STAT 416
Test 2
Fall 2021

November 4, 2021

1. A random variable X has the cumulative distribution function:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \left(\frac{1}{9}\right)\left(2x^2 - \frac{x^3}{3}\right), & 0 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

Calculate $Var[X]$.

Solution:

We need to test if this is a mixed distribution. To do this we test the endpoints of the continuous function.

$F_X(0) = 0$ so there is no discontinuity at point 0. $F_X(3) = 1$ so there is no discontinuity at point 10. Therefore, this is not a mixed distribution.

$$E[X] = \int_0^3 x \cdot f_X(x) \cdot dx = \int_0^3 x \cdot \left(\frac{4x - x^2}{9}\right) \cdot dx = \int_0^3 \left(\frac{4x^2 - x^3}{9}\right) \cdot dx = \left[\frac{4x^3}{27} - \frac{x^4}{36}\right]_0^3 = \frac{7}{4}$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{4x - x^2}{9}$$

$$E[X^2] = \int_0^3 x^2 \cdot f_X(x) \cdot dx = \int_0^3 x^2 \cdot \left(\frac{4x - x^2}{9}\right) \cdot dx = \int_0^3 \left(\frac{4x^3 - x^4}{9}\right) \cdot dx = \left[\frac{x^4}{9} - \frac{x^5}{45}\right]_0^3 = \frac{18}{5}$$

$$Var[X] = E[X^2] - \left(E[X]\right)^2 = \frac{18}{5} - \left(\frac{7}{4}\right)^2 = 0.5375$$

2. A bowl has four tiles numbered 1, 2, 3, and 4. You randomly draw two tiles from the bowl and note the sum. Note that there is no replacement.

Let the random variable S be the sum of the two tiles.

Calculate $Var[S]$.

Solution:

		Draw of First Tile			
		1	**2**	**3**	**4**
Draw of Second Tile	**1**	X	3	4	5
	2	3	X	5	6
	3	4	5	X	7
	4	5	6	7	X

The values in red are the totals of the two tiles.

$$p(3)=\frac{1}{6};\ p(4)=\frac{1}{6};\ p(5)=\frac{2}{6};\ p(6)=\frac{1}{6};\ p(7)=\frac{1}{6}$$

$$E[S]=(3)\left(\frac{1}{6}\right)+(4)\left(\frac{1}{6}\right)+(5)\left(\frac{2}{6}\right)+(6)\left(\frac{1}{6}\right)+(7)\left(\frac{1}{6}\right)=5$$

$$E[S^2]=(3)^2\left(\frac{1}{6}\right)+(4)^2\left(\frac{1}{6}\right)+(5)^2\left(\frac{2}{6}\right)+(6)^2\left(\frac{1}{6}\right)+(7)^2\left(\frac{1}{6}\right)=\frac{80}{3}$$

$$Var[S]=E[S^2]-\left(E[S]\right)^2=\frac{80}{3}-(5)^2=\frac{5}{3}$$

3. A random variable X has the cumulative distribution function:

$$F_X(x) = \begin{cases} 0, & x < 6 \\ \dfrac{x^2 - x}{90}, & 6 \le x < 9 \\ 1, & x \ge 9 \end{cases}$$

Calculate $Var[X]$.

Solution:

We need to test if this is a mixed distribution. To do this we test the endpoints of the continuous function.

$F_X(6) = 1/3$ while $F_X(5.9999999)$ is 0 so there is a discontinuity at 6 so $p(6) = 1/3 - 0 = 1/3$. $F_X(8.99999999999) = 0.8$ but $F_X(9) = 1$ so there is a discontinuity at 9. The cdf jumps from 0.8 to 1 at 9. This means that $p(9) = 1 - 0.8 = 0.2$.

$$E[X] = (6)p(6) + \int_6^9 x \cdot f_X(x) \cdot dx + (9)p(9) = (6)\left(\frac{1}{3}\right) + \int_6^9 x\left(\frac{2x-1}{90}\right)dx + (9)(0.2)$$

$$= 2 + \int_6^9 \left(\frac{2x^2 - x}{90}\right)dx + 1.8 = 3.8 + \left[\frac{x^3}{135} - \frac{x^2}{180}\right]_6^9 = 3.8 + 3.55 = 7.35$$

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{2x-1}{90}$$

$$E[X^2] = (6)^2 p(6) + \int_6^9 x^2 \cdot f_X(x) \cdot dx + (9)^2 p(9) = (36)\left(\frac{1}{3}\right) + \int_6^9 x^2\left(\frac{2x-1}{90}\right)dx + (81)(0.2)$$

$$= 12 + \int_6^9 \left(\frac{2x^3 - x^2}{90}\right)dx + 16.2 = 28.2 + \left[\frac{x^4}{180} - \frac{x^3}{270}\right]_6^9 = 28.2 + 27.35 = 55.55$$

$$Var[X] = E[X^2] - \left(E[X]\right)^2 = 55.55 - (7.35)^2 = 1.5275$$

4. The moment generating function for the random variable X is $M_X(t) = \frac{1}{1+t}$.

Calculate the $Var[X]$.

Solution:

$$E[X] = M'_X(t)\,|_{t=0} = -(1+t)^{-2}\Big|_{t=0} = -1$$

$$E[X^2] = M''_X(t)\,|_{t=0} = -(-2)(1+t)^{-3}\Big|_{t=0} = 2$$

$$Var[X] = 2 - (-1)^2 = 1$$

5. Apple iPhones are inspected by three qualify control inspectors at the factory before being put into a store. The iPhone is released to a store only if two inspectors or all three inspectors approve the phone.

