STAT/MATH 416
Fall, 2021
Test 1

September 23, 2021

1. You are given that $P[A]=0.5$ and $P[A\bigcup B]=0.7$.

   a. If A and B are mutually disjoint, calculate $P[B]$.

   Solution:

   $$P[A\bigcup B]=P[A]+P[B]-\underbrace{P[A\bigcap B]}_{\text{Zero since disjoint}}$$

   $$0.7=0.5+P[B]-0 ==> P[B]=0.7-0.5=0.2$$

   b. If A and B are independent, calculate $P[B]$.

   Solution:

   $$P[A\bigcup B]=P[A]+P[B]-P[A\bigcap B]=P[A]+P[B]-\underbrace{P[A]P[B]}_{\text{Independence}}$$

   $$0.7=0.5+P[B]-(0.5)P[B] ==> P[B]=0.4$$

2. Let X be the random variable with the probability density function of $f_X(x) = \frac{3x^2}{\theta^3}$ for $0 < x < \theta,$ and otherwise.

You are also given that $P[X > 1] = \frac{7}{8}$.

a. Determine θ .

Solution:

$$\int_1^\theta f_X(x)dx = \int_1^\theta \frac{3x^2}{\theta^3}dx = \frac{7}{8} ==> \left[\frac{x^3}{\theta^3}\right]_1^\theta = \frac{\theta^3}{\theta^3} - \frac{1}{\theta^3} = \frac{7}{8} ==> 1 - \frac{7}{8} = \frac{1}{\theta^3}$$

$\theta = 2$

b. Calculate $P[1.2 < X < 1.5]$.

Solution:

$$\int_{1.2}^{1.5} f_X(x)dx = \int_{1.2}^{1.5} \frac{3x^2}{2^3}dx \left[\frac{x^3}{8}\right]_{1.2}^{1.5} = \frac{1.5^3 - 1.2^3}{8} = 0.205875$$

3. A survey of students is taken to determine the probability of subscribing to streaming services. Let Event H be that a student subscribes to Hulu. Let Event D be that a student subscribes to Disney+. Let Event N be that a student subscribes to Netflix. Further, $P[H]=P[D]$ and $P[H^C]=1.5P[H]$.

   The survey also finds that the probability of being subscribed to both Hulu and Disney+ is 0.25, being subscribed to both Disney+ and Netflix is 0.30, and being subscribed to both Hulu and Netflix is 0.28.

   The probability of being subscribed to all three is 0.2 and of being subscribed to none of these three is 0.1.

   Determine the probability of being subscribed to Netflix.

   Solution:

   Using the Inclusion-Exclusion Formula

$$P[H \cup D \cup N] = P[H] + P[D] + P[N] - P[H \cap D] - P[H \cap N] - P[D \cap N] + P[H \cap D \cap N]$$

$$0.9 = 0.4 + 0.4 + P[N] - 0.25 - 0.28 - 0.30 + 0.20 = P[N] + 0.17$$

$$P[N] = 0.9 - 0.17 = 0.73$$

$$\boxed{\begin{array}{l} P[H \cup D \cup N] = 1 - P[(H \cup D \cup N)^C] = 1 - \underbrace{0.1}_{Given} = 0.9 \\ \\ P[H] + P[H^C] = 1 \text{==>} P[H] + 1.5P[H] = 1 \text{==>} P[H] = 0.4 \\ \\ P[D] = P[H] = 0.4 \\ \\ P[H \cap D] = 0.25;\quad P[H \cap N] = 0.28;\quad P[D \cap N] = 0.3 \\ \\ P[H \cap D \cap N] = 0.2 \end{array}}$$

4. A committee of three freshman and three sophomores are being selected from a group of 10. The group of 10 includes six freshman and four sophomores. However, two of the freshman (Tim and Alice) are mad at each other and will not serve together.

Determine the number of possible six person committees given that Tim and Alice cannot both be on the committee.

