MATH 373
Test 1
Spring 2018
February 27, 2018

1. Emily is saving for her retirement. She invests 100 at the beginning of each month for 40 years into an account earning an annual effective interest rate of 9%.

   Calculate the amount that Emily will have at the end of 40 years.

   Solution:

   $$Amount = (100)\ddot{s}_{\overline{480}|}$$

   Since payments are monthly, we need the monthly effective interest rate of $\frac{i^{(12)}}{12}$.

   $$\left(1+\frac{i^{(12)}}{12}\right)^{12} = 1+i = 1.09 \implies \frac{i^{(12)}}{12} = (1.09)^{1/12} - 1 = 0.007207323$$

   $$(100)\ddot{s}_{\overline{480}|} = (100)\left(\frac{(1.007207323)^{480} - 1}{0.007207323}\right)(1.007207323) = 424,964.87$$

2. You are given that $a(t) = \alpha + \beta t^3$. You are also given that $\delta_{10} = 0.25$.

Calculate i_5.

Solution:

$$a(0) = 1 ==> \alpha + \beta(0) = 1 ==> \alpha = 1$$

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{3\beta t^2}{1+\beta t^3} ==> \delta_{10} = 0.25 = \frac{3\beta(10)^2}{1+\beta(10)^3} ==> 0.25 + 250\beta = 300\beta$$

$$==> 0.25 = 50\beta ==> \beta = 0.005$$

$$i_5 = \frac{a(5)-a(4)}{a(4)} = \frac{1+(0.005)(5)^3 - [1+0.005(4)^3]}{1+0.005(4)^3} = 0.23106$$

3. Rodgers Bank makes five year loans to college students. Rodgers wants to receive an annual rate of 5% compounded continuously to compensate for deferred consumption. Additionally, Rodgers expects that inflation will occur at an annual rate of 3% compounded continuously over the next five years. However, since the inflation rate could be higher, Rodgers would like to receive an annual rate of 0.5% compounded continuously as compensation for the inflation risk.

   Additionally, Rogers expects 10% of the loans to default with a loan recovery rate of 48%. Rogers adds a charge for defaults to the other components of the interest rates to compensate for the expected defaults.

   Determine the annual rate compounded continuously that Rodgers should charge for defaults. Your answer should be accurate to five decimal places.

   Solution:

$$\text{Rate without Defaults} = 0.05 + 0.03 + 0.005 = 0.085$$

$$\text{Rate with Defaults} = 0.05 + 0.03 + 0.005 + \delta_s = 0.085 + \delta_s$$

Total Cash Received without defaults must equal total cash received with defaults.

$$e^{(0.085)(5)} = 0.9e^{(0.085+\delta_s)(5)} + (0.1)(0.48)e^{(0.085+\delta_s)(5)} = (0.948)e^{(0.085+\delta_s)(5)}$$

$$\frac{e^{(0.085)(5)}}{e^{(0.085)(5)}} = \frac{(0.948)e^{(0.085+\delta_s)(5)}}{e^{(0.085)(5)}} ==> 1 = (0.948)e^{(\delta_s)(5)} ==> e^{(\delta_s)(5)} = \frac{1}{0.948}$$

$$(\delta_s)(5) = \ln\left(\frac{1}{0.948}\right) ==> \delta_s = 0.01068$$

4. Maddie is repaying a loan with 10 payments. The payment at time t is $10,000t$. In other words, the first payment is 10,000. The second payment is 20,000. The payments continue to increase until a payment of 100,000 is made in the 10th year.

   The annual effective interest rate on the loan is 5%.

   Maddie makes all the payments as scheduled except that she forgets to make the payment of 50,000 in the fifth year.

   Determine the outstanding loan balance on Maddie's loan right after the payment of 80,000 is made in the 8th year.

   Solution:

   The Outstanding Loan Balance at time 8 is the unpaid payments valued at time 8

$$OLB = (50,000)(1.05)^{3} + (90,000)(1.05)^{-1} + (100,000)(1.05)^{-2} = 234,298.48$$

5. Sadi is receiving an increasing annuity from his grandparents. Sadi will receive payments at the end of each month for the next 10 years. The first payment is 1000. The second payment is 1100. The third payment is 1200. The payments continue to increase in the same pattern until a payment of 12,900 is made at the end of the 120th month.

   Sadi invests each payment in the Chen Fund earning an interest rate of 6% compounded monthly. At the end of 10 years, Sadi takes the money in the Chen Fund and uses it to purchase a perpetuity due with monthly payments of P. The payment is based on an interest rate of 6% compounded monthly.

