1. You complete an experiment by rolling a six sided fair die.

   The following events are defined:

   i. A = {1, 2, 3}

   ii. B = {4, 5, 6}

   iii. C = {1}

   iv. D = {1, 3, 5}

   v. E = {2, 4, 6}

   a. Determine $A \bigcup D$

      Solution:

      $A \bigcup D = \{1, 2, 3\} \bigcup \{1, 3, 5\} = \{1, 2, 3, 5\}$

   b. Determine $A \bigcap D$

      Solution:

      $A \bigcap D = \{1, 2, 3\} \bigcap \{1, 3, 5\} = \{1, 3\}$

   c. Determine $A \bigcap B$

      Solution:

      $A \bigcap B = \{1, 2, 3\} \bigcap \{4, 5, 6\} = \varnothing$

   d. Determine A^C

      Solution:

      $A^C = \{4, 5, 6\}$

   e. Determine $B^C \bigcap C$

      Solution:

      $B^C \bigcap C = \{1, 2, 3\} \bigcap \{1\} = \{1\}$

f. Determine $(B \cup C)^C$

Solution:

$$(B \cup C)^C = (\{4,5,6) \cup \{1\})^C = (\{1,4,5,6\})^C = \{2,3\}$$

Or Using Demorgan's Law

$$(B \cup C)^C = B^C \cap C^C = \{1,2,3\} \cap \{2,3,4,5,6\} = \{2,3\}$$

g. State whether the following statements are true of false. If false, state why it is false.

1. Events A and B are disjoint events.

   True

2. Events A, B, and C are mutually disjoint events.

   False. They are disjoint events but not mutually disjoint.

3. $D \cap E$ is an empty set, but not a null set

   False. It is both an empty set and a null set. These terms are synonymous.

4. $C \subset A$

   True.

2. You are given:

   i. $P[A] = 0.5$

   ii. $P[B] = 0.4$

   iii. $P[A \cap B] = 0.3$

a. Calculate $P[A \cup B]$

Solution:

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 0.5 + 0.4 - 0.3 = 0.6$$

b. Calculate $P[A^C \cup B^C]$

Solution:

$$P[A^C \cup B^C] = P[(A \cap B)^C] = 1 - P[A \cap B] = 1 - 0.3 = 0.7$$

c. Calculate $P[A^C \cap B^C]$

Solution:

$$P[A^C \cap B^C] = P[(A \cup B)^C] = 1 - P[A \cup B] = 1 - 0.6 = 0.4$$

d. Calculate $P[A^C \cap B]$

Solution:

$$P[B] = P[B \cap A] + P[B \cap A^C] = P[A \cap B] + P[A^C \cap B]$$

$$P[A^C \cap B] = P[B] - P[A \cap B] = 0.4 - 0.3 = 0.1$$

e. Calculate $P[A^C \cup B]$

Solution:

$$P[A^C \cup B] = P[A^C] + P[B] - P[A^C \cap B]$$

$$= (1 - P[A]) + P[B] - P[A^C \cap B]$$

$$= (1 - 0.5) + 0.4 - 0.1 = 0.8$$

General Comments and Notes

An easy way to solve most of this question is with our table. This will give you the direct answers to c. and d. and help with others.

	$P[A]$ $= 0.5$ given	$P[A^C]$ $= 1 - P[A] = 1 - 0.5 = 0.5$
$P[B]$ $= 0.4$ given	$P[A \cap B]$ $= 0.3$ given	$P[A^C \cap B]$ $= P[B] - P[A \cap B] = 0.4 - 0.3 = 0.1$
$P[B^C]$ $= 1 - P[B] = 1 - 0.4 = 0.6$	$P[A \cap B^C]$ $= P[A] - P[A \cap B] = 0.5 - 0.3 = 0.2$	$P[A^C \cap B^C]$ $= P[A^C] - P[A^C \cap B] = 0.5 - 0.1 = 0.4$

3. Let E and F be events such that $P[E]=1/2,\ P[F]=1/2,$ and $P[E^C \cap F^C]=1/3.$

Calculate $P[E \cup F^C]$.

Solution:

$$P[E \cup F^C] = P[E] + P[F^C] - P[E \cap F^C]$$

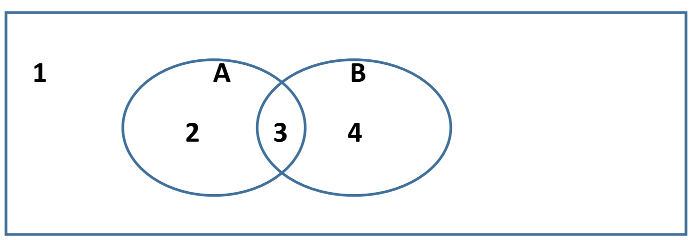
$$P[E] + (1 - P[F]) - P[E \cap F^C] = 1/2 + (1-1/2) + 1/6 = 5/6$$

To get $P[E \cap F^C]$, use the table below:

	$P[E]$ $= 0.5$ given	$P[E^C]$ $= 1 - P[E] = 1 - 0.5 = 0.5$
$P[F]$ $= 0.5$ given	$P[E \cap F]$	$P[E^C \cap F]$
$P[F^C]$ $= 1 - P[F] = 1 - 0.5 = 0.5$	$P[E \cap F^C]$ $= P[F^C] - P[E^C \cap F^C] = 1/2 - 1/3 = 1/6$	$P[E^C \cap F^C] = 1/3$ given

4. You are given $P[A \cup B] = 0.7$ and $P[A \cup B^C] = 0.9$. Calculate $P[A]$.

Solution:



Area 2 + Area 3 is Event A. Area 3 and 4 is Event B.

Area 1 + Area 2 + Area 3 +Area 4 = 1 since this is the entire sample space.

Area 2 + Area 3 +Area 4 is $P[A \cup B] = 0.7$ which is given.

Area 1 + Area 2 + Area 3 is $P[A \cup B^C] = 0.9$ which is given

This means that Area 4 must be 1 – 0.9 = 0.1

$$P[A] = \text{Area 2 + Area 3} = (\text{Area 2 + Area 3 +Area 4}) - \text{Area 4} = 0.7 - 0.1 = 0.6$$

5. A marketing survey indicates that 60% of the population own an automobile, 30% own a house, and 20% own both. Calculate the probability that a person chosen at random owns an automobile or a house but not both.

Solution:

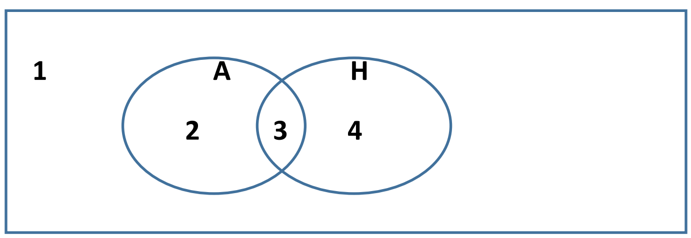
Let A represent the Automobile Owners and H be the Home Owners.

$P[A]-P[A\cap H]$ are the Auto Owners with no house.

$P[H]-P[A\cap H]$ are the House Owners with no auto.

$Answer: P[A]-P[A\cap H]+P[H]-P[A\cap H]=0.6+0.3-2(0.2)=0.5$

Or Using a Venn Diagram:



Area 2 + Area 3 is Event A. Area 3 and 4 is Event H.

Area 2 + Area 3 is $P[A]=0.6$ which is a given.

Area 3 + Area 4 is $P[B]=0.3$ which is a given.

Area 3 is $P[A\cap B]=0.2$ which is a given.

Therefore Area 2 = 0.6 – 0.2 = 0.4 and Area 4 = 0.3 – 0.2 = 0.1

People who only an auto or only a home are Area 2 and Area 4 = 0.4 + 0.1 = 0.5

6. A visit to a doctor's office results in a lab work 40% of the time. A visit to a doctor's office results in a referral to a specialist 30% of the time. A visit to a doctor's office results in no lab work and no specialist referral 35% of the time.

   What is the probability that a visit to the doctor's office will result in both lab work and a referral to a specialist?

   Solution:

   Let L be the set of people who need lab work and let S be the set of people who need a Specialist.

   We are given:

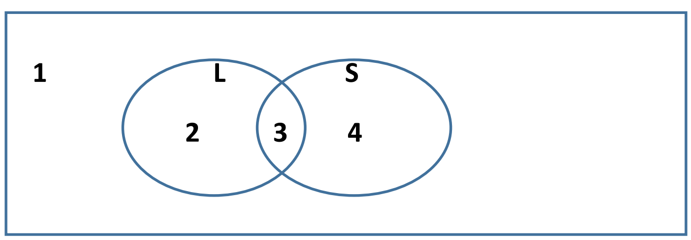
   $P[L] = 0.4 \quad \text{and} \quad P[S] = 0.3 \quad \text{and} \quad P[(L \cup H)^C] = 0.35.$

   We want $P[(L \cap S)]$

   $P[(L \cup S)] = P[L] + P[S] - P[L \cap S] ==> P[(L \cap S)] = P[L] + P[S] - P[L \cup S]$

   $= P[(L \cap S)] = P[L] + P[S] - \left(1 - P[(L \cup H)^C]\right) = 0.4 + 0.3 - \left(1 - 0.35\right) = 0.05$

   Or



   Area 2 + Area 3 is Event L. Area 3 and 4 is Event S.

   Area 2 + Area 3 is $P[L] = 0.4$ which is a given.

   Area 3 + Area 4 is $P[S] = 0.3$ which is a given.

   Area 1 is $P[(L \cup H)^C] = 0.35$ which is given.

   Area 2 + Area 3 +Area 4 = 1 – Area 1 = 0.65

   Answer = Area 3 = Area 2 + Area 3 + Area 3 + Area 4 – (Area 2 + Area 3 + Area 4) = 0.4 + 0.3 – 0.65 = 0.05.

7. An employer offers employees the following coverages:

    a. Major Medical Coverage

    b. Vision Coverage

    c. Dental Coverage.

Employees who enroll are required to enroll in at least two coverages. You may elect to have zero coverages. You are given:

    a. The probability of enrolling in Major Medical is 80%.

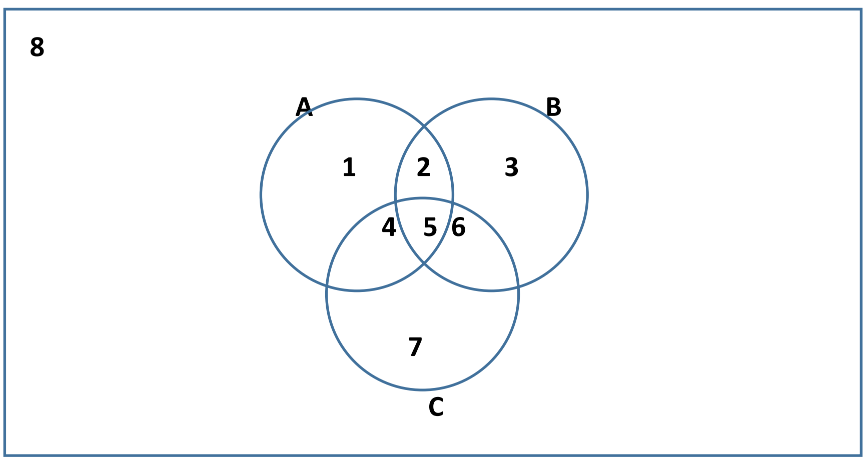
    b. The probability of enrolling in Vision is 40%

    c. The probability of enrolling in Dental is 70%

    d. The probability of enrolling in all three is 20%

Calculate the probability of enrolling in zero coverages.

Solution:



Let A be Major Medical, B be Vision, and C be Dental.

Area 1 = Area 3 = Area 7 = 0 since a person must buy two coverages if they buy any.

Area 5 = All three coverages = 0.2 which is given.

