflation on investments, consider the following illustration. Suppose that Marcel has $\$D$ now and that he could currently purchase u units of some good with his money. That is, each dollar could be used to purchase $\frac{u}{D}$ units of the good. If the inflation rate over one unit of time is r, then one unit of time later, the cost of the u units would be $(1+r)\$D$ and one dollar would buy $\frac{u}{(1+r)\$D}$ units. If Marcel invests the $\$D$ for one unit of time at an effective interest rate i, then it will have grown to $(1+i)\$D$, which would then be able to buy $[(1+i)\$D]\frac{u}{(1+r)\$D}$ units. So, Marcel's purchasing power has changed from u units to $\frac{1+i}{1+r}u$ units. Since this occurred over a time interval of one unit, Marcel's **inflation adjusted** (or **real**) interest rate j on his investment satisfies

(1.14.2) $$\boxed{1 + j = \frac{1+i}{1+r},}$$

[8]In the United States, the Consumer Price Index is calculated by the Bureau of Labor Statistics.

and this is equivalent to

(1.14.3) $$\boxed{j = \frac{i-r}{1+r}.}$$

When inflation is expected, investors will require higher interest rates because they are interested in actual growth in their buying power, not just a growth in the number of dollars that they possess. Borrowers who anticipate a depreciation in the value of a dollar should be willing to agree to repay more future dollars than they would otherwise [see Problem (1.14.4)]. The fact that interest rates reflect predicted inflation rather than actual inflation is a complication to those studying the correlation between interest rates and inflation rates.

In practice, of course, one cannot possibly know what the inflation rate will be in the future, and this makes investment decisions difficult. For example, suppose Marcel intends to only make investments for which the inflation-adjusted interest rate is at least the number j' and that he is presented with an investment opportunity with nonadjusted interest rate i. Does this investment meet Marcel's criterion? The answer to this question depends on the inflation rate for the period of the investment. That is to say, the assessment is contingent upon what happens in the future, and the correctness of the response is therefore uncertain. If Marcel *believes* that the inflation rate for this period will be r', then by (1.14.3), he *believes* that his inflation-adjusted interest rate will be $\frac{i-r'}{1+r'}$. Assuming Marcel acts based on his belief, he will only invest if his anticipated real interest rate is at least j', that is if

(1.14.4) $$\frac{i-r'}{1+r'} \geq j'.$$

Inequality (1.14.4) is equivalent to

(1.14.5) $$\boxed{i \geq j' + r' + r'j'.}$$

EXAMPLE 1.14.6 Anticipated inflation

Problem: Tamilla wishes to invest \$10,000 for one year provided that she can anticipate a 4% growth in her buying power. She forecasts that the inflation rate for the upcoming year will be 3%.

(a) What is the lowest rate at which she would be willing to make a loan?

(b) What rate of growth in her buying power does she look forward to if she is able to loan out her money for exactly one year at an annual effective interest rate of 8%?

(c) What was her actual growth in buying power if the inflation rate for the year was 3.5% and she was able to loan out her money for exactly one year at an annual effective interest rate of 8%?

Solution

(a) Tamilla's inflation prediction amounts to setting $r' = .03$, and her desired growth rate puts $j' = .04$. According to inequality (1.14.5), the lowest rate at which she should make a loan is $j' + r' + r'j' = .04 + .03 + (.03)(.04) = .0712 = 7.12\%$.