   Determine P.

   Solution:

   First we must find the amount of money that Sadi will have at the end of 10 years. Then we will use that amount of money to purchase the perpetuity due. Since payments are monthly we need $\frac{i^{(12)}}{12} = \frac{0.06}{12} = 0.005$.

$$\text{AV at time 10} = (P\,\&\,Q)(1.005)^{120} = \left[1000a_{\overline{120|}} + \frac{100}{0.005}\left\{a_{\overline{120|}} - 120v^{120}\right\}\right](1.005)^{120}$$

$$= \left[1000\left(\frac{1-(1.005)^{-120}}{0.005}\right) + \frac{100}{0.005}\left\{\left(\frac{1-(1.005)^{-120}}{0.005}\right) - 120(1.005)^{-120}\right\}\right](1.005)^{120}$$

$$= 1,041,466.28$$

$$P\ddot{a}_{\overline{\infty|}} = 1,041,466.28 ==> P\left(\frac{1}{0.005}(1.005)\right) = 1,041,466.28 ==> P = 5181.42$$

6. Tom invests 10,000 in the Pham Fund for 10 years.

   a. The Pham Fund pays simple interest at an annual rate of 5% during the first 3 years.

   b. The Pham Fund pays an interest rate equivalent to a discount rate of 8% convertible quarterly during the next five years.

   c. The Pham Fund pays a force of interest of 10% during the last two years of Tom's investment.

   Determine the amount of money that Tom has at the end of 10 years.

   Solution:

$$Amount = (10,000)[1+(0.05)(3)]\left[1-\frac{0.08}{4}\right]^{-(4)(5)}\left[e^{(0.1)(2)}\right] = 21,039.49$$

7. Tague opens a new bank account with a deposit of 3000 on February 1, 2018. On April 1, 2018, the account is worth 3100 and he deposits another 2000 into the account. On July 1, 2018, he withdraws 3500 to go on vacation. After the withdrawal, Tague had 1700 left in the account. He closes the account on November 30, 2018 by withdrawing 1850.

   Estimate, using the simple interest method learned in class, the annual effective dollar weighted return earned by Tague on the bank account.

   Solution:

   You can treat the 3000 deposit at time 0 as the amount at time 0 or as a contribution at time 0. Additionally, you can treat the 1850 at the end as the ending balance or as a negative contribution at that time. We will treat them as balances but if you treat them as contributions, you will get the same answer.

$$A = 3000; B = 1850; C = 2000 - 3500 = -1500$$

$$A + C + I = B \Longrightarrow 3000 - 1500 + I = 1850 \Longrightarrow I = 1850 - 3000 + 1500 = 350$$

$$j = \frac{350}{3000 + 2000(1 - 2/10) - 3500(1 - 5/10)} = 0.12281$$

$$1 + i = (1 + j)^{\frac{1}{T}} = (1.12281)^{\frac{1}{10/12}} = (1.12281)^{12/10} = 1.149122 \Longrightarrow i = 0.149122$$

8. James invests 10,000 at the end of each year for 15 years into the Marshall Fund. The Marshall Fund pays an annual effective interest rate of 6%. At the end of each year, James takes the interest earned in the Marshall Fund and transfers it to the Reber Fund. The Reber Fund pays an annual effective interest rate of 8%.

   Determine the total amount that James has in the Marshall and Reber Funds combined at the end of 15 years.

   Solution:

   Since James does not invest any money until the end of the first year, the amount of interest in the first year is 0. During the second year, the Marshall Fund earns (10,000)(0.06) = 600 which gets transfered to the Reber Fund. At the end of the third year, (2)(10,000)(0.06) = 1200 gets transfered to the Reber Fund. This continues until (14)(10,000)(0.06) = 8400 is transfered at the end of the 15th year.

   The amount in the Marshall Fund will be (15)(10,000) = 150,000

   The amount in the Reber Fund will be the accumulated value of the interest using 8%.

$$\text{Reber Fund} = (P\,\&\,Q)(1.08)^{14} = \left[600a_{\overline{14}|} - \frac{600}{0.08}\left\{a_{\overline{14}|} - 14(1.08)^{-14}\right\}\right](1.08)^{14}$$

$$= \left[600\left(\frac{1-(1.08)^{-14}}{0.08}\right) - \frac{600}{0.08}\left\{\left(\frac{1-(1.08)^{-14}}{0.08}\right) - 14(1.08)^{-14}\right\}\right](1.08)^{14} = 91{,}140.85$$

$$Total = 150,000 + 91,140.85 = 241,140.85$$

9. Chrissy is the beneficiary of the Maple Trust. The Maple Trust will make payments to Chrissy at the beginning of each year for the next 25 years. The first payment is 1000. The second payment is $(1000)(1.07)^1$. The third payment is $(1000)(1.07)^2$. The payments continue to increase with each payment being 107% of the prior payment.