P[A] = 0.8 = Area 1 + Area 2 + Area 4 + Area 5 = 0 + Area 2 + Area 4 + 0.2

⇨ Area 2 + Area 4 = 0.6

P[B] = 0..4 = Area 2 + Area 3 + Area 5 + Area 6 = Area 2 +0 + 0.2 + Area 6

⇨ Area 2 + Area 6 = 0.2

P[C] = 0.7 = Area 4 + Area 5 + Area 6 + Area 7 = Area 4 + 0.2 + Area 6 + 0

⇨ Area 4 + Area 6 = 0.5

We now have three equations and three unknowns and we can solve for

Area 2 = 0.15   Area 4 = 0.45   Area 6 = 0.05

We need $P[(A \cup B \cup C)^C] = 1 - P[(A \cup B \cup C)]$

= 1 – Sum of Area 1 through Area 7 = 1 – 0.85 = 0.15

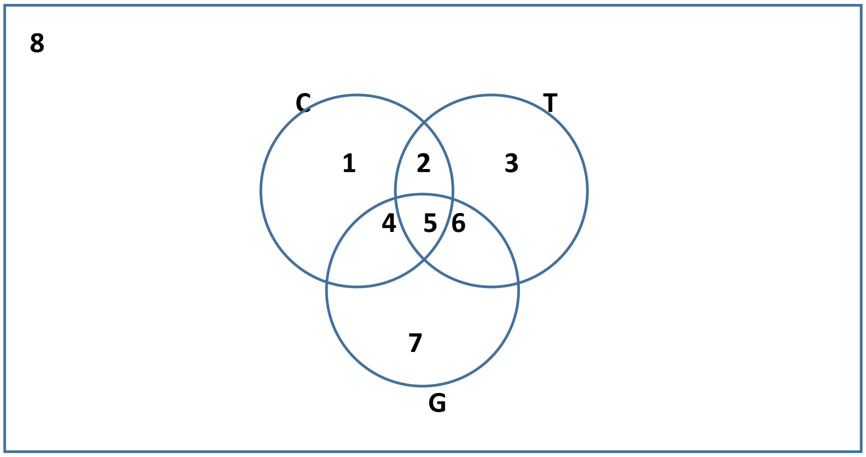
8. In a survey of 120 high school students, the following data was obtained:

   a. 60 students participated in Cross Country
   b. 56 students participated in Track
   c. 42 students participated in Golf
   d. 34 students participated in both Cross Country and Track
   e. 20 students participated in both Track and Golf
   f. 16 students participated in Cross Country and Golf
   g. 6 students participated in all three sports

   Calculate:

   i. The number of students who did not participate in any of these sports.
   ii. The number of students who did Track only.
   iii. The number of students who did Cross Country and Track, but not Golf.

Solution:



Let C be Cross Country, T be Track, and G be Golf.

Area 5 = All three sports = 6

Area 2 and Area 5 is $C \cap T$ = 34 ➔ Area 2 = 34 – 6 = 28

Area 5 and Area 6 is $T \cap G$ = 20 ➔ Area 6 = 20 – 6 = 14

Area 4 and Area 5 is $C \cap G$ = 16 ➔ Area 4 = 16 – 6 = 10

Event C = 60 = Area 1 + Area 2 + Area 4 + Area 5 ➔ Area 1 = 60 – 28 – 10 – 6 = 16

Event T = 56 = Area 2 + Area 3 + Area 5 + Area 6 ➔ Area 3 = 56 – 28 – 6 – 14 = 8

Event G = 42 = Area 4 + Area 5 + Area 6 + Area 7 ➔ Area 7 = 42 – 10 – 6 – 14 = 12

Total of Areas 1 through 7 = 16 + 28 + 8 + 10 + 6 + 14 + 12 = 94

Area 8 = 120 – 94 = 26   Which is the answer to Part i.

Part ii = Track only = Area 3 = 8

Part iii = $T \cap C \cap G^C$ = Area 2 = 28

9. You roll two fair dice. Calculate the probability of the following events:

a. Total of the two dice is 8.

Solution:

$$P[Event]=\frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}}=\frac{\{2,6\},\{3,5\},\{4,4\},\{5,3\},\{6,2\}}{6\times 6}=\frac{5}{36}$$

b. You roll a 4 and a 2.

Solution:

$$P[Event]=\frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}}=\frac{\{2,4\},\{4,2\}}{6\times 6}=\frac{2}{36}=\frac{1}{18}$$

c. The total of the two dice is less than 6.

Solution:

$$P[Event]=\frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}}$$

$$=\frac{\{1,1\},\{1,2\},\{1,3\},\{1,4\},\{2,1\},\{2,2\},\{2,3\},\{3,1\},\{3,2\},\{4,1\}}{6\times 6}=\frac{10}{36}=\frac{5}{18}$$

The other way to do the numerator is:

Die 1 = 1, four values for Die 2
Die 1 = 2, three values for Die 2
Die 1 = 3, two values for Die 2
Die 1 = 4, one value for Die 2

$$4+3+2+1=10$$

d. The number on each die is the same.

Solution:

$$P[Event]=\frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}}=\frac{6}{6\times 6}=\frac{1}{6}$$

10. A license plate contains a letter followed by five numeric digits that can be 0 to 9.

    a. What is the size of the sample space if digits can be repeated.

       Solution:

       There are 26 possible letters for the first item and 10 possible numbers for the next five items on the license plate.

       $$(26)(10)^5 = 2,600,000$$

    b. What is the size of the sample space if digits cannot be repeated.

       Solution:

       There are 26 possible letters for the first item and 10 possible numbers for the next item. There are only 9 possible numbers for the third item, etc.

       $$(26)(10)(9)(8)(7)(6) = 786,240$$

11. A bowl has five balls. They are red, green, blue, purple, and yellow. You draw all five balls out of the bowl.

    How many possible orders are there?

    Solution:

    $$5! = 120$$

12. A bowl has five balls. They are red, green, blue, purple, and yellow. You draw all three balls out of the bowl without replacement.

    How many possible orders are there?

    Solution:

    $$(5)(4)(3) = 60$$

13. A bowl has five tiles which each tile being numbered. The tiles are numbered 1 to 5. You draw three tile out of the bowl without replacement.

    What is the probability of getting a number greater than 350?

    Solution:

    If you select a 3 first, then a five second, then you can select the 1, 2, 4. This is 3 possibilities. If you select the 4 first, then you can select the second digit from the four remaining number and the third digit from the 3 remaining numbers. This is (4)(3) = 12 possibilities. If you select the 5 first, there are also 12 possibilities so the total possibilities is 27. The total in the sample space is (5)(4)(3) = 60.

    Therefore, the probability is 27/60.

14. A bowl has 10 balls. Four balls are purple, three balls are red, two balls are blue and one ball is green. You draw three balls without replacement.

   a. Determine the number of unique arrangements.

      Solution:

      Assuming that we had at least three balls of each color, the answer would be easy. It would be (4)(4)(4) = 64. But we only have one green ball and 2 blue balls. Therefore there are certain combinations that we cannot get – for example – GGG or BBB. We must determine these combinations and subtract from 64. If G is first, then there is only 3 colors left for the second ball and 3 colors left for the third ball. Thus we have (3)(3) (instead of (4)(4)) so we need to subtract 7 which is 16 – 9. If we have P, R, or B taken first and G taken second, then there are only three colors that can be taken third. Therefore there are only (3)(G)(3) versus (3)(G)(4) so we need to subtract 3 which is 12 – 9. Finally, if BB are the first two, then there are only 3 possibilities for the third ball because all the blue balls have been selected so it is BB(3) instead of BB(4) so we need to subtract 1 which is 4 – 3.

      That means there are 64 – 7 – 3 – 1 = 53 unique arrangements.

      The alternative way to solve this is to add up all the possibilities:

      (R)(R or P or B)(4) = (1)(3)(4) = 12

      (R)(G)(3) = (1)(1)(3) = 3

      (P)(R or P or B)(4) = (1)(3)(4) = 12

      (P)(G)(3) = (1)(1)(3) = 3

      (B)(R or P)(4) = (1)(2)(4) = 8

      (B)(G)(3) = (1)(1)(3) = 3

      (B)(B)(3) = (1)(1)(3) = 3

      G(3)(3) = (1)(3)(3) = 9

      12 + 3 +12 + 3 + 8 + 3 +3 + 9 = 53

   b. Determine the probability that you will draw 2 purple and one red ball.

      Solution:

$$P[Event] = \frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}} = \frac{\binom{4}{2}\binom{3}{1}}{\binom{10}{3}} = \frac{18}{120} = 0.15$$

15. A bowl contains 4 blue, 5 white, 6 red and 7 green balls. You chose a sample of 5 balls without replacement. How many different ways can you select a sample that has every color represented?

Solution:

We need to pick 1 apiece from 3 colors and 2 from the fourth color. The number of ways to do this is:

$$\binom{4}{2}\binom{5}{1}\binom{6}{1}\binom{7}{1}+\binom{4}{1}\binom{5}{2}\binom{6}{1}\binom{7}{1}+\binom{4}{1}\binom{5}{1}\binom{6}{2}\binom{7}{1}+\binom{4}{1}\binom{5}{1}\binom{6}{1}\binom{7}{2}$$

$$=(6)(5)(6)(7)+(4)(10)(6)(7)+(4)(5)(15)(7)+(4)(5)(6)(21)=7560$$

16. A softball team has 10 players which consists of five males and five females. All players bat. In determining a batting order, a female must bat first. Furthermore, successive batters must be of the opposite gender.

Determine the number of possible batting orders.

Solution:

We must select a female as the starting point. Then each time we must select someone from the other gender. Therefore it is:

$$(5)(5)(4)(4)(3)(3)(2)(2)(1)(1)=14,400$$

17. A committee of five people is randomly chosen from a list of that contains 7 men and 3 females.

Calculate the probability that the committee will have 3 men and 2 women?

Solution:

$$P[Event]=\frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}}=\frac{\binom{7}{3}\binom{3}{2}}{\binom{10}{5}}=\frac{5}{12}$$

18. What is the probability that a hand of five cards chosen at random and without replacement from a standard 52 card deck will have a king of spades, exactly one other king, and exactly two queens?

Solution:

$$P[Event] = \frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}} = \frac{\binom{3}{1}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}} = \frac{(3)(6)(44)}{2,598,960} = 0.000305$$

The king of spades is given so it does not factor into the numerator. The $\binom{3}{1}$ reflects chosing one additional king out of the 3 remaining kings. The $\binom{4}{2}$ reflects selecting of two queens out of the four queens. Finally, $\binom{44}{1}$ reflects chosing the fifth card which is not a king or queen. This card is out of 44 as there are 44 cards that are not kings or queens.

19. A bowl has 6 balls. Three balls are red, two balls are blue and one ball is green. You randomly draw three balls without replacement.

Determine the probability that at least one color is **not** drawn.

Solution:

$$P(\text{At least one color is not drawn}) = 1 - P(\text{Every color is drawn})$$

$$= 1 - \frac{\binom{3}{1}\binom{2}{1}\binom{1}{1}}{\binom{6}{3}} = 1 - \frac{(3)(2)(1)}{\left[\frac{(6)(5)(4)}{(3)(2)(1)}\right]} = 1 - \frac{6}{20} = \frac{7}{10}$$

20. A bowl has 10 balls. Five balls are purple, three balls are red and two balls are blue. You randomly draw three balls **with** replacement.

    What is the probability that the three balls are include one of each color?

    Solution:

    Probability of getting a purple ball is 5/10. Probability of getting a red ball is 3/10. The probability of getting a blue ball is 2/10. Then, there are 3! orders of selection. Therefore:

    $$\left(\frac{5}{10}\right)\left(\frac{3}{10}\right)\left(\frac{2}{10}\right)(3!) = 0.18$$

21. If four fair die are rolled, what is the probability of obtaining two identical even numbers and two identical odd numbers.

    Solutions:

    $$P[Event] = \frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}} = \frac{\binom{4}{2}(3)(3)}{6^4} = \frac{54}{1296} = \frac{1}{24}$$

    There are 3 odd numbers and 3 even numbers. The $\binom{4}{2}$ reflects that there are 4 dice and 2 must be even and 2 must be odd.

22. An employer has 30 employees. There are 5 employees that are older than 60 and 25 employees that are 60 or younger. Two employees are selected at random.