(b) If Tamilla is able to loan out her money at the higher rate of 8%, she foresees that her buying power will grow at the rate

$$\frac{i - r'}{1 + r'} = \frac{.08 - .03}{1.03} = \frac{5}{103} \approx .048543689 \approx 4.85\%.$$

(c) If Tamilla underestimated the rate of inflation and it actually grew at 3.5%, her actual growth in buying power was at the rate

$$\frac{i - r}{1 + r} = \frac{.08 - .035}{1.035} = \frac{45}{1035} \approx .043478261 \approx 4.35\%.$$

■

We note that the answer to question (c) of Example (1.14.1) was found using Equation (1.14.3). Had we ignored the denominator $1 + r$ of $\frac{i-r}{1+r}$, $i - r$ would have given us a good approximation to Tamilla's inflation-adjusted rate of interest, namely 4.5% rather than the true rate that was slightly below 4.35%. More generally, in times of low inflation, using the numerator $i - r$ as an estimate to j will not lead to great inaccuracies. *However*, when there is high inflation, it is critical that one include the denominator for a meaningful analysis.

EXAMPLE 1.14.7 Illustrating the importance of the denominator of Equation (1.14.3) in economies with high inflation.

Problem: Gustavo noted that the annual inflation rate was very high, some **months** in excess of 20%. He therefore usually converted his salary, paid in Brazilian Cruzados, to a more stable currency. However, one January 1st, Gustavo loaned his sister 20,000 Cruzados for one year at an effective interest rate of 620%. If the annual rate of inflation for the year was 600%, what was Gustavo's real interest rate on this family loan?

Solution According to (1.14.3), the real monthly rate of interest is $\frac{6.20-6}{1+6} \approx .028571429$. Thus, Gustavo had a modest return of about 2.86%, not the large 20% gain that would result if one ignored the denominator. ■

Since i is only equal to the real interest rate j if $r = 0$ [see Problem (1.14.2)], the interest rate i may be referred to as a **nominal** interest rate. Beware that this is a different use of the term "nominal interest rate" than was introduced in Section (1.10). In this book, except when explicitly stated to the contrary, we will ignore inflation. In real life problems, the stated effective interest rate should always be replaced by the real interest rate j if it is known, or by the anticipated real interest rate j' if you are dealing with growth in the future. If the actual inflation rate significantly exceeds the anticipated one, the real interest rate is quite likely to be negative. However, $1 + j$ is equal to the quotient $\frac{1+i}{1+r}$ and is hence positive except if there is depreciation exceeding 100% for the period under consideration. In periods where the actual rate of inflation is de-accelerating, overestimates of the inflation rate are common, and these result in real interest rates that are unexpectedly high.

1.15 PROBLEMS, CHAPTER 1

(1) Call a local bank. Learn what accounts and rates are available with an initial deposit of $1,000
   (a) if you want access to your money at any time.
   (b) if you are willing to invest your money for a fixed term. (Ask about Certificates of Deposit.)

(2) Interest is a rent for the use of money. Learn about rates charged for the rental of another commodity, e.g., vacuum cleaners, house, car. Then write a clear paragraph describing the rental situation. Be sure to include the item to be rented, the approximate value of the item at the beginning of the rental period, the length of the rental period, the price paid for the rental, and estimates of the depreciation (or appreciation) and maintenance costs during the rental period.

(3) In western cultures, the charging of interest is now well accepted as fair business practice. This was not always the case. Learn about the history of interest, and then write an essay describing what you have learned. You might include Aristotle's views on interest, the views of various religious groups (over the ages) regarding interest, and Henry VIII of England's contribution to the acceptance of interest.

(4) Learn what "usury" means. Write an essay in which you either support or argue against laws limiting the rates of interest that lenders may charge.

(5) Learn the word meaning "interest" in several foreign languages. Discover the etymologies of these words, and try to think of English words with the same roots. Write a clear paragraph (or paragraphs) describing what you have discovered.

(6) [following Section (1.14)] Inflation rates vary over time and from country to country. Pick a country and describe its inflation history. Discuss political and social developments that accompanied periods of remarkable inflation or deflation.

(7) The United States Treasury issues savings bonds for debt financing. These include EE bond and I bonds. Learn about how each of these are purchased and make returns to the investor. Write a short prospectus for each type of bonds. Include a paragraph discussing their relative advantages and disadvantages.