   Calculate the present value of the payments at an annual effective interest rate of 6%.

   Solution:

   $$PV = 1000 + 1000(1.07)(1.06)^{-1} + ... + 1000(1.07)^{24}(1.06)^{-24}$$

   $$= \frac{1000 - 1000(1.07)^{25}(1.06)^{-25}}{1-(1.07)(1.06)^{-1}} = 28,045.94$$

10. A US Treasury Bill matures for 100,000 at the end of 200 days and has a price of P_{TBILL}^{US}.

A Canadian Treasury Bill matures for 100,000 at the end of 200 days and has a price of P_{TBILL}^{CAN}.

The Quoted Rate for both the US and the Canadian Treasury Bill is 7.25%.

Calculate $P_{TBILL}^{CAN} - P_{TBILL}^{US}$.

Solution:

Canadian

$$\text{Quoted Rate} = \left(\frac{365}{\text{Days to Maturity}}\right)\left(\frac{\text{Interest}}{\text{Price}}\right) \text{==>}\ 0.0725 = \left(\frac{365}{200}\right)\left(\frac{100,000 - \text{Price}}{\text{Price}}\right)$$

$$\text{==>}\ (0.0725)\left(\frac{200}{365}\right)(\text{Price}) = 100,000 - \text{Price} \text{==>}\ \text{Price} = \frac{100,000}{1 + (0.0725)\left(\frac{200}{365}\right)} = 96,179.18$$

United States

$$\text{Quoted Rate} = \left(\frac{360}{\text{Days to Maturity}}\right)\left(\frac{\text{Interest}}{\text{Maturity Value}} = \right) \text{=>}\ 0.0725 = \left(\frac{360}{200}\right)\left(\frac{100,000 - \text{Price}}{100,000}\right)$$

$$\text{==>}\ (0.0725)\left(\frac{200}{360}\right)(100,000) = 100,000 - \text{Price} \text{==>}\ \text{Price} = 100,000 - (0.0725)\left(\frac{200}{360}\right)(100,000)$$

$$= 95,972.22$$

$$Answer = 96,179.18 - 95,972.22 = 206.96$$

11. Kayla borrows 10,000 from Alex. Alex tells Kayla that she can repay him using one of the following two options:

    a. Pay a single payment 26,017.40 at the end of 10 years; or

    b. Make monthly payments of P for the next 10 years.

    Both payment options are equivalent which means that Kayla will be paying the same interest rate.

    Determine P .

Solution:

Option a:

$$10,000(1+i)^{10} = 26,017.40 ==> i = \left(\frac{26,017.40}{10,000}\right)^{1/10} - 1 = 0.100339$$

Option b:

For this option since payments are monthly, we need $\frac{i^{(12)}}{12}$.

$$\left(1+ \frac{i^{(12)}}{12}\right)^{12} = 1+i = 1.100339 ==> \frac{i^{(12)}}{12} = (1.100339)^{1/12} - 1 = 0.008$$

$$Pa_{\overline{120|}} = 10,0000 ==> P\left(\frac{1-(1.008)^{-120}}{0.008}\right) = 10,000 ==> P = \frac{10,0000}{\frac{1-(1.008)^{-120}}{0.008}} = 129.95$$

12. Carl has 10,000 to invest. He has the following options:

   a. Invest 10,000 in the White Fund. Under this investment, Carl will receive 20,000 at the end of 10 years. Carl used the Rule of 72 to estimate that annual effective interest rate that he will earn is i.

   b. Loan 10,000 to Patrick at a interest rate of i. Under the loan, Patrick would repay P at time 3 and $2P$ at time 7.

Note: The i in part a. is the same as the i in part b.

Determine P.

Solution:

Using the Rule of 72, money doubles in $\frac{72}{i}$ years.

$$==> i = \frac{72}{10} = 0.072$$

$$10,000 = P(1.072)^{-3} + 2P(1.072)^{-7}$$

$$P = \frac{10,000}{(1.072)^{-3} + 2(1.072)^{-7}} = 4899.41$$