    What is the probability that exactly one of the two selected employees is over 60?

    Solutions:

    $$P[Event] = \frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}} = \frac{\binom{25}{1}\binom{5}{1}}{\binom{30}{2}} = \frac{(25)(5)}{\left[\frac{(30)(29)}{2}\right]} = \frac{25}{87}$$

23. In a neighborhood with 10 houses, k houses are not insured.

A tornado randomly damages 3 of the houses. The probability that none of the damaged houses are insured is 1/120.

Calculate the probability that at most one of the damaged houses is insured.

Solution:

$$P(\text{None is insured}) = \frac{\binom{k}{3}}{\binom{10}{3}} = \frac{\binom{k}{3}}{120} = \frac{1}{120} \text{ ==> } k = 3$$

$$P(\text{One is insured}) = \frac{\binom{3}{1}\binom{7}{1}}{\binom{10}{3}} = \frac{21}{120}$$

$$P(\text{At least one is insured}) = P(\text{None is insured}) + P(\text{One is insured})$$

$$= \frac{21+1}{120} = \frac{22}{120}$$

24. A professor approaches 12 undergraduate students independently, each of whom is equally likely to want to participate in research projects. Six are interested only in applied statistics research, four are interested only in actuarial science research, and two are interested only in data science research.

After the talking with the students, six students state that they want to participate in the research.

Calculate the probability that two participate in applied statistics research, two participate in actuarial research, and 2 participate in data science research.

Solution:

$$P[Event] = \frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}} = \frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{\binom{12}{6}} = \frac{(15)(6)(1)}{924} = \frac{15}{154}$$

25. Roll a six sided fair die. Let A be the event that the outcome on the dice is an even number. Let B be the event that the outcome on the dice is 4 or smaller. Let C be the event that the outcome on the dice is 3 or larger.

a. Are A and B independent?

Solution:

If A and B independent, then

$$P[A \cap B] = P[A]P[B]$$

$$A = \{2,4,6\} ==> P[A] = \frac{3}{6}$$

$$B = \{1,2,3,4\} ==> [B] = \frac{4}{6}$$

$$A \cap B = \{2,4\} ==> P[A \cap B] = \frac{2}{6} = \frac{1}{3}$$

$$P[A]P[B] = \left(\frac{3}{6}\right)\left(\frac{4}{6}\right) = \frac{12}{36} = \frac{1}{3}$$

$$\therefore Independent$$

b. Are B and C independent?

Solution:

If B and C independent, then

$$P[B \cap C] = P[B]P[C]$$

$$B = \{1,2,3,4\} ==> [B] = \frac{4}{6}$$

$$C = \{3,4,5,6\} ==> P[C] = \frac{4}{6}$$

$$B \cap C = \{3,4\} ==> P[B \cap C] = \frac{2}{6}$$

$$P[B]P[C] = \left(\frac{4}{6}\right)\left(\frac{4}{6}\right) = \frac{16}{36} = \frac{4}{9}$$

$$\therefore NOT\ \ Independent$$

26. You roll two six sided fair die – one is red and one is green. Event A is that the red die is a 6. Event B is that the sum of the two die exceeds 8.

a. Demonstrate that these events are not independent.

Solution:

If A and B independent, then

$$P[A \cap B] = P[A]P[B]$$

$$P[A] = \frac{6}{36}$$

$$[B] = \frac{10}{36}$$

$$P[A \cap B] = \frac{4}{36}$$

$$P[A]P[B] = \left(\frac{6}{36}\right)\left(\frac{10}{36}\right) = \frac{5}{108}$$

$$\therefore NOT \;\; Independent$$

We can create a box to help us get these probabilities:

Red	1						
	2						
	3						B
	4					B	B
	5				B	B	B
	6	A	A	A&B	A&B	A&B	A&B
		1	2	3	4	5	6

Green

b. Calculate $P(B \mid A)$.

Solution:

$$P[B \mid A] = \frac{P[A \cap B]}{P[A]} = \frac{4/36}{1/6} = \frac{2}{3}$$

27. A dresser drawer contains five gloves – One pair of blue gloves, one pair of red gloves, and a single white glove. Suppose that Olivia is looking for two matching blue gloves.

Olivia randomly simultaneously pulls two gloves out of the drawer without looking in the drawer. Let Event B be the event that at least one of the gloves is blue. Let Event A be the event that both gloves are blue.

Calculate the conditional probability of Event A given that Event B occurs.

Solution:

$$P[A\,|\,B] = \frac{P[A \cap B]}{P[B]} = \frac{0.1}{0.7} = \frac{1}{7}$$

$$P[B] = 1 - P[B^C] = 1 - P[\text{Neither Gloves Is Blue}] = 1 - \left(\frac{3}{5}\right)\left(\frac{2}{4}\right) = 1 - 0.3 = 0.7$$

$$P[A \cap B] = P[\text{Both Gloves Are Blue}] = \left(\frac{2}{5}\right)\left(\frac{1}{4}\right) = 0.1$$

28. You roll two pair of fair dice. Let B be the event that the two dice have different values. Given the B occurs, calculate the conditional probability of A, the event that the sum of the two dice is an even number.

Solution:

$$P[A\,|\,B] = \frac{P[A \cap B]}{P[B]} = \frac{12/36}{30/36} = 0.4$$

$$P[B] = 1 - P[B^C] = 1 - \frac{6}{36} = \frac{30}{36}$$

$P[A \cap B] = \frac{12}{36}$ >>> 18 out of 36 rolls result in an even number but 6 have the same number on both die.

29. You roll two pair of fair dice. Let B be the event that the sum of the two dice is 9 or larger. Given the B occurs, calculate the conditional probability of A, the event that the sum of the two dice is exactly 10.

Solution:

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{3/36}{10/36} = 0.3$$

$$P[B] = \frac{10}{36}$$

$$P[A \cap B] = \frac{3}{36}$$

30. At Purdue, 40% of the students live on campus and 60% live off campus. Students who live on campus arrive to class on time 85% of the time. Students who live off campus arrive to class on time 70% of the time.

A randomly selected student arrived on time. What is the probability that the student lives on campus.

Solution:

Let A be the event that the student lives on campus and B be the event that the student arrived on time. We use Bayes Theorem and we need:

$$P[A|B] = \frac{P[B|A]P[A]}{P[B|A]P[A] + P[B|A^C]P[A^C]} = \frac{(0.85)(0.4)}{(0.85)(0.4) + (0.7)(0.6)} = \frac{17}{38}$$

31. There are two coins in a hat. One coin has two heads. The other coin has one head and one tail. Shannon reaches into the hat and pulls out a coin. The side facing up is heads. What is the probability that the side facing down is a head also?

Solution:

Let B be the event that the side facing up is a head. Let A be the event that the side facing down is a head. $P[B] = 3/4$ since that are four equally likely sides that couple be facing up and three are heads. $P[A \cap B] = 1/2$. Both are true only if we chose the two headed coin. Then:

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{1/2}{3/4} = 2/3$$

32. Let A, B, C be three events. You are given:

$$P[A] = 0.3 \qquad\qquad P[B] = 0.2$$

$$P[C \mid A] = 0.6 \qquad\qquad P[C \mid B] = 0.8$$

You are also given that $C \cap A$ and $C \cap B$ are mutually exclusive.

Calculate $P[C \cap (A \cup B)]$.

Solution:

$P[C \cap (A \cup B)] = P[(C \cap A) \cup (C \cap B)] = P[(C \cap A)] + P(C \cap B)]$
since these are mutually exclusive.

$$P[C \mid A] = \frac{P[C \cap A]}{P[A]} ==> P[C \cap A] = P[C \mid A]P[A] = (0.6)(0.3) = 0.18$$

$$P[C \mid B] = \frac{P[C \cap B]}{P[B]} ==> P[C \cap B] = P[C \mid B]P[B] = (0.8)(0.2) = 0.16$$

$$P[(C \cap A)] + P(C \cap B)] \; = 0.18 + 0.16 = 0.34$$

33. Let A, B, C be three events. The events are pairwise independent but not mutually independent. The probability of each event is 1/3 and the probability that all three occur is 1/10. Calculate the probability that none of them occur.

Solution:

$$P[A] = P[B] = P[C] = 1/3 \text{ and } P[A \cap B \cap C] = 1/10$$

$$\text{We need } P[(A \cup B \cup C)^C] = 1 - P[(A \cup B \cup C)]$$

We are given

$$= 1 - \{P[A] + P[B] + P[C] - P[A \cap B] - P[B \cap C] - P[C \cap A] + P[A \cap B \cap C]\}$$

$$= 1 - \{1/3 + 1/3 + 1/3 - \underbrace{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}_{\substack{\text{Since the events are pairwise independent} \\ P[A \cap B] = P[A]P[B] \quad \text{etc.}}} + 1/10\} = \frac{7}{30}$$

34. A die is rolled three times in succession. What is the probability that the outcome on each roll is less than that on the previous roll and the product of the three outcomes is an even number?

Solution:

This is just a counting problem

$$P[Event] = \frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}} = \frac{19}{6^3} = \frac{19}{216}$$

The numerator is:

$\{3,2,1\},\{4,2,1\},\{4,3,1\},\{4,3,2\},\{5,4,3\},\{5,4,2\},\{5,4,1\},\{5,3,2\},\{5,2,1\},\{6,5,4\},$
$\{6,5,3\},\{6,5,2\},\{6,5,1\},\{6,4,3\},\{6,4,2\},\{6,4,1\},\{6,3,2\},\{6,3,1\},\{6,2,1\}$

It is simplified if we recognize that because the product is even, we must have one even number among the three rolls.

35. A box contains 4 red balls and 6 white balls. A sample of size 3 is drawn without replacement from the box. What is the probability of obtaining one red ball and two white balls, given that at least 2 white balls are drawn in the sample.

Solution:

Let A be the event that we get 1 red and 2 white balls. Let B be the event that at least 2 balls are white. We need $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{\left[\binom{6}{2}\binom{4}{1}\right] \div \binom{10}{3}}{\left[\binom{6}{2}\binom{4}{1} + \binom{6}{3}\right] \div \binom{10}{3}} = \frac{3}{4}$

We can also do this by counting: $\frac{\{WWR\}\{WRW\}\{RWW\}}{\{WWR\}\{WRW\}\{RWW\}\{WWW\}} = \frac{3}{4}$

36. Let A, B, C be three events such that A and B are independent, B and C are mutually exclusive, and $P[A]=\frac{1}{4}$, $P[B]=\frac{1}{6}$, and $P[C]=\frac{1}{2}$.

Calculate $P[(A\cap B)^C \cup C]$.

Solution:

$$\underbrace{P[(A\cap B)^C \cup C] = P[(A\cap B)^C]}_{\substack{\text{Since B and C are mutually exclusive, then} \\ C\subset B^C \subset (A\cap B)^C}} = \underbrace{1-P[(A\cap B)] = 1-P[A]P[B]}_{\text{Due to Independence}} = \frac{23}{24}$$

37. A researcher examines medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease. Of those 312 men, 102 died from causes related to heart disease.

Calculate the probability that a man randomly selected from this group dies of caused related to heart disease, given that neither of his parents suffered from heart disease.

Solution:

Let A be the event that the man selected died of causes related to heart disease. Let B be the event that neither of his parents suffered from heart disease. We need $P[A|B]$.

We are given:

$$P[A]=\frac{210}{937}; P[B^C]=\frac{312}{937}; and \ \ P[A|B^C]=\frac{102}{312}$$

$$P[A|B]=\frac{P[A\cap B]}{P[B]}=\frac{P[A]-P[A\cap B^C]}{1-P[B^C]}=\frac{\frac{210}{937}-\frac{102}{937}}{1-\frac{312}{937}}=\frac{108}{625}$$

$$P[A|B^C]=\frac{P[A\cap B^C]}{P[B^C]} ==> P[A\cap B^C]=P[B^C]P[A|B^C]=\left(\frac{312}{937}\right)\left(\frac{102}{312}\right)=\frac{102}{937}$$

38. You are given:

   a. An automobile owner is twice as likely to purchase collision coverage as disability coverage.

   b. The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.

   c. The probability that an automobile owner purchases both collision and disability coverage is 15%

   Calculate the probability that an automobile owner purchases neither collision nor disability coverage.