(1.3) Accumulation and amount functions

(1) Given that $A_K(t) = \frac{1{,}000}{100-t}$ for $0 \le t < 100$, find K and $a(20)$.

(2) If you invest \$2,000 at time 0 and $a(t) = 1 + .04t$, how much will you have at time 5?

(3) Suppose that an account is governed by a quadratic accumulation function $a(t) = \alpha t^2 + .01t + \beta$ and the interest rate i_1 for the first year is 2%. Find α, β, and the interest rate for the fourth year i_4.

(4) Suppose $a(t) = \alpha t^2 + \beta t + \gamma$. If \$100 invested at time 0 accumulates to \$152 at time 4 and \$200 invested at time 0 accumulates to \$240 at time 2, find the accumulated value at time 8 of \$1,600 invested at time 6. [HINT: First find the unknowns α and β. Then determine how much money X that you would need to deposit at time 0 in order to have an accumulation of \$1,600 at time 6. The desired answer is the accumulated value of X at time 8.]

(5) It is known that for each positive integer k, the amount of interest earned by an investor in the k-th period is k. Find the amount of interest earned by the investor from time 0 to time n, n a fixed positive integer.

(6) It is known that for each positive integer k, the amount of interest earned by an investor in the k-th period is 2^k. Find the amount of interest earned by the investor from time 0 to time n, n a fixed positive integer.

(7) Let $A_K(t) = 3t^2 + 2t + 800$. Show that the sequence of interest rates $\{i_n\}$ is decreasing for $n \ge 17$.

(8) Prof. Oops reports to his class three facts: (a) $a(t) = \alpha t^2 + \beta t + 1$; (b) \$1,000 invested at time 0 accumulates to \$1,200 at time 2; (c) \$1,000 invested at time 0 accumulates to \$10,000 at time 4. Explain why all three of these statements cannot simultaneously be true. Give an example of an accumulation function such that facts (b) and (c) hold.

(1.4) Simple interest

(1) How much interest is earned in the fourth year if \$1,000 is invested under simple interest at an annual rate of 5%? What is the balance at the end of the fourth year?

(2) In how many years will \$500 accumulate to \$800 at 6% simple interest?

(3) The monthly simple interest rate is .5%. What is the yearly simple interest rate?

(4) Find the yearly simple interest rate so that \$1,000 invested at time 0 will grow to \$1,700 in eight years.

(5) At a particular rate of simple interest, \$1,200 invested at time $t = 0$ will accumulate to \$1,320 in T years. Find the accumulated value of \$500 invested at the same rate of simple interest and again at $t = 0$, but this time for $2T$ years.

(6) A loan is made at time 0 at simple interest at a rate of 5%.
   (a) In which period is this equivalent to an effective rate of $\frac{1}{23}$?
   (b) What is the effective interest rate for the interval [4,6]?

(7) [BA II Plus Calculator] Use the Date worksheet and what you know about leap years to calculate the number of days Albert Einstein lived. His date of birth was March 14, 1879, and he died on April 18, 1955. [HINT: The calculator can count the days between March 14, 1979 and December 31, 2049. It can also count the days between January 1, 1950 and April 18, 1955.]

(1.5) Compound interest

(1) Alice invests \$2,200. Her investment grows according to compound interest at an annual effective interest rate of 4% for T years, at which time it has accumulated to \$8,000. Find T.

(2) Elliott received an inheritance from his Aunt Ruth when she died on his fifth birthday. On his eighteenth birthday, the inheritance has grown to \$32,168. If the money has been growing by compound interest at an annual effective interest rate of 6.2%, find the amount of money Aunt Ruth left to Elliott.

(3) Horatio invests money in an account earning compound interest at an unknown annual effective interest rate i. His money doubles in nine years. Find i.

(4) How much interest is earned in the fourth year by \$1,000 invested under compound interest at an annual effective interest rate of 5%?

