MATH 373
Spring 2019
Test 2

March 26, 2019

1. Shobana takes out a 30 year mortgage loan to buy a new house. The loan is for 1,000,000 and will be repaid with level monthly payments. The interest rate is 9% compounded monthly.

   Determine the principal in the 240th payment.

   Solution:

   The best way to do this is with the calculator

   $\boxed{N} \leftarrow (30)(12) = 360$

   $\boxed{I/Y} \leftarrow \dfrac{0.09}{12}(100) = 0.75$

   $\boxed{PV} \leftarrow -1{,}000{,}000$

   $\boxed{CPT}\boxed{PMT} \Rightarrow 8{,}046.226169$

   $\boxed{2nd}\boxed{Amort}$

   $\boxed{P1} \leftarrow 240$

   $\boxed{P2} \leftarrow 240$

   $\boxed{\downarrow}\boxed{\downarrow}\,\mathrm{Pr}\,in ==> 3257.92$

2. Assad invests money invests 10,000 at the beginning of the each year for 8 years into Fund A. Fund A pays an annual effective interest rate of 6%.

   At the end of each year, Assad withdraws the interest earned in Fund A and deposits into Fund B which is earning an annual effective interest rate of 8.25%.

   Determine the total amount that Assad will have at the end of 8 years taking into account both Fund A and Fund B.

Solution:

Fund A pays all interest to Fund B. Therefore, at the end of 8 years, all that is left in Fund A are the deposits which are $(10,000)(8)=80,000$. The interest paid into Fund B is $(10{,}000)(0.06)=600$ at the end of the first year, $(2)(10{,}000)(0.06)=1200$ at the end of the second year, ... until $(8)(10{,}000)(0.06)$ is paid at the end of the 8th year. The amount in Fund B at the end of 8 years using the P&Q formula is:

$$\left[600a_{\overline{8}|0.0825}+\frac{600}{0.0825}\left(a_{\overline{8}|0.0825}-8(1.0825)^{-8}\right)\right](1.0825)^{8}$$

$$\left[600\left[\frac{1-(1.0825)^{-8}}{0.0825}\right]+\frac{600}{0.0825}\left(\left[\frac{1-(1.0825)^{-8}}{0.0825}\right]-8(1.0825)^{-8}\right)\right](1.0825)^{8}=26,317.42$$

$$Total=80,000+26,317.42=106,317.42$$

3. Dylan is receiving a continuous annuity that pays at a rate of $1000t+500$ at time t for 25 years.

   Calculate the present value at $\delta = 0.09$.

   Solution:

$$\int_0^{25} (1000t+500)v^t\,dt = \int_0^{25} (1000t)v^t\,dt + \int_0^{25} (500)v^t\,dt = 1000\left(\bar{I}\bar{a}\right)_{\overline{25|}} + 500\bar{a}_{\overline{25|}}$$

$$= 1000\left(\frac{\dfrac{1-e^{-(25)(0.09)}}{0.09} - 25e^{-(25)(0.09)}}{0.09}\right) + 500\left(\frac{1-e^{-(25)(0.09)}}{0.09}\right)$$

$$= 81{,}166.98 + 4970.00 = 86{,}136.98$$

Use this information to answer questions 4 and 5.

Jaiying borrows 300,000 from Nick Bank to be repaid over the next 10 years. The loan is a sinking fund loan.

Under the loan, Jaiying will pay the bank interest at the end of each year at an annual effective interest rate of i . This payment will be I .

Additionally, Jaiying will make a deposit into a sinking fund at the end of each year such that at the end of 10 years, the amount in the sinking fund will exactly repay the loan principle of 300,000. The deposit into the sinking fund will be D. The Sinking Fund will earn an annual effective interest rate of 6.25%.

Under this loan, $I = D$.

4. Determine i .

Solution:

$$I = iL = i(300,000)$$

$$D = \frac{L}{s_{\overline{10}|}} = \frac{300,000}{\left(\frac{(1.0625)^{10} - 1}{0.0625}\right)} = 22,494.54$$

$$I = D ==> i(300,000) = 22,494.54 ==> i = \frac{22,494.54}{300,000} = 0.07498$$

5. Calculate the amount in the sinking fund right after the 7th deposit.

Solution:

$$Answer = Ds_{\overline{7}|} = (22,494.54)\frac{(1.0625)^7 - 1}{0.0625} = 190,260.82$$

6. Eston will buy one of the following set of payments:

    i. A perpetuity with increasing payments. The payments are 100 at the end of the first year, 1100 at the end of the second year, 2100 at the end of the third year, etc. Payments continue forever with each payment being 1000 greater than the previous payment.

    ii. A 20 year annuity due with annual increasing payments. The first payment is Q. The second payment is $Q(1.08)$. The third payment is $Q(1.08)^2$. The payments continue for 20 years with each payment being 108% of the prior payment.

Both the perpetuity and annuity have a present value of 249,575.30 at an interest rate of i.

Determine Q.