   Solution:

   Let A be the event that an auto owner purchases collision coverage and B be the event that an auto owner purchases disability coverage. We are given:

   $$P[A]=2P[B]; P[A\cap B]=\underbrace{P[A]P[B]}_{Independence}=0.15$$

   We need $P[(A\cup B)^C]=1-P[(A\cup B)]=1-\{P[A]+P[B]-P[A\cap B]\}$

   $$P[A]P[B]=0.15=2P[B]P[B] ==> P[B]=\sqrt{\frac{0.15}{2}}=\sqrt{0.075} ==> P[A]=2\sqrt{0.075}$$

   $$=1-\left\{2\sqrt{0.075}+\sqrt{0.075}-0.15\right\}=0.32842$$

39. You are studying the prevalence of three health risk factors denoted by A, B and C within a population of women. For each of the three factors, the probability is 0.1 that a women in the population has this risk but not the other risk factors. For any two of the three factors, the probability is 0.12 that she has exactly these two factors but not the third factor. The probability that a woman has all three risk factors given that she has A and B is 1/3.

Calculate the probability that a woman has none of the three factors given that she does not have factor A .

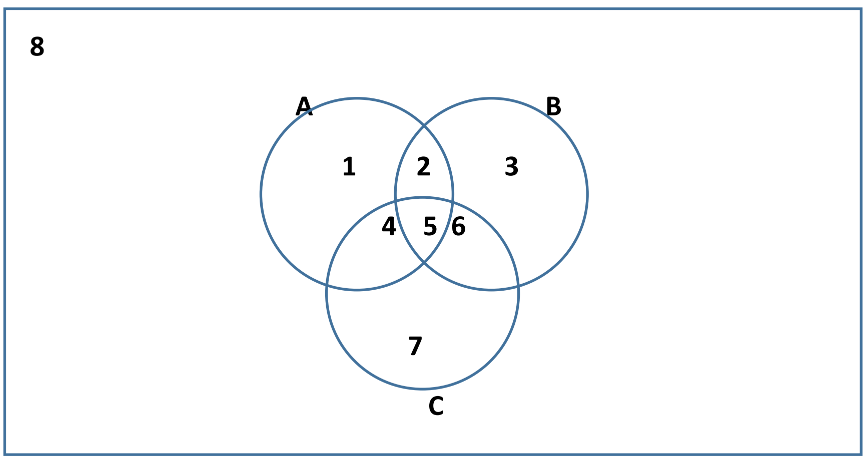
Solution:

We are given:

$$P[A \cap B^C \cap C^C] = P[A^C \cap B \cap C^C] = P[A^C \cap B^C \cap C] = 0.1$$

$$P[A \cap B \cap C^C] = P[A \cap B^C \cap C] = P[A^C \cap B \cap C] = 0.12$$

$$P[(A \cap B \cap C) | (A \cap B)] = 1/3$$



Area 1 = Area 3 = Area 7 = 0.1

Area 2 = Area 4 = Area 6 = 0.12

Area 5/(Area 2 +Area 5) = 1/3 ➔ Area 5 = 0.06

$$P[A^C \cap B^C \cap C^C | A^C] = \frac{P[A^C \cap B^C \cap C^C]}{P[A^C]} = \frac{1 - P[A \cup B \cup C]}{1 - P[A]}$$

$$= \frac{1 - Areas\ 1\text{-}7}{1 - Areas\ 1\text{-}3} = \frac{1 - (0.1 + 0.12 + 0.1 + 0.12 + 0.1 + 0.12 + 0.06}{1 - 0.40} = \frac{7}{15}$$

40. Aiden rolls two six sided die. Let D be the random variable representing the absolute value of the difference between the value of each die. Let M be the random variable representing the minimum value on the two die.

a. State the possible values of D.

Solution:

{0,1,2,3,4,5}

b. State the possible value of M.

Solution:

{1,2,3,4,5,6}

c. Calculate $P(D=2)$.

Solution:

$$P[Event] = \frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}} = \frac{2\times[\{1,3\},\{2,4\},\{3,5\},\{4,6\}]}{6^2} = \frac{2}{9}$$

d. Calculate $P(M=2)$.

Solution:

$$P[Event] = \frac{\text{Number of Outcomes in Event}}{\text{Number of Outcomes in Sample Set}} = \frac{\{2,2\}+2\times[\{2,3\},\{2,4\},\{2,5\},\{2,6\}]}{6^2} = \frac{1}{4}$$

e. Calculate $F_D(d)$.

Solution:

Using the technique in part c, we can find:

$$P[D=0]=p(0)=\frac{6}{36};\ \ P[D=1]=p(1)=\frac{10}{36};\ \ P[D=2]=p(2)=\frac{8}{36}$$

$$P[D=3]=p(3)=\frac{6}{36};\ \ P[D=4]=p(4)=\frac{4}{36};\ \ P[D=5]=p(5)=\frac{2}{36}$$

$$F_D(d)=P[D\le d]=\begin{cases} 0, & \text{for } x<0 \\ 6/36, & \text{for } 0\le x<1 \\ 6/36+10/36=16/36, & \text{for } 1\le x<2 \\ 16/36+8/36=24/36, & \text{for } 2\le x<3 \\ 24/36+6/36=30/36, & \text{for } 3\le x<4 \\ 30/36+4/36=34/36, & \text{for } 4\le x<5 \\ 34/36+2/36=36/36, & \text{for } x\ge 5 \end{cases}$$

41. In a company, 10% of the employees in their first five years earn 100,000 or more while 20% of employees with more than 5 years with the company earn 100,000 or more. Overall, 13% of employees earn 100,000 or more.

The company has 10,000 employees. Determine the number of employees who are in their first five years with the company.

Solution:

Total Employees earning 100,000 = $(10,000)(0.13)=1300$

Then, by the law of total probabilility,

$(x)(0.1)+(10,000-x)(0.2)=1300$ where x is the number of employees in first 5 years.

$x=7000$

42. In a doctor's practice, 20% of patients have high blood pressure and 30% of patients have high cholesterol. Of the patients with high blood pressure, 25% have high cholesterol.

A patient is selected at random.

Calculate the probability that the patient has high blood pressure, given that the patient has high cholesterol.

Solution:

Let A be the event of high blood pressure and B be the event of high cholesterol.

$$P[B \mid A] = \frac{P[A \cap B]}{P[A]} \text{ ==> } 0.25 = \frac{P[A \cap B]}{0.2} \text{ ==> } P[A \cap B] = 0.05$$

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{0.05}{0.30} = \frac{1}{6}$$

43. A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease.

Calculate the probability that a person has the disease given the test indicates the presence of the disease.

Solution:

Let Y be the event of a positive test and D be the event of having the disease.

$$P[D|Y] = \frac{P[Y|D]P[D]}{P[Y|D]P[D] + P[Y|D^C]P[D^C]}$$

$$= \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.005)(0.99)} = 0.657439$$

44. Mihika takes a test consisting of multiple choice questions with five answer choices for each question. Mihika may or may not know the answer to the question. If she knows the answer, she will answer the question correctly. Otherwise, she will randomly guess the answer.

For a given question, the conditional probability that that Mihika knows the answer to the question, given that Mihika answered it correctly, is 0.824. Find the probability that Mihika knows the answer to the question.

Solution:

Let A be the event of answering correctly and B be the event of knowing the question. We know:

$$0.824 = P[B|A] = \frac{P[A|B]P[B]}{P[A|B]P[B] + P[A|B^C]P[B^C]} = \frac{(1)P[B]}{(1)P[B] + (0.2)(1 - P[B])}$$

$$\Longrightarrow \frac{5P[B]}{1 + 4P[B]} = 0.824 \Longrightarrow P[B] = \frac{0.824}{1.704}$$

45. A continuous random variable X has a density function:

$$f_X(x) = \begin{cases} \dfrac{1}{(1+x)^2}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Calculate $P(2 < X < 5)$.

Solution:

$$P(2 < X < 5) = \int_2^5 f_X(x)dx = \int_2^5 (1+x)^{-2}dx = -(1+x)^{-1}\Big]_2^5 = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

46. For the random variable X, you are given the probability density function:

$$f_X(x) = \begin{cases} cx^2, & 1 \leq x \leq 4 \\ 0, & \text{elsewhere} \end{cases}$$

Calculate $P(2 \leq X \leq 3)$.

Solution:

First we must determine c.

$$\int_1^4 f_X(x)dx = 1 ==> \int_1^4 cx^2dx = \frac{cx^3}{3}\Bigg]_1^4 = c\left[\frac{64-1}{3}\right] = 21c ==> c = \frac{1}{21}$$

$$P(2 \leq X \leq 3) = \int_2^3 f_X(x)dx = \frac{x^3}{3(21)}\Bigg]_2^3 = \frac{27-8}{63} = \frac{19}{63}$$

47. The distribution of the size of claims paid under an insurance policy has PDF of:

$$f_X(x) = \begin{cases} cx^{\alpha}, & 0 < x < 5 \\ 0, & \text{elsewhere} \end{cases} \quad \text{for } \alpha > 0 \text{ and } c > 0$$

For a randomly selected claim, the probability that the size of the claim is less than 3.75 is 0.4871.

Calculate the probability that the size of a randomly selected claim is greater than 4.

Solution:

$$\text{We know } \; f_X(x) = \begin{cases} cx^{\alpha}, & 0 < x < 5 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and } F_X(x) = 1 \;\text{ and }\; F(3.75) = 0.4871$$

$$F_X(x) = \int_0^x cs^{\alpha} ds = \frac{cx^{\alpha+1}}{\alpha+1} ==> \frac{c(5)^{\alpha+1}}{\alpha+1} = 1 ==> c = \frac{\alpha+1}{(5)^{\alpha+1}}$$

$$==> \frac{c(3.75)^{\alpha+1}}{\alpha+1} = 0.4871 ==> \left(\frac{\alpha+1}{(5)^{\alpha+1}}\right)\frac{(3.75)^{\alpha+1}}{\alpha+1} = 0.4871 ==> \left(\frac{3.75}{5}\right)^{\alpha+1} = 0.4871$$

$$==> (\alpha+1)\ln(0.75) = \ln(0.4871) ==> \alpha = 1.50028$$

$$c = \frac{1.5+1}{(5)^{2.5}}$$

$$==> Ans = 1 - F(4) = 1 - \frac{c(4)^{\alpha+1}}{\alpha+1} = 1 - \left(\frac{1.5+1}{(5)^{2.5}}\right)\left(\frac{4^{2.5}}{2.5}\right) = 1 - \left(0.8\right)^{2.5} = 0.4276$$

48. The lifetime of a car has a continuous distribution on the interval (0,40) with a PDF that is proportional to $(10+x)^{-2}$ on the interval.

Calculate the probability that the lifetime of the car is less than 6.

Solution:

We can simply divide the integral of the expression $(10+x)^{-2}$ to 6 by the integral to 40, since the integral to 40 is the reciprical of the proportionality constant.

$$\int_0^6 (10+u)^{-2}\,du = -(10+u)^{-1}\Big|_0^6 = \frac{1}{10}-\frac{1}{16} = 0.0375$$

$$\int_0^{40} (10+u)^{-2}\,du = -(10+u)^{-1}\Big|_0^{40} = \frac{1}{10}-\frac{1}{50} = 0.08$$

$$P[X<6] = \frac{0.0375}{0.08} = 0.46875$$

49. The lifetime of a car has a continuous distribution on the interval (0,40) with a PDF that is equal to $12.5(10+x)^{-2}$ on the interval.

Determine $S_X(x)$.

Solution:

$$S_X(x) = 1-F_X(x) = 1-[1.25-12.5(10+x)^{-1}] = 12.5(10+x)^{-1}-0.25$$

$$F_X(x) = \int_0^x 12.5(10+x)^{-2}\,dx = \left[-12.5(10+x)^{-1}\right]_0^x = 1.25-12.5(10+x)^{-1}$$

50. A discrete random variable N has the following probability function:

$$f_N(n) = \begin{cases} p, & n=1 \\ 2p, & n=3 \\ 1-3p, & n=5 \end{cases}$$

You are also given that $E[N] = 3.8$.