(5) At a certain rate of compound interest, money will double in α years, money will triple in β years, and money will increase tenfold in γ years. At this same rate of compound interest, \$5 will increase to \$12 in n years. Find integers a, b, and c so that $n = a\alpha + b\beta + c\gamma$.

(6) Sean deposits \$826 in a savings account that earns interest at Increasing Rates Bank. For the first three years the money is on deposit, the annual effective interest rate is 3%. For the next two years the annual effective interest rate is 4%, and for the following five years the annual effective interest rate is 5%. What is Sean's balance at the end of ten years?

(7) For a fourteen-year investment, what level annual effective rate of interest gives the same accumulation as an annual effective rate of interest of 5% for eight years followed by a monthly effective rate of interest .6% for six years?

(8) Suppose you invest \$2,500 in a fund earning 10% simple interest annually. After two years you have the option of moving your money to an account that pays compound interest at an annual effective rate of 7%. Should you move your money to the compound interest account
   (a) if you wish to liquidate in five more years?
   (b) if you are confident your money will stay on deposit for a total of ten years?

(9) In 1963, an investor opened a savings account with $\$K$ earning simple interest at an annual rate of 2.5%. Four years later, the investor closed the account and invested the accumulated amount in a savings account earning 5% compound interest. Determine the number of years (since 1963) necessary for the balance to reach $\$3K$.

(10) On March 1, 1993, Mr. Hernandez deposited \$4,200 into an account that used a 4% annual effective interest rate when the balance was under \$5,000 and a 5.5% annual effective interest rate when the balance is at least \$5,000. Mr. Hernandez withdrew \$1,000 on March 1, 1999. If there were no other deposits or withdrawals, find Mr. Hernandez's account balance on March 1, 2003.

(11) In terms of accumulation functions, the condition that there should never be an advantage or disadvantage to closing and immediately reopening one's account means that for all positive real numbers s and t, $a(s+t) = a(s)a(t)$. Assume that $a(t)$ is differentiable for all $t \geq 0$ and differentiable from the right at $t = 0$. These conditions on derivatives amount to the accumulation function being continuous and not having any sudden changes in direction.

(a) Use the definition of the derivative to show that

$$a'(s) = a(s) \lim_{h \to 0} \frac{a(h) - 1}{h}.$$

(b) Show that $a'(s) = a(s)a'(0)$ where $a'(0)$ is a right-handed derivative.

(c) Note that $\frac{d}{ds} \ln a(s) = \frac{a'(s)}{a(s)}$. Use (b) to show that

$$\int_0^t \frac{d}{ds} \ln a(s) ds = a'(0)t.$$

(d) Deduce $\ln a(t) = \ln a(t) - \ln a(0) = a'(0)t$ from (c).

(e) Show that $a'(0) = \ln(1+i)$.

(f) Show that $a(t) = (1+i)^t$.

(1.6) Effective discount rates/ Interest in advance

(1) Antonio borrows \$3,000 for one year at an annual discount rate of 8%. How much extra money does he have the use of?

(2) Grace borrows $\$X$ for one year at a discount rate of 6%. She has use of an extra \$2,400. Find X.

(3) Jonathan borrows \$1,450 for one year at a discount rate of D. He has the use of an extra \$1,320. Find D and the annual interest rate that this is equivalent to.

(4) The amount of interest earned on $\$K$ for one year is \$256. The amount of discount paid on a one year loan "for $\$K$," transacted on a discounted basis at a discount rate that is equivalent to the interest rate of the first transaction, is \$236. Find K.

(5) A savings account earns compound interest at an annual effective interest rate i. Given that $i_{[2,4.5]} = 20\%$, find $d_{[1,3]}$.

(1.7) Discount functions/ The time value of money

(1) Suppose money grows according to the simple interest accumulation function $a(t) = 1 + .05t$. How much money would you need to invest at time 3 in order to have \$3,200 at time 8?

(2) Find the value at $t = 6$ of \$4,850 to be paid at time 12 if $a(t) = (1 - .04t)^{-1}$.