Solution:

$$PV = \frac{100}{i} + \frac{1000}{i^2} = 249,575.30$$

$$100i + 1000 = 249,575.30i^2 \text{ ==> } 249,575.30i^2 - 100i - 1000 = 0$$

$$i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-100) \pm \sqrt{(-100)^2 - 4(249,575.30)(-1000)}}{(2)(249,575.30)} = 0.0635$$

$$PVii = Q + Q(1.08)(1.0635)^{-1} + \ldots + Q(1.08)^{19}(1.0635)^{-19}$$

$$= \frac{Q - Q(1.08)^{20}(1.0635)^{-20}}{1 - (1.0635)(1.08)^{-1}} = 10,738.46$$

7. Ian can purchase any of the following perpetuities. All three perpetuities have the same price at an annual interest rate of i.

   a. Perpetuity A pays continuously at a rate of 20,000 per year.

   b. Perpetuity B pays continuously at a rate of 1000t at time t.

   c. Perpetuity C has quarterly payments at the end of each quarter. Each payment in the first year is P. Each payment in the second year is $2P$. Each payment in the third year is $3P$. The payments continue to increase in the same pattern.

   Determine P.

   Solution:

$$A = B ==> \frac{20,000}{\delta} = \frac{1000}{\delta^2} ==> 20,000\delta = 1000 ==> \delta = 0.05$$

$$PVofA = \frac{20,000}{\delta} = \frac{20,000}{0.05} = 400,000$$

$$PVofC = P\left(\frac{(1+i)}{i \cdot \frac{i^{(4)}}{4}}\right)$$

$$e^{\delta} = 1 + i ==> i = 0.051271096$$

$$1 + i = \left(1 + \frac{i^{(4)}}{4}\right)^4 ==> \frac{i^{(4)}}{4} = (1.051271096)^{1/4} - 1 = 0.01257845$$

$$400,000 = P\left(\frac{1.051271096}{(0.051271096)(0.01257845)}\right) ==> P = 245.38$$

8. Owen invests quarterly deposits for 5 years with the Fairfield Fund. The deposits are made at the beginning of each quarter. The deposits during the first year are 10,000. The deposits during the second year are 20,000. The deposits during the third year are 30,000. The payments continue in the same pattern with payments increasing by 10,000 each year.

   Fairfield Fund pays an annual effective interest rate of 7.5%.

   Determine how much Owen will have in the Fairfield Fund at the end of 5 years.

   Solution:

   First we note that the payments are level during each year but increase year to year. Therefore, we know that it is the Formula That Does Not Follow The Rules. Secondly, we note that it is an annuity due. Finally, we note that we need the accumulated value.

   We are given $i = 0.075$ and we will need $\dfrac{i^{(4)}}{4}$.

$$(1+i) = \left(1+\frac{i^{(4)}}{4}\right)^4 \implies \frac{i^{(4)}}{4} = (1.075)^{1/4} - 1 = 0.018244601$$

$$AV = (10{,}000)\left(\frac{\left[\dfrac{1-(1.075)^{-5}}{0.075}\right](1.075) - 5(1.075)^{-5}}{0.018244601}\right)(1.018244601)(1.075)^5 = 694{,}296.90$$

9. James has a loan for 100,000 which will be repaid with level annual payments of Q for n years. The interest rate on the loan is 5.43%.

   The principle in the 20th payment is 60.90.

   Determine Q .

   Solution:

   $$\text{Interest in the First Payment} = (\text{L})(\text{i}) = (100{,}000)(0.0543) = 5430$$

   $$\text{Principle in the first payment} = (\text{principle in the 20th Payment})(1.0543)^{-(20-1)}$$

   $$= 60.90(1.0543)^{-19} = 22.30$$

   $$Q = 5430 + 22.30 = 5452.30$$

10. Flo is making payments continuously at a rate of $1000t+500$ at time t for the next 10 years.

Using a discount function of $1-0.006t^2$, calculate the present value of Flo's payments.

Solution:

$$PV=\int_0^{10}(1000t+500)(1-0.006t^2)dt$$

$$=\int_0^{10}(1000t+500-6t^3-3t^2)dt$$

$$\left[500t^2+500t-\frac{6}{4}t^4-t^3\right]_0^{10}=39,000$$

11. Ben has a loan for 50,000 which will be repaid with three non-level annual payments. The interest rate on the loan is an annual effective interest rate of 10%.

    The first loan repayment is 30,000. The second loan repayment 20,000. The third loan repayment is P. The loan is repaid after the third payment.

    Calculate the principle in the second payment.

Solutions:

k	Payment	Interest	Principle	OLB
0				50,000
1	30,000	(50,000)(0.10)=5000	30,000-5000=25,000	50,000-25,000=25,000
2	20,000	(25,000)(0.10)=2500	20,000-2500=17,500	

12. Leen is repaying a loan with quarterly payments for 5 years. The first payment is 1000. The second payment is 1600. The third payment is 2200. The payments continue to increase in the same pattern until the last payment is made.

The loan has an interest rate of 8% compounded quarterly.

Determine the amount of the loan. (The amount of the loan is the present value of the payments.)

Solution:

Every payment is increasing so we have the p&Q formula.

$$i^{(4)} = 0.08 ==> \frac{i^{(4)}}{4} = 0.02$$

$$PV = 1000a_{\overline{20|}} + \frac{600}{0.02}\left[a_{\overline{20|}} - 20v^{20}\right]$$

$$= (1000)\left(\frac{1-(1.02)^{-20}}{0.02}\right) + \frac{600}{0.02}\left[\left(\frac{1-(1.02)^{-20}}{0.02}\right) - 20(1.02)^{-20}\right]$$

$$= 103{,}111.63$$