Calculate p.

Solution:

$$E[N] = 3.8 = \sum x \cdot p(x) = (1)p + (3)(2p) + 5(1-3p)$$

$$==> p + 6p + 5 - 15p = 3.8 ==> 8p = 1.2 ==> p = 0.15$$

51. John rolls a die and will be paid 10 minus the number on the die as a reward.

Calculate the expected value of John's reward.

Solution:

Let the random variable R be the amount of the award.

$$E[R] = \sum_{r=1}^{6} (10-r) \cdot p(r)$$

$$= (9)\left(\frac{1}{6}\right) + (8)\left(\frac{1}{6}\right) + (7)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{6}\right) + (5)\left(\frac{1}{6}\right) + (4)\left(\frac{1}{6}\right) = \frac{39}{6} = 6.5$$

52. Calculate $8 + 16 + 32 + 64 + ... + 8192$.

Solution:

This is a geometric summary. The first term in 8. The ratio is 2.

$$Answer = \frac{FirstTerm - NextTermAfterLast}{1 - Ratio} = \frac{8 - (8192)(2)}{1-2} = 16{,}376$$

53. Calculate $8+14+20+26+...+596$.

Solution:

This is an arithmetic summary. The first term in 8. The ratio is 2.

$$Answer = \left[\frac{FirstTerm + LastTerm}{2}\right](NumberOfTerms) = \left(\frac{8+596}{2}\right)\left(\frac{596-8}{6}+1\right) = 29,898$$

54. Ethan rolls a six sided fair die until he gets a 6. Let N be the random variable that is the number of the roll on which Ethan gets the first 6.

Calculate $E[N]$.

Solution:

$$E[N] = (1)(1/6) + (2)(5/6)(1/6) + (3)(5/6)^2(1/6) + ...$$

$$\underbrace{\frac{1/6}{(1-5/6)^2}}_{\text{Based on Formula Derived in Class}} = 6$$

55. The length of a staff meeting, in hours, is a random variable having the following probability density function:

$$f_X(x) = \begin{cases} 5(x-0.5)^4 & 0.5 \le x \le 1.5 \\ 0, & \text{otherwise} \end{cases}$$

Calculate the expected length of the staff meeting.

Solution:

$$E[X] = \int_{0.5}^{1.5} x \cdot f_X(x) \cdot dx = \int_{0.5}^{1.5} x[5(x-0.5)^4]dx$$

$$= 5\int_{0.5}^{1.5} x[(x^4 - 2x^3 + 1.5x^2 - 0.5x + 0.0625)]dx$$

$$= 5\left[\frac{x^6}{6} - \frac{2x^5}{5} + \frac{1.5x^4}{4} - \frac{0.5x^3}{3} + \frac{0.0625x^2}{2}\right]_{0.5}^{1.5}$$

$$= 5\left[0.2671875 - 0.000520833\right] = \frac{4}{3}$$

or using u substitution

$$u = x - 0.5 \text{ ==> } du = dx \text{ ==> } x = u + 0.5$$

Integral becomes $\int_{0.0}^{1.0} (u+0.5)[5u^4]du$

$$= \int_{0.0}^{1.0} (5u^5 + 2.5u^4)du = \left[\frac{5u^6}{6} + \frac{2.5u^5}{5}\right]_0^1 = \frac{5}{6} + \frac{1}{2} - 0 - 0 = \frac{4}{3}$$

56. Let X be the continuous random variable with density function

$$f_X(x) = \begin{cases} \theta x + 1.5\theta^{1.5}x^2 & 0 \le x \le 1/\sqrt{\theta} \\ \\ 0, & \text{otherwise} \end{cases}$$

Where $\theta > 0$.

What is the expected value of X in terms of θ .

Solution:

$$E[X] = \int_0^{1/\sqrt{\theta}} x \cdot f_X(x) \cdot dx = \int_0^{1/\sqrt{\theta}} x[\theta x + 1.5\theta^{1.5}x^2]dx$$

$$\int_0^{1/\sqrt{\theta}} [\theta x^2 + 1.5\theta^{1.5}x^3]dx = \left[\frac{\theta x^3}{3} + \frac{1.5\theta^{1.5}x^4}{4}\right]_0^{1/\sqrt{\theta}} = \frac{1}{3\sqrt{\theta}} + \frac{1.5}{4\sqrt{\theta}} - 0 - 0 = \frac{17}{24\sqrt{\theta}}$$

57. Let X be a continuous random variable with density function:

$$f_X(x) = \begin{cases} \frac{|x|}{10} & -2 \le x \le 4 \\ \\ 0, & \text{otherwise} \end{cases}$$

What is the expected value of X .

Solution:

$$E[X] = \int_{-2}^{4} x \cdot f_X(x) \cdot dx = \int_{-2}^{4} x\left(\frac{|x|}{10}\right)dx = \int_{-2}^{0} \frac{-x^2}{10}dx + \int_0^4 \frac{x^2}{10}dx$$

$$= \left[\frac{-x^3}{30}\right]_{-2}^{0} + \left[\frac{x^3}{30}\right]_0^4 = 0 - \frac{8}{30} + \frac{64}{30} - 0 = \frac{28}{15}$$

58. Let X be a continuous random variable with a density function:

$$f_X(x) = \begin{cases} \dfrac{p-1}{x^p}, & x > 1 \\ 0, & \text{otherwise} \end{cases}$$

Calculate the value of p such that $E[X] = 2.$

Solution:

$$E[X] = \int_1^\infty x \cdot f_X(x) \cdot dx = \int_1^\infty x \left[\frac{p-1}{x^p} \right] dx = \int_1^\infty \frac{p-1}{x^{p-1}} dx$$

$$= \left[\frac{(p-1)x^{2-p}}{2-p} \right]_1^\infty \text{ <== At this point, we need to make an assumption that } p > 2.$$

If $p \le 2,$ the integral goes to infinity so we will make this assumption and solve for p.

$$0 - \frac{p-1}{2-p} = \frac{p-1}{p-2} \text{ ==> } \frac{p-1}{p-2} = 2 \text{ ==> } p = 3 \text{ which is consistent with our assumption.}$$

59. A box contains 10 balls, of which 3 are red, two are yellow and 5 are blue. Five balls are randomly selected with replacement.

Calculate the probability that fewer than 2 of the selected balls are red.

Solution:

Let $A =$ Fewer than 2 of the selected balls are red.

$$P[A] = (0.7)^5 + \binom{5}{1}(0.7)^4(0.3) = 0.52822$$

60. A plane has 30 seats. The probability that an particular passenger does not show up for a flight is 10% and independent of other passengers. The airline sells 32 tickets for a flight.

Calculate the probability that more passengers show up for the flight that there are seats.

Solution:

$$P[32 \text{ or } 31 \text{ passengers}] = (0.9)^{32} + \binom{32}{31}(0.9)^{31}(0.1) = 0.15642$$

61. Defective items on an assembly line occur independently with a probability of 0.05. A random sample of 100 items are selected. Calculate the probability that the first item sampled is not defective given that at least 99 of the items are not defective.

Solution:

Let $A =$ First is not defective and $B =$ at least 99 are not defective.

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{(0.95)\left[\binom{99}{98}(0.95)^{98}(0.05) + (0.95)^{99}\right]}{\left[\binom{100}{99}(0.95)^{99}(0.05) + (0.95)^{100}\right]}$$

$$= \frac{(99)(0.05) + 0.95}{(100)(0.05) + 0.95} = \frac{5.9}{5.95} = 0.991597$$

62. Let X be a continuous random variable with density function

$$f_X(x) = \begin{cases} \frac{1}{30}\, x(1+3x), & 1 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Calculate $E[1/X]$.

Solution:

$$E[1/X] = \int_1^3 \left(\frac{1}{x}\right) \cdot f_X(x) \cdot dx = \int_1^3 \left(\frac{1}{x}\right)\left[\frac{1}{30}\, x(1+3x)\right] dx = \int_1^3 \left[\frac{1}{30}(1+3x)\right] dx$$

$$\left[\frac{1}{30}\left(x + \frac{3x^2}{2}\right)\right]_1^3 = \frac{16.5}{30} - \frac{1}{12} = \frac{7}{15}$$

63. You are given two random variables X and Y.

X is has a probability mass function of:

$$\begin{array}{ll} 0.2 & \text{if } x=3 \\ 0.3 & \text{if } x=8 \\ 0.5 & \text{if } x=10 \\ 0 & \text{elsewhere} \end{array}$$

Y has a probability density function of $f_Y(y)=\dfrac{y^3}{64}$ for $0\le y\le 4$.

Calculate $E[4X+5Y]$.

Solution:

$$E[X]=(3)(0.2)+(8)(0.3)+(10)(0.5)=8$$

$$E[Y]=\int_0^4 y\cdot f_Y(y)\cdot dy=\int_0^4 y\frac{y^3}{64}dy=\left[\frac{y^5}{(64)(5)}\right]_0^4=\frac{4^5}{(64)(5)}=3.2$$

$$E[4X+5Y]=4E[X]+5E[Y]=(4)(8)+(5)(3.2)=48$$

64. You roll a fair 6 sided die three times. Let X be the number of 6s that are rolled.

a. Calculate the $E[X]$ using the definition of Expectation.

Solution:

$$E[X]=\sum_{x=0}^{3}x\cdot p_x=0\binom{3}{0}\left(\frac{5}{6}\right)^3+1\binom{3}{1}\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^1+2\binom{3}{2}\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^2+3\binom{3}{3}\left(\frac{1}{6}\right)^3=\frac{1}{2}$$

b. Calculate the $E[X]$ using the Indicator random variables.

$X=I_1+I_2+I_3$ where I_t is a 1 if you roll a 6 on the t^{th} roll and a zero otherwise.

$$E[X]=E[I_1]+E[I_2]+E[I_3]=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}$$

65. A cookie jar that has five cookies of which 3 are chocolate chip and 2 are oatmeal. Gayathri randomly selects a cookie from the cookie jar. After she selects a cookie, the cookie jar is replenished by adding one cookie which is the same type as the cookie that she selected. Gayathri selects two more cookies where the same process is followed. (This means that this a problem with replacement.)

Let X be the number of chocolate chip cookies that Gayathri selects.

a. Calculate the $E[X]$.

Solution:

Let S be the number of chocolate chip cookies. We will use indicator variables.

$$S = I_1 + I_2 + I_3$$

$$E[S] = E[I_1 + I_2 + I_3] = E[I_1] + E[I_2] + E[I_3] = \frac{3}{5} + \frac{3}{5} + \frac{3}{5} = \frac{9}{5}$$

b. Calculate the $E[X]$ if there is no replacement of the cookies.

Solution:

Let S be the number of chocolate chip cookies. We will use indicator variables.

$$S = I_1 + I_2 + I_3$$

$$E[S] = E[I_1 + I_2 + I_3] = E[I_1] + E[I_2] + E[I_3] = \frac{3}{5} + \frac{3}{5} + \frac{3}{5} = \frac{9}{5}$$

$$\text{Proof of E}[I_2] = \frac{\overset{C,C}{(3)(2)} + \overset{O,C}{(2)(3)}}{(5)(4)} = \frac{3}{5}$$

$$\text{Proof of E}[I_3] = \frac{\overbrace{(3)(2)(1)}^{C,C,C} + \overbrace{(2)(3)(2)}^{O,C,C} + \overbrace{(3)(2)(2)}^{C,O,C} + \overbrace{(2)(1)(3)}^{O,O,C}}{(5)(4)(3)} = \frac{3}{5}$$

66. You are given that the random variable X is has a probability mass function of:

$$\begin{array}{ll} 0.2 & \text{if } x=3 \\ 0.3 & \text{if } x=8 \\ 0.5 & \text{if } x=10 \\ 0 & \text{elsewhere} \end{array}$$

Calculate the $Var[X]$.