   For each inspector, the probability of approving the phone when in it defective is 0.2.

   Calculate the probability that a flawed iPhone makes it to a store.

   Solution:

   $X \sim Binomial(3, 0.2)$

   $$Answer = p(2) + p(3) = \binom{3}{2}(0.2)^2(0.8) + \binom{3}{3}(0.2)^3(0.8)^0$$

   $$= 0.096 + 0.008 = 0.104$$

6. The number of vehicles in one hour using the Toll Bridge in Michigan between the United States and Canada is a Poisson process with a parameter $\lambda = 50$.

The Toll Bridge charges 3 for an automobile without a trailer, 5 for an automobile with a trailer, and 15 for a semi-truck.

The traffic on the bridge consists of 40% automobiles without a trailer, 10% of automobiles with a trailer, and 50% semi-trucks.

a. Calculate the probability that less than two trucks cross the bridge in a one minute period.

Solution:

50 vehicles cross the bridge in an hour with 50% being trucks. This means that there are 25 trucks per hour. Here we are talking about trucks per minute so that mean the expected number is 25/60.

$$\text{Answer} = p(0) + p(1) = \frac{e^{-25/60}(25/60)^0}{0!} + \frac{e^{-25/60}(25/60)^1}{1!} = 0.659241 + 0.274684 = 0.933924$$

Let R be the expected revenue generated by the Toll Bridge in a 24 hour period.

b. Calculate $E[R]$.

Solution:

Expected Trucks are (50 vehicles per hour)(0.50 for trucks)(24 hours) = 600
Expected Autos without trailers are (50 vehicles per hour)(0.40 for trucks)(24 hours) = 480
Expected Autos with trailers are (50 vehicles per hour)(0.10 for trucks)(24 hours) = 120

$$E[R] = (600)(15) + (480)(3) + (120)(5) = 11{,}040$$

7. The random variable X has a continuous uniform distribution from a to b. You are given $E[X]=6617$ and $Var[X]=\frac{28,976,689}{3}$.

Find the probability that $X > 5000$.

Solution:

$$E[X]=\frac{b+a}{2}=6617 ==> b+a=13,234$$

$$Var[X]=\frac{(b-a)^2}{12}=\frac{28,976,689}{3} ==> (b-a)^2=(28,976,689)(4)$$

$$==> b-a=\sqrt{(28,976,689)(4)}=10,766 ==> b=10,766+a$$

$$b+a=10,766+2a=13,234 ==> a=1234$$

$$b=10,766+a=10,766+1234=12,000$$

$$\Pr(X>5000)=\frac{b-x}{b-a}=\frac{12,000-5000}{12,000-1234}=0.650195$$

8. The random variable Y has a discrete uniform distribution over possible values of 1, 2, 3, and 4.

Calculate $E\left[\frac{1}{Y}\right]$.

Solution:

$Y = \{1, 2, 3, 4\}$

$p(1) = p(2) = p(3) = p(4) = 0.25$

$$E\left[\frac{1}{Y}\right] = \left(\frac{1}{1}\right)p(1) + \left(\frac{1}{2}\right)p(2) + \left(\frac{1}{3}\right)p(3) + \left(\frac{1}{4}\right)p(4)$$

$$= \left(\frac{1}{1}\right)(0.25) + \left(\frac{1}{2}\right)(0.25) + \left(\frac{1}{3}\right)(0.25) + \left(\frac{1}{4}\right)(0.25) = 0.520833$$

9. There are 20 balls in a bowl. 5 are red and 15 are purple. You draw balls without replacement.

What is the probability that you get the 3rd purple ball on the 5th draw?

Solution:

Since this is without replacement it is not Negative Binomial. We must use first principles.

Out of the first 4 balls, 2 must be red and two must be purple and the last ball must be purple.

$$\overbrace{\frac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}}}^{\text{Probability of 2 red and 2 purple out of the first 4 draws.}} \overbrace{\left(\frac{13}{16}\right)}^{\text{Probability that 5th ball is purple.}} = \frac{(105)(10)}{4845}\left(\frac{13}{16}\right) = 0.176084$$

10. A bowl has 15 tiles labeled 1, 2, 3, …, 14, 15. A tile is selected at random from the bowl. A success is considered a value in excess of 11. If the tile is less than 11, it is replaced in the bowl and a tile is drawn again at random. If the tile is greater than 11, you stop the process.

Let N be the number of the trial on which you draw the first tile greater than 11.

a. Calculate the probability that $N = 4$.

Solution:

This is a geometric distribution since it is first success with replacement.

$$p = \frac{4}{15}$$

$$p(4) = (1-p)^{4-1}p = \left(1 - \frac{4}{15}\right)^3 \left(\frac{4}{15}\right) = 0.105165$$

b. Calculate the $E[N]$.

Solution:

$$E[N] = \frac{1}{p} = \frac{15}{4} = 3.75$$

c. Calculate the $Var[N]$.

Solution:

$$Var[N] = \frac{1-p}{p^2} = \frac{11/15}{(4/15)^2} = 10.3125$$

Bonus: What animal is in the picture with my grandson?

It was a moose.