(3) Frances Morgan purchased a house for \$156,000 on July 31, 2002. If real estate prices rose at a compound rate of 6.5% annually, how much was the home Frances bought worth on July 31, 1998?

(4) A payment of $\$X$ two years from now along with a payment of $\$2X$ four years from now repays a debt of \$6,000 at 6.5% annual effective compound interest. Find X.

(5) What is the present value of \$5,000 due in ten years assuming money grows according to compound interest and the annual effective rate of interest is 4% for the first three years, 5% for the next two years, and 5.5% for the final five years?

(6) Show that if the growth of money is governed by compound interest at an annual effective interest rate $i > 0$, then the sum of the current value of a payment of $\$K$ made n periods ago and a payment of $\$K$ to be made n periods from now is greater than $\$2K$. More generally, what must be true about the operative accumulation function $a(t)$ in order that the stated conclusion holds?

(7) You have two options to repay a loan. You can repay \$6,000 now and \$5,940 in one year, or you can repay \$12,000 in 6 months. Find the annual effective interest rate(s) i at which both options have the same present value.

(8) Two projects have equal net present values when calculated using a 6% annual effective interest rate. Project 1 requires an investment of \$20,000 immediately and will return \$8,000 at the end of one year and \$15,000 at the end of two years. Project 2 requires investments of \$10,000 immediately and \$X in two years. It will return \$3,000 at the end of one year and \$14,000 at the end of three years. Find the difference in the net present values of the two projects if they are calculated using a 5% annual effective interest rate.

(1.8) Simple discount

(1) If money grows according to simple discount at an annual rate of 5%, what is the value at time 4 of \$3,460 to be paid at time 9?

(2) Sylvia invests her money in an account earning interest based on simple discount at a 2% annual rate. What is her effective interest rate in the fifth year?

(3) Suppose you can invest \$1,000 in a fund earning simple discount at an annual rate of 8% or in a fund earning simple interest at an annual rate of 12%. How long must you invest your money in order for the simple discount account to be preferable?

(4) On July 1, 1990, John invested \$300 in an account that earned 8% simple interest. On July 1, 1993 he closed this account and deposited the liquidated funds in a new account earning $q\%$ simple discount. On July 1, 1998, John had a balance of \$520 in the simple discount account. How much interest did he earn between July 1, 1993 and July 1, 1994?

(5) Suppose you invest \$300 in a fund earning simple interest at 6%. Three years later you withdraw the investment (principal and interest) and invest it in another fund earning 8% simple discount.

(a) How much time (including the three years in the simple interest account) will be required for the original \$300 to accumulate to \$650?

(b) At what annual effective rate of compound interest would \$300 accumulate to \$650 in the same amount of time?

(1.9) Compound discount

(1) A savings account starts with \$1,000 and has a level annual effective discount rate of 6.4%. Find the accumulated value at the end of five years.

(2) Latisha wishes to obtain \$4,000 to pay her college tuition now. She qualifies for a loan with a level annual effective discount rate of 3.5%.

(a) How much will she have to repay if the loan term is six years?

(b) What is the annual effective interest rate of Latisha's loan?

(3) The annual effective interest rate on Mustafa's loan is 6.8%. What is the equivalent effective quarterly discount rate on the loan?

(4) An account is governed by compound interest. The interest for three years on \$480 is \$52. Find the amount of discount for two years on \$1000.

(5) An account is governed by compound interest. The discount for three years on \$2,120 is \$250. Find the amount of interest for two years on \$380.

(6) An account with amount an initial amount $\$B$ earns compound interest at an annual effective interest rate i. The interest in the third year is \$426 and the discount in the seventh year is \$812. Find i.

(7) The amount of interest on $\$X$ for two years is \$320. The amount of discount on $\$X$ for one year is \$148. Find the annual effective interest rate i and the value of X.