Solution:

$$E[X]=(3)(0.2)+(8)(0.3)+(10)(0.5)=8$$

$$E[X^2]=(3)^2(0.2)+(8)^2(0.3)+(10)^2(0.5)=71$$

$$Var[X]=E[X^2]-\left(E[X]\right)^2=71-(8)^2=7$$

67. You are given that the random variable Y has a probability density function of

$f_Y(y)=\dfrac{y^3}{64}$ for $0\le y\le 4$.

Calculate the $Var[Y]$.

Solution:

$$E[Y]=\int_0^4 y\cdot f_Y(y)\cdot dy=\int_0^4 y\frac{y^3}{64}dy=\left[\frac{y^5}{(64)(5)}\right]_0^4=\frac{4^5}{(64)(5)}=3.2$$

$$E[Y^2]=\int_0^4 y^2\cdot f_Y(y)\cdot dy=\int_0^4 y^2\frac{y^3}{64}dy=\left[\frac{y^6}{(64)(6)}\right]_0^4=\frac{4^6}{(64)(6)}=\frac{32}{3}$$

$$Var[Y]=E[Y^2]-\left(E[Y]\right)^2=\frac{32}{3}-(3.2)^2=\frac{32}{75}$$

68. You have a standard deck of playing cards. There are 52 cards. There are 4 suits. Each suit has cards numbered from 2 to 10 and then four face cards (jack, queen, king and ace). You randomly draw a card from the deck. I will pay you the number on the card. If you draw a face card, I will pay you 15 unless it is an ace in which case you will pay me 100.

Let X be the random variable representing the amount of the payment.

a. Calculate the $E[X]$ from your standpoint.

Solution:

$$p_2 = p_3 = ... = p_{10} = p_J = p_Q = p_K = p_A = \frac{1}{13}$$

$$E[X] = (2)\left(\frac{1}{13}\right) + (3)\left(\frac{1}{13}\right) + ... + (9)\left(\frac{1}{13}\right) + (10)\left(\frac{1}{13}\right) + (15)\left(\frac{3}{13}\right) + (-100)\left(\frac{1}{13}\right) = -\frac{1}{13}$$

b. Calculate the $Var[X]$.

Solution:

$$E[X^2] = (2)^2\left(\frac{1}{13}\right) + (3)^2\left(\frac{1}{13}\right) + ... + (9)^2\left(\frac{1}{13}\right) + (10)^2\left(\frac{1}{13}\right) + (15)^2\left(\frac{3}{13}\right) + (-100)^2\left(\frac{1}{13}\right) = 850.6923$$

$$Var[X] = E[X^2] - (E[X])^2 = 850.6923 - (-1/13)^2 = 850.686$$

69. You are given that $E[X] = 10$ and $Var[X] = 40$. Calculate $E[(X+5)^2]$.

Solution:

$$Var[X] = 40 = E[X^2] - \left(E[X]\right)^2 = E[X^2] - (10)^2 \implies E[X^2] = 140$$

$$E[(X+5)^2] = E[X^2] + 10E[X] + 25 = 140 + (10)(10) + 25 = 265$$

70. You are given that $E[X]=2,\ E[X^3]=9,\ and\ E[(X-2)^3]=0$.

Calculate $Var[X]$ and the standard deviation of X.

Solution:

$$E[(X-2)^3]=E[X^3-6X^2+12X-8]=E[X^3]-6E[X^2]+12E[X]-8$$

$$=9-6E[X^2]+12(2)-8=0 ==> E[X^2]=\frac{25}{6}$$

$$Var[X]=E[X^2]-\left(E[X]\right)^2=\frac{25}{6}-2^2=\frac{1}{6}$$

$$\text{Standard Deviation}=\sqrt{\frac{1}{6}}$$

71. A recent study indicates that the annual cost of maintaining a car in West Lafayette averages 200 with a variance of 200. West Lafayette has just levied a 20% tax on car maintenance so the cost of everything if 120% of the prior cost.

Determine the expected value and variance of car maintenance in West Lafayette with the new tax in place.

Solution:

Let X be the random variable before the tax. The cost after the tax is 1.2X.

$$E[1.2X]=1.2E[X]=1.2(200)=240$$

$$Var[1.2X]=(1.2)^2Var[X]=1.44(200)=288$$

72. A random variable X has the cumulative distribution function:

$$F_X(x) = \begin{cases} 0, & x < 2 \\ \dfrac{x^2 - 2x + 2}{10}, & 2 \le x < 4 \\ 1, & x \ge 4 \end{cases}$$

Calculate $Var[X]$.

Solution:

$F_X(1.999999999....) = 0$ but $F_X(2) = 0.2$ so we have a mixture distribution with a weight of 0.2 at 2. We also note that $F_X(3.999999999....) = F_X(2) = 1$ so the function is continuous otherwise.

For values from 2 to 4, $f_X(x) = \dfrac{d}{dx} F_X(x) = \dfrac{2x-2}{10} = \dfrac{x-1}{5}$

$$E[X] = (2)p(2) + \int_2^4 x \cdot f_X(x) \cdot dx = (2)(0.2) + \int_2^4 x\left[\frac{x-1}{5}\right] \cdot dx = 0.4 + \left[\frac{x^3}{15} - \frac{x^2}{10}\right]_2^4$$

$$= 0.4 + \frac{64}{15} - \frac{16}{10} - \frac{8}{15} + \frac{4}{10} = \frac{44}{15}$$

$$E[X^2] = (2)^2 p(1) + \int_2^4 x^2 \cdot f_X(x) \cdot dx = (2)^2(0.2) + \int_2^4 x^2\left[\frac{x-1}{5}\right] \cdot dx = 0.8 + \left[\frac{x^4}{20} - \frac{x^3}{5}\right]_2^4$$

$$= 0.8 + \frac{4^4}{20} - \frac{4^3}{15} - \frac{2^4}{20} + \frac{2^3}{15} = \frac{136}{15}$$

$$Var[X] = E[X^2] - (E[X])^2 = \frac{136}{15} - \left(\frac{44}{15}\right)^2 = \frac{104}{225}$$

73. The moment generating function for X is $M(t)=c+0.4e^{t}+0.3e^{2t}+0.2e^{at}$ where c is a constant. The mean of the random variable is 2.

Determine a .

Solution:

$$M'(0)=0.4+0.6+0.2a=2$$

$$a=5$$

74. Let the random variable X have a moment generating function of $M(t)=e^{3t+t^2}$.

Calculate $E[X^2]$.

Solution:

$$M'(t)=(3+2t)e^{3t+t^2}$$

$$M''(t)=\frac{d}{dt}(3+2t)e^{3t+t^2}=2e^{3t+t^2}+(3+2t)^2e^{3t+t^2}$$

$$M''(0)=E[X^2]=2+9=11$$

75. Let the random variable X have a moment generating function of $M(t)=\left(\frac{2+e^t}{3}\right)^9$.

Calculate $Var[X]$.

Solution:

$$M'(t)=9\left(\frac{2+e^t}{3}\right)^8\left(\frac{e^t}{3}\right) ==> M'(0)=3$$

$$M''(t)=(9)(8)\left(\frac{2+e^t}{3}\right)^7\left(\frac{e^t}{3}\right)^2+9\left(\frac{2+e^t}{3}\right)^8\left(\frac{e^t}{3}\right) ==> M''(0)=8+3=11$$

$$Var[X]=11-3^2=2$$

76. The damage to a house as a result of a tornado has a random variable X with a moment generating function of:

$$M(t)=\frac{1}{(1-2500t)^4}$$

Determine the standard deviation of the amount of damage.

Solution:

$$M(t)=\frac{1}{(1-2500t)^4}=(1-2500t)^{-4}$$

$$M'(t)=(-4)(1-2500t)^{-5}(-2500)=(10,000)(1-2500t)^{-5} \text{ ==> } M'(0)=E[X]=10,000$$

$$M''(t)=(10,000)(-5)(1-2500t)^{-6}(-2500)=(10,000)(12,500)(1-2500t)^{-6}$$

$$\text{==> } M''(0)=E[X^2]=125,000,000$$

$$Var[X]=125,000,000-(10,000)^2=25,000,000$$

$$\text{==> Standard Deviation }=\sqrt{25,000,000}=5000$$

77. You are given that the Probability Generating Function for the random variable X is

$$P_X(t) = \frac{t^1}{6} + \frac{t^2}{3} + \frac{t^3}{3} + \frac{t^4}{6}$$

a. Calculate $p(x)$ for $x = 1, 2, 3, 4$

Solution:

$$P_X^{(n)}(t)\,|_{t=0} = n!\,p(n)$$

$$P_X'(t)\,|_{t=0} = 1!\,p(1) = \left[\frac{1}{6} + \frac{2t}{3} + \frac{3t^2}{3} + \frac{4t^3}{6}\right]_{t=0} = \frac{1}{6} \text{ ==> } p(1) = \frac{1}{6}$$

$$P_X''(t)\,|_{t=0} = 2!\,p(2) = \left[0 + \frac{2}{3} + \frac{6t^1}{3} + \frac{12t^2}{6}\right]_{t=0} = \frac{2}{3} \text{ ==> } 2!\,p(2) = \frac{2}{3} \text{ ==> } p(2) = \frac{1}{3}$$

$$P_X^{(3)})\,|_{t=0} = 3!\,p(3) = \left[0 + 0 + \frac{6}{3} + \frac{24t}{6}\right]_{t=0} = \frac{6}{3} \text{ ==> } 3!\,p(3) = 2 \text{ ==> } p(3) = \frac{1}{3}$$

$$P_X^{(4)})\,|_{t=0} = 4!\,p(4) = \left[0 + 0 + 0 + \frac{24}{6}\right]_{t=0} = \frac{24}{6} \text{ ==> } 4!\,p(4) = \frac{24}{6} \text{ ==> } p(4) = \frac{1}{6}$$

b. Calculate $E[X]$.

Solution:

$$E[X] = (1)\,p(1) + (2)\,p(2) + (3)\,p(3) + (4)\,p(4)$$

$$= (1)\left(\frac{1}{6}\right) + (2)\left(\frac{1}{3}\right) + (3)\left(\frac{1}{3}\right) + (4)\left(\frac{1}{6}\right) = 2.5$$

c. Calculate $Var[X]$.

Solution:

$$E[X^2] = (1)^2 p(1) + (2)^2 p(2) + (3)^2 p(3) + (4)^2 p(4)$$

$$= (1)^2 \left(\frac{1}{6}\right) + (2)^2 \left(\frac{1}{3}\right) + (3)^2 \left(\frac{1}{3}\right) + (4)^2 \left(\frac{1}{6}\right) = \frac{43}{6}$$

$$Var[X] = \frac{43}{6} - (2.5)^2 = \frac{11}{12}$$

78. The random variable X is distributed as a discrete uniform distribution and can take on values of 1, 2, 3, …, 100.

a. Calculate $E[X]$.

Solution:

Under a discrete uniform distribution from 1, 2, to 100, $E[X]=\frac{100+1}{2}=50.5$.

We could also do this from first principles.

$$E[X]=\sum_x xp(x)=(1)\left(\frac{1}{100}\right)+(2)\left(\frac{1}{100}\right)+(3)\left(\frac{1}{100}\right)+....+(100)\left(\frac{1}{100}\right)$$

$$=\frac{1+2+3+....+100}{100}=\frac{\frac{1+100}{2}(100)}{100}=50.5$$

b. Calculate $Var[X]$.

Solution:

Under a discrete uniform distribution from 1, 2, to 100,

$$Var[X]=\frac{100^2-1}{12}=\frac{33333}{4}=833.25$$

79. The random variable X is distributed as a discrete uniform distribution and can take on values of 0, 1, 2, 3, …, 100.

a. Calculate $E[X]$.

Solution:

Let $Y = X + 1$. Then, $Y = \{1, 2, ..., 101\}$

$$E[Y] = \frac{101+1}{2} = 51$$

$$X = Y - 1 ==> E[X] = E[Y-1] = E[Y] - 1 = 51 - 1 = 50$$

Or

Under a discrete uniform distribution from 1, 2, to 100, $E[X] = \frac{100+1}{2} = 50.5$.

