Math 373
Test 2
Spring 2018

March 27, 2018

1. Each of the following two bonds has a price of 50,000 and the same yield rate:

    a. A zero coupon bond that matures for 100,000 at the end of 10 years.

    b. A 10 year bond with a maturity value of 80,000 and semi-annual coupons of X.

    Determine X.

    Solution:

    Bond a.

    $$50,000 = 100,000(1+i)^{-10} \implies 0.5 = (1+i)^{-10} \implies 2 = (1+i)^{10} \implies i = (2)^{0.1} - 1 = 0.0717735$$

    Bond b.

    We need $\frac{i^{(2)}}{2}$ since coupons are semi-annually.

    $$\left(1+\frac{i^{(2)}}{2}\right)^2 = 1+i = 1.0717735 \implies \frac{i^{(2)}}{2} = (1.0717735)^{0.5} - 1 = 0.035264924$$

    $$50,000 = Xa_{\overline{20|}} + 80,000(1.035264924)^{-20} = X\left(\frac{1-(1.035264924)^{-20}}{0.035264924}\right) + 80,000(1.035264924)^{-20}$$

    $$X = \frac{50,000 - 80,000(1.035264924)^{-20}}{\left(\frac{1-(1.035264924)^{-20}}{0.035264924}\right)} = 705.30$$

2. A 20 year bond is sold at a discount of 380. The bond has a par value of F and a maturity of $F+400$. The bond pays semi-annual coupons at a rate of 6% compounded semi-annually.

   The bond is purchased to yield 6.5% compounded semi-annually.

   Determine the price of the bond.

   Solution:

$$C - P = Discount = 380 ==> P + 380 = C$$

$$C = F + 400 ==> F + 400 = P + 380 ==> P = F + 20$$

$$P = Fra_{\overline{40}|} + Cv^{40} ==> F + 20 = F(0.03)\left(\frac{1-(1.0325)^{-40}}{0.0325}\right) + (F+400)(1.0325)^{-40}$$

$$F + 20 = F(0.666253) + 0.27822592F + 111.2903679$$

$$F(1 - 0.666253 - 0.27822592) = 111.2903679 - 20$$

$$F = 1644.25$$

$$P = F + 20 = 1664.25$$

3. Kevin invests in the White Fund. He makes deposits at the end of each month for 15 years. During the first year, each monthly deposit is 1000. During second year, each monthly deposit is 1100. During the third year, each monthly deposit is 1200. The deposits continue in the same pattern until each monthly deposits during the 15th year is 2400.

   The White Fund pays an interest rate of 12% compounded monthly.

   Calculate the amount that Kevin will have at the end of 15 years.

   Solution:

For this problem, we must split the payments into level payments and payments that allow us to use the formula that does not follow the rules. The level payments will be the amount of the first payment less the amount of the increase $=1000-100=900$.

We need both i and $\frac{i^{(12)}}{12}$. We are given $i^{(12)}=0.12 \Longrightarrow \frac{i^{(12)}}{12}=0.01$ and $i=(1.01)^{12}-1=0.12682503$

$$AV = PV(1.12682503)^{15}$$

$$AV = \left[900a_{\overline{180}|0.01} + 100\left(\frac{\ddot{a}_{\overline{15}|0.12682503} - 15(1.12682503)^{-15}}{0.01}\right)\right](1.12682503)^{15}$$

$$= \left[900\left(\frac{1-(1.01)^{-180}}{0.01}\right) + 100\left(\frac{\left\{\frac{1-(1.12682503)^{-15}}{0.12682503}\right\}(1.12682503) - 15(1.12682503)^{-15}}{0.01}\right)\right](1.12682503)^{15}$$

$$= 743,492.78$$

4. Carl has a 20 year sinking fund loan of 250,000. Under the sinking fund loan, Carl will pay the interest at the end of each year on the loan. Additionally, Carl will make a deposit into a sinking fund at the end of each year. The deposit into the sinking fund will be such that the amount in the sinking fund will exactly repay the loan at the end of 20 years.

   The annual effective interest rate on the loan is 12% and the sinking fund pays an annual effective interest rate of 9%.