(1.10) Nominal rates of interest and discount

(1) Suppose we have compound interest and $d^{(4)} = 8\%$. Find equivalent rates d, $d^{(3)}$, i, and $i^{(6)}$.

(2) The annual effective interest rate on Rogelio's loan is 6.6%. What is the equivalent nominal discount rate convertible monthly on the loan? What is the effective monthly discount rate?

(3) Suppose we have compound interest and an effective monthly interest rate of 0.5%. Find equivalent rates $i^{(12)}$, i, and d.

(4) Find the accumulated value of \$2,480 at the end of twelve years if the nominal interest rate was 2% convertible monthly for the first three years, the nominal rate of discount was 3% convertible semiannually for the next two years, the nominal rate of interest was 4.2% convertible once every two years for the next four years, and the annual effective rate of discount was .058 for the last three years.

(5) Given equivalent rates $i^{(m)} = .0469936613$ and $d^{(m)} = .046773854$, find m.

(6) Given that

$$1 - \frac{d^{(n)}}{n} = \frac{1 + \frac{i^{(7)}}{7}}{1 + \frac{i^{(6)}}{6}},$$

find n.

(7) Let m be a positive real number. Suppose interest is paid once every m years at a nominal interest rate $i^{(\frac{1}{m})}$. This means that the borrower pays interest at an effective rate of $\frac{i^{(\frac{1}{m})}}{\frac{1}{m}} = mi^{(\frac{1}{m})}$ per m year period.

(a) Find an expression for $i^{(\frac{1}{m})}$ in terms of i.

(b) If $i^{(\frac{2}{3})} = .06$, find i.

(c) Define $d^{(\frac{1}{m})}$ to be the nominal discount rate payable once every m years. This means that the borrower pays discount at an effective rate of

$$\frac{d^{(\frac{1}{m})}}{\frac{1}{m}} = md^{(\frac{1}{m})} \quad \text{per } m \text{ year period.}$$

Find a formula that gives $d^{(\frac{1}{m})}$ in terms of $i^{(\frac{1}{m})}$, and a formula that gives $d^{(\frac{1}{m})}$ in terms of d.

(1.11) A friendly competition (Constant force of interest)

(1) Suppose $d^{(4)} = 3.2\%$. Find δ.

(2) Given that $\delta = .04$, find the accumulated value of \$300 five years after it is deposited.

(3) You have a choice of depositing your money in account A which has an annual effective interest rate of 5.2%, account B which has an effective monthly rate of .44%, or account C that is governed by force of interest $\delta = .0516$. Which account should you choose? Which account would give you the lowest accumulation?

(4) Let $i(x) = x[(1+i)^{\frac{1}{x}} - 1]$, $x > 1$. Note that $i(m) = i^{(m)}$.

(a) Find the derivative $i'(x)$ and show that the condition that $i(x)$ is decreasing is equivalent to $(1+i)^{\frac{1}{x}}\left(1 - x^{-1}\ln(1+i)\right) < 1$.

(b) Show that the condition $i(x)$ is decreasing is equivalent to the equation

$$\ln\left[(1+i)^{\frac{1}{x}}(1 - x^{-1}\ln(1+i))\right] < \ln 1 = 0.$$

(c) Let $z = \frac{1}{x}\ln(1+i)$. Show that $i(x)$ is decreasing is equivalent to $z + \ln(1-z) < 0$ and therefore to $1 - z < e^{-z}$.

(d) Show that $1 - z < e^{-z}$ follows from Taylor's theorem with remainder.

(e) Explain how this problem allows us to conclude that if $p > m > 1$, then $i^{(p)} < i^{(m)}$.

(1.12) Force of interest

(1) Given that the force of interest is $\delta_t = .05 + .006t$, find the accumulated value after three years of an investment of \$300 made at

(a) time 0.

(b) time 4.

(2) Given that the force of interest is $\delta_t = \frac{2t}{1+t^2}$, find the effective rate of discount for the sixth year.