Solution:

There are many ways to do this. One way is to consider all possible arrangements and then subtract the ones that have both Tim and Alice:

$$\underset{\textit{Freshman}}{\binom{6}{3}}\ \underset{\text{Sophomores}}{\binom{4}{3}} - \underset{\text{Tim and Alice}}{\binom{2}{2}}\ \underbrace{\binom{4}{1}}_{\text{One other Freshman}}\ \underbrace{\binom{4}{3}}_{\text{Sophomores}} = \left(\frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}\right)(4)-(1)(4)(4)=80-16=64$$

Another way is to add up the possible arrangements:

$$\underbrace{\binom{2}{1}}_{\text{One of Tim and Alice}}\ \underbrace{\binom{4}{2}}_{\text{Two other Freshman}}\ \underbrace{\binom{4}{3}}_{\text{Sophomores}} + \underbrace{\binom{2}{0}}_{\text{Niether Tim nor Alice}}\ \underbrace{\binom{4}{3}}_{\text{Three other Freshman}}\ \underbrace{\binom{4}{3}}_{\text{Sophomores}} = (2)\left(\frac{4\cdot 3}{2\cdot 1}\right)(4)+(1)(4)(4)=48+16=64$$

5. Upon arrival at a hospital's emergency room, patients are classified as either critical, serious, or stable. Over the last year, 10% of the patients have been classified at critical, 30% were serious, and the rest were stable.

   Additionally, 40% of the critical patients died. In other words, given that a patient is critical, the probability of death is 40%. Also, 10% of the serious patients died and 1% of the stable patients died.

   Given that a patient survived, what is the probability that the patient was classified as serious upon arrival?

   Solution:

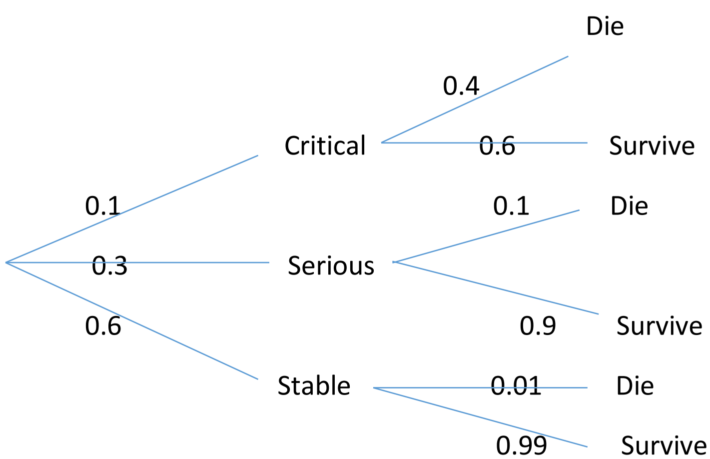
   We can do this using Bayes or we can use a tree.

   Bayes: Let Event A be alive, Event C be classified as Critical, Event S be classified at Serious, and Event B be classified as staBle.

$$P[S|A] = \frac{P[A|S]P[S]}{P[A|S]P[S] + P[A|C]P[C] + P[A|B]P[B]}$$

$$\frac{(0.9)(0.3)}{(0.9)(0.3) + (0.6)(0.1) + (0.99)(0.6)} = 0.29221$$

Tree:



$$P[S|A] = \frac{(0.9)(0.3)}{(0.9)(0.3) + (0.6)(0.1) + (0.99)(0.6)} = 0.29221$$

6. A standard deck of 52 playing cards has four suits – hearts, diamonds, clubs, or spades. Each suit has 13 cards. These 13 cards are identified by numbers or letters. Each suit has cards labeled as 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, and A.

   What is the probability of randomly drawing five cards without replacement from the deck and having three of the cards have the same number or letter while the other two cards are different from each other and different from the three of a kind? (This is known as a three of a kind in poker.)