   Carl make deposits into the sinking fund for 12 years.

   Determine the amount that Carl has in the sinking fund right after the 12th deposit.

   Solution:

$$D = \frac{250,000}{s_{\overline{20}|}} = \frac{250,000}{\left(\dfrac{(1.09)^{20} - 1}{0.09}\right)} = 4886.62$$

$$Amount = Ds_{\overline{12}|} = 4886.62\left(\frac{(1.09)^{12} - 1}{0.09}\right) = 98,420.02$$

5. A loan which is being repaid with level annual payments at an interest rate of 8%. The interest in the 10th payment is 846.41. The principle in the 20th payment is 609.92.

   Calculate the amount of the loan.

   Solution:

$$P_{10} = P_{20}(1.08)^{-10} = 609.92(1.08)^{-10} = 282.51$$

$$Q = P_{10} + I_{10} = 846.41 + 282.51 = 1128.92$$

$$P_1 = 282.51(1.08)^{-9} = 141.326$$

$$I_1 = 1128.92 - 141.326 = 987.594$$

$$I_1 = iL \text{ ==> } 987.594 = (0.08)(L) \text{ ==> } L = 12,344.93 \approx 12,345$$

6. Keyi wants to borrow 100,000 for the next ten years. She has the following two loan options:

   a. Bank Cao offers a 10 year sinking fund loan. Under the sinking fund loan, Keyi will pay the interest at the end of each year on the loan. Additionally, Keyi will make a deposit into a sinking fund at the end of each year. The deposit into the sinking fund will be such that the amount in the sinking fund will exactly repay the loan at the end of 10 years.

      The annual effective interest rate on the loan is 8% and the sinking fund pays an annual effective interest rate of 6%.

   b. Bank Soultz offers a 10 year amortization loan at an annual effective interest rate of 9%. The loan will be paid with level annual payments.

   Determine which loan Keyi should accept and demonstrate why.

   Solution:

   Option a:

$$I = (100,000)(0.08) = 8000$$

$$D = \frac{100,000}{s_{\overline{10}|}} = \frac{100,000}{\left(\frac{(1.06)^{10} - 1}{0.06}\right)} = 7586.80$$

$$\text{Total Payment} = I + D = 8000 + 7586.80 = 15,586.80$$

   Option b.

$$Q = \frac{100,000}{a_{\overline{10}|}} = \frac{100,000}{\left(\frac{1-(1.09)^{-10}}{0.09}\right)} = 15,582.01$$

   Keyi should select Option b because the total annual payment is smaller.

7. Aisling has a 30 year loan with non-level annual payments. The payment at the end of each odd numbered year (year 1, 3, 5, …, 29) will be 5000. The payment at the end of each even numbered year (year 2, 4, 6, …, 30) will be 10,000.

   The interest rate on the loan is an annual effective interest rate of 6%.

   Determine the principle in the payment of 5000 at the end of the 27th year.

   Solution:

   To find the principle in the 27th payment, we need to find the OLB after the 26th payment.

$$OLB_{26} = 5000(1.06)^{-1} + 10,000(1.06)^{-2} + 5000(1.06)^{-3} + 10,000(1.06)^{-4} = 25,735.98$$

$$I_{27} = (25,735.98)(0.06) = 1544.16$$

$$P_{27} = 5000 - 1544.16 = 3455.84$$

8. A 20 year continuous annuity pays at the rate of $300+10t$ at time t.

Calculate the present value of this annuity at a force of interest of 7%.

Solution:

$$PV = 300\bar{a}_{\overline{20|}} + 10(\bar{I}\bar{a})_{\overline{20|}} =$$

$$300\left(\frac{1-e^{-(20)(0.07)}}{0.07}\right)+10\left[\frac{\left(\frac{1-e^{-(20)(0.07)}}{0.07}\right)-20e^{-(20)(0.07)}}{0.07}\right]$$

$$= 3228.87 + 832.99 = 4061.86$$

9. A 20 year bond has a maturity value of 100,000 and semi-annual coupons that are not level. The two coupons paid during the first year are 500 each. The two coupons paid during the second year are 1000 each. The coupons continue increasing in the same pattern until the two coupons paid during the 20th year are 10,000 each.