(3) Given that the force of interest is $\delta_t = \frac{t^2}{1+t^3}$, find the present value at time 0 of \$300 to be paid at time $t = 4$.

(4) Given that $a(t) = e^{.03t+.002t^2}$, find δ_2.

(5) Given that $a(t) = (1+.02)^t(1+.03t)(1-.05t)^{-1}$, find δ_3.

(6) Fund 1 accumulates with a discount rate of 2.4% convertible monthly. Fund 2 accumulates with a force of interest $\delta_t = \frac{t}{12}$ for all $t \geq 0$. At time 0, \$100 is deposited in each fund. Determine all later times at which the two funds have equal holdings, assuming that there are no further contributions to either fund.

(7) As in Problem (1.5.8), suppose you invest \$2,500 in a fund earning 10% simple interest. Further suppose that you have the option at any time of closing this account and opening an account earning compound interest at an annual effective interest rate of 7%. At what instant should you do so in order to maximize your accumulation at the end of five years? How about if you wish to maximize the accumulation at the end of ten years?

(8) (a) Fund I grows according to simple interest at rate r. Find the force of interest $\delta_t^{(I)}$ acting on fund I at time t.

(b) Fund D grows according to simple discount at rate s. Find the force of interest $\delta_t^{(D)}$ acting on fund D at time t.

(c) Suppose $r > s$. Find all t such that $\delta_t^{(I)} = \delta_t^{(D)}$.

(9) Fund A has a balance of \$600 at time $t = 0$ and its growth is determined by a force of interest

$$\delta_t^{(A)} = \frac{.08}{1+.08t}.$$

Fund B has a balance of \$300 at time $t = 0$, and its growth is determined by a force of interest $\delta_t^{(B)} = .01t$. Fund C has amount function $A^{(C)}(t) = A^{(A)}(t) + 2A^{(B)}(t)$ where $A^{(A)}(t)$ is the amount function giving the growth of fund A and $A^{(B)}(t)$ is the amount function giving the growth of fund B. The force of interest for fund C is $\delta_t^{(C)}$. Find $\delta_4^{(C)}$.

(10) Mr. Valdez has \$10,000 to invest at time $t = 0$, and three ways to invest it. Investment account I is governed by compound interest with an annual effective discount rate of 3%. Investment account II has force of interest equal to $\frac{.04}{1+.05t^2}$. Investment account III is governed by the accumulation function $a^{\text{III}}(t) = (1 - .005t^2)^{-1}$. Mr. Valdez can transfer his money between the three investments at any time. What is the maximum amount he can accumulate at time $t = 5$? [HINT: At all times, Mr. Valdez wishes to have his money in the account that has the greatest force of interest at that moment. Therefore, begin by determining the force of interest function for each of the investment accounts. Next decide for which time interval Mr. Valdez should have his money in each of the accounts. Assume that he accordingly moves his money to maximize his return. You will then need the accumulation functions for the accounts in order to determine Mr. Valdez's balance at $t = 5$. Remember to use Important Fact (1.7.4).]

(1.13) Note for those who skipped Section (1.11) and (1.12)

(1) What rate of annual effective interest does John earn if his money is invested in a bank that credits interest continuously at a rate of 3.75%?

(2) Andrea is upset because her bank does not compound interest continuously. Instead, they use a nominal discount rate of 4% compounded quarterly. To what rate of continuous compounding is that equivalent?

(1.14) Inflation

(1) Inflation is forecast to be at an annual rate of 3% for the next year.

(a) What will the real rate of interest be if the forecast holds true, and the stated effective rate for the year is 4.2%?

(b) What will the real rate be if the actual rate of inflation is 4.6%?

(2) Show that the nominal interest rate i and the real interest rate j are equal if and only if the inflation rate is zero.

(3) The nominal rate of discount is 3% convertible quarterly. The inflation-adjusted (effective) rate of interest is 1.24%. Find the rate of inflation.