   Solution:

$$Answer = \frac{\overset{\text{This is getting three cards out of 4 so getting the three of a kind.}}{\binom{4}{3}}\ \overset{\text{There are 13 numbers or letters from which to get the pair.}}{(13)}\ \overbrace{\binom{4}{1}\binom{4}{1}}^{\text{This is selecting one card of each number or letter for the other two cards in hand.}}\ \overbrace{\binom{12}{2}}^{\text{There are twelve numbers or letters for the remaining 2 cards and you need 2 different numbers or letters.}}}{\underbrace{\binom{52}{5}}_{\text{You are Picking five cards out of 52 and the order does not matter.}}} = \frac{54,912}{2,598,960} = 0.02113$$

Or

$$\frac{\overbrace{(52)(3)(2)(48)(44)}^{\text{There are 52 cards that you can pick with the first card, then 3 cards to pick to get the second card that matches, then 2 to get the third card, then 48 cards which do not match the three of a kind, then 44 that do not match the three of a kind or the other card.}}\ \overbrace{\left(\frac{5!}{3!2!}\right)}^{\text{This accounts for the possible orders. You are picking 5 cards, of which three must be the same and two must be different.}}}{(52)(51)(50)(49)(48)} = \frac{6,589,440}{311,875,200} = 0.02113$$

7. The John Purdue Club is a group of people who are Purdue Sports Fans. Of the members of the John Purdue Club, 60% have basketball tickets and 40% have football tickets. Of the members with basketball tickets, 1/3 have football tickets also.

   A member of the John Purdue Club is selected at random.

   Calculate the probability that the member has basketball tickets given that the member has football tickets.

   Solution:

   We can do this easily using Bayes.

   Bayes: Let Event B be having basketball tickets and Event F being having football tickets.

$$P[B \mid F] = \frac{P[B \cap F]}{P[F]} = \frac{0.2}{0.4} = 0.5$$

$$\boxed{\begin{array}{l} P[F] = 0.4 \quad \text{given} \\ \\ P[B \cap F] = P[F \mid B]P[B] = (1/3)(0.6) = 0.2 \end{array}}$$

8. A Covid test is 90 percent effective in detecting Covid when the person actually has Covid. However, the test also has a 5% false positive rate when a healthy person is tested. (In other words, if a healthy person is tested, the probability is 5% that the test will indicate that the healthy person has Covid.) If 8% of the population actually has Covid, what is the probability that a person actually has Covid given a positive test?

Solution:

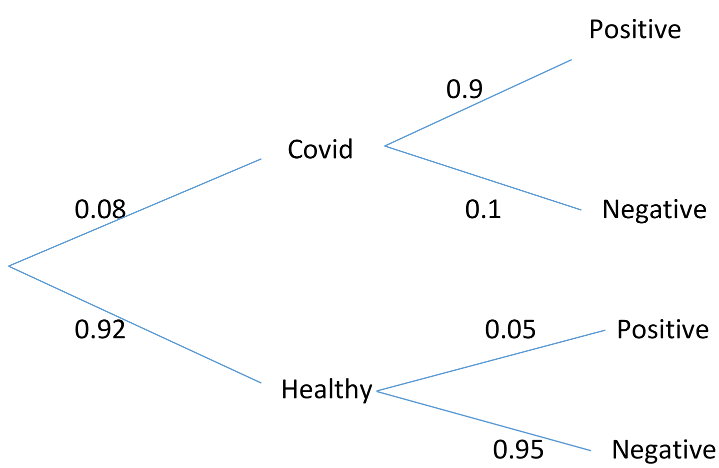
We can do this using Bayes or we can use a tree.

Bayes: Let Event C be has Covid, Event H be Healthy (not Covid), Event P be a Positive test and Event N be a Negative test.

$$P[C \mid P] = \frac{P[P \mid C]P[C]}{P[P \mid C]P[C] + P[P \mid C^C]P[C^C]}$$

$$\frac{(0.9)(0.08)}{(0.9)(0.08) + (0.5)(0.92)} = 0.61017$$

Tree:



$$P[C \mid P] = \frac{(0.9)(0.08)}{(0.9)(0.08) + (0.5)(0.92)} = 0.61017$$