   The bond is purchased to yield 8% convertible semi-annually.

   Determine the price of this bond.

   Solution:

   The price of the bond is the PV of future cash flows. The coupons follow a pattern that will require us to use the formula that does not follow the rules so we will need both $\frac{i^{(2)}}{2}$ and i. We are given $i^{(2)} = 0.08$ so $\frac{i^{(2)}}{2} = 0.04$ and $i = (1.04)^2 - 1 = 0.0816.$

$$\text{Price} = 500\left(\frac{\ddot{a}_{\overline{20}|0.0816} - 20(1.0816)^{-20}}{0.04}\right) + 100,000(1.04)^{-40}$$

$$= 500\left(\frac{\frac{1-(1.0816)^{-20}}{0.0816}(1.0816) - 20(1.0816)^{-20}}{0.04}\right) + 100,000(1.04)^{-40} = 99,932.28$$

10. Caroline has just been named the beneficiary of the Chen Trust Fund. The balance in the Chen Trust Fund is 1,000,000.

Caroline can choose one of the following options as her payout from the trust fund:

a. A continuous perpetuity that will pay at a rate of $10,000t$ at time t; or

b. An 30 year increasing annuity with quarterly payments at the end of each quarter. The first payment will be P. The second payment will be $2P$. The third payment will be $3P$. The payments will continue to increase in the same pattern until the last payment is made.

The two options have a present value of 1,000,000 based on a force of interest of δ.

Determine P.

Solution:

Option a:

$$1,000,000 = \frac{10,000}{\delta^2} \text{ ==> } \delta = 0.1$$

Option b:

Since payments are quarterly, we need $\frac{i^{(4)}}{4}$. $e^{0.1} = \left(1+\frac{i^{(4)}}{4}\right)^4$.

$$\frac{i^{(4)}}{4} = e^{0.25} - 1 = 0.025315121$$

$$1,000,000 = Pa_{\overline{120|}} + \frac{P}{0.025315121}\left(a_{\overline{120|}} - 120(1.025315121)^{-120}\right)$$

$$P = \frac{1,000,000}{a_{\overline{120|}} + \frac{1}{0.025315121}\left(a_{\overline{120|}} - 120(1.025315121)^{-120}\right)} = 778.66$$

11. A 25 year bond has annual coupons of 1000 and a maturity value of C .

The interest in the 10th coupon is 978.87. The interest in the 20th coupon is 957.45.

The book value at the end of the 22 year after the coupon is paid is 13,069.15.

Determine C .

Solution:

$$P_{10} = 1000 - 978.87 = 21.13$$

$$P_{20} = 1000 - 957.45 = 42.55$$

$$P_{10}(1+j)^{10} = P_{20} ==> 21.13(1+i)^{10} = 42.55 ==> i = \left(\frac{42.55}{21.13}\right) - 1 = 0.0725$$

Book Value is the present value of future cash flows.

$$B_{22} = 13,069.15 = 1000a_{\overline{3}|} + C(1.0725)^{-3} ==> C = \frac{13,069.15 - 1000\left(\frac{1-(1.0725)^{-3}}{0.0725}\right)}{(1.0725)^{-3}} = 12,900$$

12. A 20 year callable bond has a maturity value equal to the par value of 20,000 and semi-annual coupons paid at a coupon rate of 7.5% convertible semi-annually. The bond may be called at the end of 12 years for a call value of 21,500. The bond may be called at the end of 15 years for a call value of 20,800. Finally, the bond may be called at the end of 18 years for a call value of 20,300.

    Yang purchased the bond at issue to yield 6% convertible semi-annually.

    Determine the price that Yang paid.

    Solution:

    Calculate the value at each call date and the maturity date and select the lowest price.

N	I/Y	PMT	FV	CPT PV
24	3	(20,000)(0.0375)=750	21,500	23,278.23
30	3	750	20,800	23,269.66
36	3	750	20,300	23,378.35
40	3	750	20,000	23,467.22

The lowest present value is the price ➔ 23,269.66