Under a discrete uniform distribution from 0, 1, 2, to 100, $E[X] = \frac{100+0}{2} = 50$.

Or

We could also do this from first principles.

$$E[X] = \sum_x xp(x) = (0)\left(\frac{1}{100}\right) + (1)\left(\frac{1}{100}\right) + (2)\left(\frac{1}{100}\right) + (3)\left(\frac{1}{100}\right) + ... + (100)\left(\frac{1}{100}\right)$$

$$= \frac{0+1+2+3+....+100}{101} = \frac{\frac{0+100}{2}(101)}{101} = 50$$

b. Calculate $Var[X]$

Solution:

Let $Y = X + 1$. Then, $Y = \{1, 2, ..., 101\}$

$$Var[Y] = \frac{101^2 - 1}{12} = 850$$

$$X = Y - 1 ==> Var[X] = Var[Y-1] = Var[Y] = 850$$

80. The random variable X is distributed as a discrete uniform distribution and can take on values of 4, 8, 12, …, 300.

a. Calculate $E[X]$.

Solution:

Let $Y = X / 4$. Then, $Y = \{1, 2, ..., 75\}$

$$E[Y] = \frac{75+1}{2} = 38$$

$$X = 4Y ==> E[X] = E[4Y] = 4E[Y] = (4)(38) = 152$$

Or

$$E[X] = \frac{300+4}{2} = 152$$

Or

We could also do this from first principles.

$$E[X] = \sum_x xp(x) = (4)\left(\frac{1}{750}\right) + (8)\left(\frac{1}{75}\right) + (2)\left(\frac{1}{75}\right) + ... + (300)\left(\frac{1}{75}\right)$$

$$= \frac{4+8+12+....+300}{75} = \frac{\frac{4+300}{2}(75)}{75} = 152$$

b. Calculate $Var[X]$

Solution:

Let $Y = X / 4$. Then, $Y = \{1, 2, ..., 75\}$

$$Var[Y] = \frac{75^2 - 1}{12} = \frac{1406}{3}$$

$$X = 4Y ==> Var[X] = Var[4Y] = (4)^2 E[Y] = (16)\left(\frac{1406}{3}\right) = 7498.67$$

81. The Purdue freshman class has 10,000 students. The probability that a student will not return to campus next year is 0.15. A student's return to school is independent of any other student's return to school.

Let R be the random variable for the number of students that will return to school.

Calculate $E[R]$ and $Var[R]$.

Solution:

$$E[R] = np = (10,000)(0.85) = 8500$$

$$Var[R] = np(1-p) = (10,000)(0.85)(0.15) = 1275$$

82. You are given that X is distributed as a binomial distribution with parameters of $n = 5$ and $p = 0.3$.

a. Calculate the probability that $X = 2 \text{ or } 4$.

Solution:

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$p(2) = \binom{5}{2} (0.3)^2 (1-0.3)^{5-2} = \left(\frac{5!}{2!3!}\right)(0.3)^2(0.7)^3 = 0.3087$$

$$p(4) = \binom{5}{4} (0.3)^4 (1-0.3)^{5-4} = \left(\frac{5!}{4!1!}\right)(0.3)^4(0.7)^1 = 0.02835$$

$$Answer = 0.3087 + 0.02835 = 0.33705$$

b. Calculate $F_X(2)$.

Solution:

$$F_X(2) = p(0) + p(1) + p(2) = 0.16807 + 0.36015 + 0.3087 = 0.83692$$

$p(2) = 0.3087$ from the previous part.

$$p(0) = \binom{5}{0} (0.3)^0 (1-0.3)^5 = 0.16807$$

$$p(1) = \binom{5}{1} (0.3)^1 (1-0.3)^4 = 0.36015$$

83. The ice cream machine in Hillenbrand Hall has a probability of breaking down in any day of 0.20. The breakdowns are independent of any other breakdowns. The machine can only breakdown once per day.

Calculate the probability that the machine breaks down two or more times in a 10 day period.

Solution:

Since the breakdowns are independent, this is a binomial distribution with $n = 10$ and $p = 0.2$.

$$\Pr(X \geq 2) = 1 - \Pr(X < 2) = 1 - p(0) - p(1) = 1 - \binom{10}{0}(0.2)^0(0.8)^{10} - \binom{10}{1}(0.2)^1(0.8)^9 = 0.6242$$

84. Two studies are being conducted on Covid vaccines. There are ten participants in each study. There is a 20% chance that a participant will not complete the study. Drop outs are independent of other drop outs.

What is the probability that at least 9 participants will complete the study for one of the two groups but both groups will not have at least 9 participants complete the study.

Solution:

We can treat each trial group as a binomial distribution with $n = 10$ and $p = 0.8$.

$$\Pr(X \geq 9) = p(9) + p(10) = \binom{10}{9}(0.8)^9(0.2)^1 + \binom{10}{10}(0.8)^{10}(0.2)^0 = 0.376$$

$$\Pr(X < 9) = 1 - 0.376 = 0.624$$

Probability that one group has at least 9 but not both groups is

$$\Pr(\text{Group 1} \geq 9) \cdot \Pr(\text{Group 2} < 9) + \Pr(\text{Group 1} < 9) \cdot \Pr(\text{Group 2} \geq 9)$$

$$= (0.376)(0.624) + (0.624)(0.376) = 0.469$$

85. You have three random variables which are distributed as follows:

i. $X \sim Binomial(4, 0.4)$

ii. $Y \sim Binomial(8, 0.4)$

iii. $Z \sim Binomial(12, 0.4)$

The random variable $S = X + Y + Z$.

a. Calculate the $E[S]$.

Solution:

$S \sim Binomial(4+8+12, 0.4)$

$E[S] = np = (24)(0.4) = 9.6$

b. Calculate the $Var[S]$.

Solution:

$Var[S] = np(1-p) = (24)(0.4)(0.6) = 5.76$

86. A bowl has four tiles numbered 1, 2, 3, and 4. You randomly draw a tile from the bowl and note the number. You replace the tile and draw again.

A success is if the sum of the two tiles exceeds 6.

N is the number of times you repeated the experiment when you have your first success.

a. Calculate the probability that $N = 4$.

Solution:

This is a geometric distribution with parameter of $p = \frac{3}{16}$ since the probability of getting a 7 is $\frac{2}{16}$ and the probability of getting an 8 is $\frac{1}{16}$.

$$p(4) = (1-p)^{4-1}p = \left(\frac{13}{16}\right)^3\left(\frac{3}{16}\right) = 0.10057$$

b. Calculate $E[N]$.

Solution:

$$E[N] = \frac{1}{p} = \frac{16}{3}$$

c. Calculate $Var[N]$.

Solution:

$$Var[N] = \frac{1-p}{p^2} = \frac{\frac{13}{16}}{\left(\frac{3}{16}\right)^2} = \left(\frac{13}{16}\right)\left(\frac{16}{3}\right)^2 = \frac{208}{9}$$

87. A part of a wellness check for faculty at Purdue, all faculty members are being tested for high blood pressure. Let X be the number of tests completed when the first person with high blood pressure is found. The $E[X]=12.5$.

Calculate the probability that the sixth person tested is the first person with high blood pressure.

Solution:

If the expected number of tests until the first positive is 12.5, then application of the geometric distribution yields the proportion of people having high blood pressure is $\frac{1}{12.5}=0.08$.

So the probability that the sixth person tested is the first person with high blood pressure is

$$(1-0.08)^5(0.08)=0.052727$$

88. A fair coin is tossed repeatedly. Calculate the probability that the third head occurs on the nth toss.

Solution:

We need two heads in $n-1$ tosses and a head on the nth toss. The probability of two heads on $n-1$ tosses is $\binom{n-1}{2}\left(\frac{1}{2}\right)^{n-1}$ and the probability of a head on the nth toss is $\left(\frac{1}{2}\right)$.

$$\text{Answer} = \binom{n-1}{2}\left(\frac{1}{2}\right)^{n}$$

89. A bowl has five tiles numbered 1, 2, 3, 4 and 5. You randomly draw a tile from the bowl and note the number. You replace the tile and draw again.

A success is if the sum of the two tiles exceeds 7.

X is the number of times you repeat the experiment when you have your third success.

a. Calculate $E[X]$.

Solution:

$$X \sim NegativeBinomial(r, p)$$

$$r = 3$$

$$p = \text{probability of a success} = \Pr(X > 7) = \frac{3+2+1}{5^2} = 0.24$$

$$E[X] = \frac{3}{p} = \frac{3}{0.24} = 12.5$$

b. Calculate $Var[X]$.

Solution:

$$Var[X] = \frac{r(1-p)}{p^2} = \frac{(3)(1-0.24)}{(0.24)^2} = 39.58333 = \frac{475}{12}$$

90. In a shipment of 20 packages, 7 packages are damaged. The packages are randomly inspected, one at a time, without replacement, until the fourth damaged package is discovered.

Calculate the probability that exactly 12 packages are inspected.

Solution:

Since this question is sampling without replacement, the negative binomial distribution is not applicable. The probability of 3 damaged packages in the first 11 is:

$$\frac{\binom{7}{3}\binom{13}{8}}{\binom{20}{11}} = 0.268189$$

The probability that the 12th package is damaged is $\frac{4}{9}$ as there are 9 packages remaining, of which 4 are damaged.

$$\text{Answer} = (0.268189)\left(\frac{4}{9}\right) = 0.119195$$

91. Let X be the number of independent Bernoulli trials performed until a success occurs. Let Y be the number of independent Bernoulli trials performed until 5 successes occur. A success occurs with a probability of p and the $Var[X] = 3/4$.

Calculate the $Var[Y]$.

Solution:

By the relationship between negative binomial and geometric distribution, we have:

$$Var[Y] = 5Var[X] = \frac{15}{4}$$

92. You are given that X is distributed as a Poisson distribution with $\lambda = 1$.

   a. Calculate the probability that $X \geq 2$.

   Solution:

   $$\Pr[X \geq 2] = 1 - \Pr[X = 0] - \Pr[X = 1] = 1 - \frac{e^{-1}(1)^0}{0!} - \frac{e^{-1}(1)^1}{1!} = 1 - 2e^{-1} = 0.26424$$

   b. Calculate that probability that $X \geq 2$ given that $X \leq 4$.

   Solution:

   $$\Pr[X \geq 2 \mid X \leq 4] = \frac{\Pr[2 \leq X \leq 4]}{\Pr[X \leq 4]} = \frac{p_2 + p_3 + p_4}{p_0 + p_1 + p_2 + p_3 + p_4}$$

   $$\frac{e^{-1}(1/2 + 1/6 + 1/24)}{e^{-1}(1 + 1 + 1/2 + 1/6 + 1/24)} = \frac{17}{65}$$

93. The number of traffic accidents per week in Remington has a Poisson distribution with a mean of 3.

   What is the probability of exactly two accidents in two weeks.

   Solutions:

   The number of accidents in one week has a poisson distribution with a λ=3, so the number of accidents for two weeks has a poisson distribution with a λ=6.

   $$\text{Answer} = p_2 = \frac{e^{-6}(6)^2}{2!} = 18e^{-6} = 0.044618$$

94. The number of power surges in an electric grid has a Poisson distribution with a mean of 1 surge every 12 hours.

   Calculate the probability that there will be no more than one power surge in a 24 hour period.

   Solution:

   The λ for 24 hours is twice the λ for 12 hours so $\lambda = 2$.

   $$\text{Answer} = p_0 + p_1 = \frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} = 3e^{-2} = 0.40601$$

95. You are given that N is distributed as Poisson. You are also given that $\Pr(N=2)=3\Pr(N=4)$.

Calculate the $Var[N]$.

Solution:

$$\Pr(N=2)=3\Pr(N=4) ==> \frac{e^{-\lambda}\lambda^2}{2!}=3\left[\frac{e^{-\lambda}\lambda^4}{4!}\right]$$

$$==> \frac{\lambda^2}{2}=\frac{\lambda^4}{8} ==> \lambda^2=4 ==> \lambda=2$$

$$Var[N]=\lambda=2$$

96. The number of people who ride an elevator in the Math Building is distributed as a Poisson distribution with an average of 60 people per hour. Students make up 80% of those riding the elevator while faculty make up the other 20%.