(4) We have noted that the effective rate i should be replaced by the inflation-adjusted rate

$$j = \frac{1+i}{1+r} - 1$$

in real life calculations. With notation as in Section (1.7), find Y such that

$$PV_{(1+j)^t}(\$X \text{ at time } n) = PV_{(1+i)^t}(\$Y \text{ at time } n).$$

(5) Money is invested in a savings account with a nominal interest rate of 2.4% convertible monthly for three years. The rate of inflation is 1.5% for the first year, 2.8% for the second year, and 3.4% for the third year. Find the percentage of purchasing power lost during the time the money is invested; that is, find p so that if you could purchase exactly u units at the time the money was invested, three years later you could purchase $u(1 - .01p)$.

(6) Suppose that there is compound interest at an annual inflation adjusted rate of 1.8% and that the annual rate of inflation is 2.3%. Find the force of interest corresponding to the stated rate of interest and the inflation-adjusted force of interest corresponding to the real rate of interest. In general, what can you say about the difference between the force of interest and the inflation-adjusted force of interest?

Chapter 1 review problems

(1) Find the accumulated value of $6,208 at the end of eight years if the nominal rate of discount is 2.3% convertible quarterly for the first two years, the nominal rate of interest is 3% convertible monthly for the next year, the annual effective rate of discount is 4.2% for the next three years, and the force of interest is .046 for the last two years.

(2) Suppose that $d^{(2)} - d = .00107584$. Find $i^{(3)} - i$.

(3) Assume that an investment is governed by compound interest at a level interest rate. The present value of $\$K$ payable at the end of two years is $1,039.98. If the force of interest is cut in half (resulting in a change in the level interest rate), the present value of $\$K$ payable at the end of two years is $1,060.78. What is the present value of $\$K$ payable at the end of two years if the annual effective discount rate is cut in half?

(4) (a) Express $\frac{d}{d\delta}v$ as a function of d.

(b) Express $\left(\frac{d}{dv}\delta\right)\left(\frac{d}{di}d\right)$ as a function of d.

(5) A borrower will have two options for repaying a loan. The *first option* is to make a payment of $6,000 on December 1, 2003 and a payment of $4,000 on December 1, 2004. The *second option* is to make a single payment of $12,000 N months after December 1, 2003. Assuming that the two options have the same value on December 1, 2003, if the interest rate is an annual effective rate of 5%, determine N.

(6) On June 3, 1977, Alan borrowed $3,000 from Chan and gave Chan a promissory note at an annual rate of simple interest of 10% and a maturity date of May 15, 1978. Exact simple interest is used to compute the repayment amount on this note. Javier purchased the promissory note from Chan on December 20, 1977 based on simple discount at an annual rate of 12%, with time measured using the "actual/actual" method. Determine Javier's purchase price, and the equivalent annual effective interest rates earned by Chan and by Javier. (In Chapter 2, such rates are called annual yield rates.)

(7) Suppose that an investment is governed by an accumulation function

$$a(t) = \begin{cases} at^2 + bt + c, & \text{for } 0 \le t \le 6 \\ (at^2 + bt + c)(1 + .05(t-6)), & \text{for } t > 6. \end{cases}$$

Further suppose that $i_3 = 50/1{,}088$ and $d_4 = 54/1{,}192$.

(a) Determine the constants a, b, and c.

(b) Find the value at $t = 3$ of $1,000 to be paid at $t = 8$.

(8) Shanon deposits $\$K$ into an account that earns compound interest at an annual effective interest rate i. She makes no further deposits or withdrawals. Her interest in the fifth year is $175.37, and the discount for the second year is $153.59. Find i.

(9) Suppose that $\delta_t = \frac{4}{t-1}$ for $2 \le t \le 8$. For $2 \le n \le 7$, let $f(n) = i_{n+1} + 1$. Write a simple formula for $f(n)$.