Calculate the probability that no faculty member rides the elevator during a 15 minute period.

Solution:

The number of riders in an hour is 60. Of these, the average number of faculty and staff is 12 per hour. This means that in a 15 minute period, the average number would be 3.

Therefore, $\lambda=3$ and $p_0=\dfrac{e^{-3}3^0}{0!}=e^{-3}=0.049787$

97. Let the random variable L be uniformly distributed between 2000 and 20,000.

a. Calculate the $E[L]$.

Solution:

$$E[L]=\frac{a+b}{2}=\frac{2000+20,000}{2}=11,000$$

b. Calculate the $Var[L]$.

Solution:

$$Var[L]=\frac{(b-a)^2}{12}=\frac{(20,000-2000)^2}{12}=27,000,000$$

c. Calculate the probability that L will exceed 15,000.

Solution:

$$S_L(15,000)=1-F_L(15,000)=1-\frac{15,000-2000}{20,000-2000}=\frac{5}{18}$$

d. Calculate the probability that L will exceed 15,000 given that it exceeds 12,000.

Solution:

$$\frac{\Pr(L>12,000)\bigcap \Pr(l>15,000)}{\Pr(L>12,000)}=\frac{\Pr(L>15,000)}{\Pr(L>12,000)}=\frac{5/18}{1-\frac{12,000-2,000}{20,000-2000}}=\frac{5}{8}$$

98. The expected inter-arrival time of busses at the Greyhound station in Indianapolis is 20 minutes.

a. Determine the parameter θ for the exponential distribution.

Solution:

$$E[X]=20=\frac{1}{\theta} ==> \theta=\frac{1}{20}$$

b. Calculate the probability that the time between busses will be at least 20 minutes.

Solution:

$$S_X(t)=e^{-\theta t} ==> S_X(20)=e^{-\frac{1}{20}(20)}=e^{-1}=0.368$$

c. Calculate the probability that the time between busses will exceed 20 minutes but will be less than 30 minutes.

Solution:

$$F_X(30) - F_X(20) = S_X(20) - S_X(30) = e^{-1} - e^{-1.5} = 0.14475$$

d. Calculate the variance of the inter-arrival time.

Solution:

$$Var[X] = \frac{1}{\theta^2} = (20)^2 = 400$$

e. If you arrive at the bus station 10 minutes after the last bus arrived, what is the expected time until the next bus arrives.

Solution:

Due to the memoryless property, it is 20 minutes.

99. The lifetime of a printer is exponentially distributed with a mean of 2 years. The printer costs 200. The manufacturer of the printer provides a guarantee that it will refund the cost of the printer if it fails in the first year. Additionally, the manufacturer will refund 50% of the cost if the printer fails in the second year. No refund is provided if the printer does not fail in the first two years.

Calculate the expected total amount of refunds for each printer sold.

Solution:

The refund for failure in the first year is 200. The refund for failure in the second year is 100.

Expected lifetime is 2 years. Therefore, $\frac{1}{\theta} = 2 \Longrightarrow \theta = 0.5$

Failure in the first year $= F_X(1) = 1 - e^{-0.5(1)} = 0.39346934$

Failure in the second year $= F_X(2) - F_X(1) = 1 - e^{-0.5(2)} - \left[1 - e^{-0.5(1)}\right]$

$= 0.632120559 - 0.39346934 = 0.238651219$

$Amount = (200)(0.39346934) + (100)(0.238651219) = 102.56$

100. The time to failure of an electronic component has an exponential distribution with a median of four hours. Remember that the median is the x such that $F_X(x) = 0.5$.

Calculate the probability that the component will work without failing for at least 5 hours.

Solution:

For an exponential distribution, $F_X(x) = 1 - e^{-\theta x}$. We are given that $F_X(4) = 1 - e^{-4\theta} = 0.5$.

$==> e^{-4\theta} = 0.5$

We need $F_X(5) = 1 - e^{-5\theta} = 1 - \left(e^{-4\theta}\right)^{5/4} = 1 - (0.5)^{5/4} = 0.42045$

101. A company has two independent electric generators. The time until failure for each generator follows an exponential distribution with a mean of 10. The company will begin using the second generator upon the failure of the first generator.

Calculate the variance of expected total time that the generators produce electricity.

Solution:

Let X be the random variable for the time of generator 1 and let Y be the random variable for the time of generator 2.

$$\theta_X = \frac{1}{10} ==> Var[X] = \frac{1}{\theta^2} = 100$$

$$\theta_Y = \frac{1}{10} ==> Var[Y] = \frac{1}{\theta^2} = 100$$

Since the generators are independent, $Var[X+Y] = Var[X] + Var[Y] = 100 + 100 = 200$

102. Calculate $\Gamma(6)$.

Solution:

$\Gamma(6) = 5! = 120$

103. The random variable X has a Gamma distribution with parameters $\alpha = 6$ and $\beta = 0.5$.

a. Calculate $E[X]$.

Solutions:

$$E[X] = \frac{\alpha}{\beta} = \frac{6}{0.5} = 12$$

b. Calculate $Var[X]$.

Solutions:

$$Var[X] = \frac{\alpha}{\beta^2} = \frac{6}{(0.5)^2} = 24$$

104. The random variable Y has Gamma distribution with a mean of 24 and a variance of 12.

Determine the parameters α and β.

Solutions:

$$E[X] = \frac{\alpha}{\beta} = 24$$

$$Var[X] = \frac{\alpha}{\beta^2} = 12 = \frac{\alpha}{\beta}\left(\frac{1}{\beta}\right) = 24\left(\frac{1}{\beta}\right) ==> 12\beta = 24 ==> \beta = 2$$

$$\frac{\alpha}{\beta} = 24 ==> \alpha = (24)\beta = (24)(2) = 48$$

105. Let X be distributed as a Gamma distribution with $\alpha = 6$ and $\beta = 3$ and Y be distributed as a Gamma distribution with $\alpha = 21$ and $\beta = 3$.

The random variable $S = X + Y$.

Calculate the $Var[S]$.

Solution:

$$S \sim Gamma(\alpha_X + \alpha_Y, \beta) = Gamma(6+21, 3) = Gamma(27, 3)$$

$$Var[S] = \frac{\alpha}{\beta^2} = \frac{27}{3^2} = 3$$

106. If $X \sim Normal(5,16)$, calculate the $\Pr(X > 7)$.

Solution:

$$\Pr(X > 7) = \Pr\left(\frac{x-\mu}{\sigma} > \frac{7-\mu}{\sigma}\right) = \Pr\left(Z > \frac{7-5}{4}\right) = \Pr\left(Z > 0.5\right)$$

$$= 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085$$

107. Let X be a normal random variable with a mean of zero and a variance of $a > 0$. Calculate the $\Pr(X^2 < a)$.

Solution:

$$\Pr(X^2 < a) = \Pr(X < \sqrt{a}) + \Pr(X > -\sqrt{a}) = \Pr(-\sqrt{a} < X < \sqrt{a})$$

$$\Pr\left(\frac{-\sqrt{a}-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{\sqrt{a}-\mu}{\sigma}\right) = \Pr\left(\frac{-\sqrt{a}-0}{\sqrt{a}} < Z < \frac{\sqrt{a}-0}{\sqrt{a}}\right) =$$

$$\Pr\left(-1 < Z < 1\right) = \Pr(Z < 1) - \Pr(Z > -1) = \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1$$

$$= 2(0.8413) - 1 = 0.6826$$

108. Let $X_1, X_2, ..., X_{36}$ be an random sample from a distribution with mean $\mu_X = 30.4$ and standard deviations of $\sigma_X = 12.$

Using the Central Limit Theorem, calculate the approximate value of $\Pr[\bar{X}_{36} > 29].$

Solution:

$$\Pr[\bar{X}_{36} > 29] = \Pr\left[\frac{\bar{X}_{36} - \mu_X}{\sigma_X / \sqrt{n}} > \frac{29 - 30.4}{12 / \sqrt{36}}\right] = \Pr\left[Z > -1.4\right]$$

$$= 1 - \Phi(-1.4) = 1 - \left[1 - \Phi(1.4)\right] = 1 - \left[1 - 0.7580\right] = 0.7580$$

109. Let $X_1, X_2, ..., X_{36}$ and $Y_1, Y_2, ..., Y_{49}$ be independent random samples from distributions with means $\mu_X = 30.4$ and $\mu_Y = 32.1$. Further, the distributions also have standard deviations of $\sigma_X = 12$ and $\sigma_Y = 14.$

Using the Central Limit Theorem, calculate the approximate value of $\Pr[\bar{X}_{36} > \bar{Y}_{49}].$

Solution:

By the CLT, $\bar{X}$ is distributed approximately $N(30.4, 12^2).$

By the CLT, $\bar{Y}$ is distributed approximately $N(32.1, 14^2).$

$\therefore \bar{X} - \bar{Y}$ is distributed approximately $N(30.4 - 32.1, 12^2 / 36 + 14^2 / 49) = N(-1.7, 8).$

$$\Pr[\bar{X} > \bar{Y}] = \Pr[\bar{X} - \bar{Y} > 0] = \Pr\left[Z > \frac{0 - -1.7}{\sqrt{8}}\right] = \Pr\left[Z > 0.601\right] = 1 - \Phi(0.601)$$

$$= 1 - 0.7257 = 0.2743$$

110. Let $X_1, X_2, \ldots, X_{100}$ be a random sample from an exponential distribution with a mean of 0.5.

Determine the approximate value of $\Pr\left[\sum_{i=1}^{100} X_i > 57\right]$ using the Central Limit Theorem.

Solution:

$$\Pr\left[\sum_{i=1}^{100} X_i > 57\right] = \Pr\left[\frac{1}{100}\sum_{i=1}^{100} X_i > 57/100\right] = \Pr\left[\bar{X}_{100} > 0.57\right] =$$

$$\Pr\left[\frac{\bar{X}_{100} - E[\bar{X}_{100}]}{\sqrt{Var[\bar{X}_{100}]}} -> \frac{0.57 - 0.5}{0.05}\right] = \Pr\left[Z > 1.4\right] = 1 - \Phi(1.4) = 0.0808$$

$E[\bar{X}_{100}] = \mu_X = 0.5$
$Var[\bar{X}_{100}] = \sigma_X^2 / 100 = (0.5)^2 / 100 ==> \sqrt{(0.5)^2 / 100} = 0.5/10 = 0.05$

111. Let $X_1, X_2, ..., X_n$ be an random sample from a distribution with mean $\mu_X = 30.4$ and standard deviations of $\sigma_X = 12.$

Using the Central Limit Theorem, determine the number of samples necessary to be 95% certain that $\bar{X}_n$ is within 1 of the true value of μ_X .

Solution:

$$\Pr[-1 < \bar{X}_n - \mu_X < 1] = 0.95$$

$$\Pr\left[\frac{-1}{12/\sqrt{n}} < \frac{\bar{X}_n - \mu_X}{\sigma_X/\sqrt{n}} < \frac{1}{12/\sqrt{n}}\right] = \Pr\left[-\frac{\sqrt{n}}{12} < Z < \frac{\sqrt{n}}{12}\right] = 0.95$$

$$\Phi\left(\frac{\sqrt{n}}{12}\right) - \Phi\left(-\frac{\sqrt{n}}{12}\right) = \Phi\left(\frac{\sqrt{n}}{12}\right) - \left[1 - \Phi\left(\frac{\sqrt{n}}{12}\right)\right] = 0.95$$

$$2\Phi\left(\frac{\sqrt{n}}{12}\right) - 1 = 0.95$$

$$1.96 = \frac{\sqrt{n}}{12} \text{ ==> } n = [(1.96)(12)]^2 = 553.19 \text{ ==> } 554$$

112. Let the random variable N be distributed as Poisson with an expected value of 81.

Use the normal distribution to approximate the probability that N will exceed 100.

Solution:

$$\Pr[N > 100] \approx \Pr\left[\frac{N-\lambda}{\sqrt{\lambda}} > \frac{100-81}{\sqrt{81}}\right] = \Pr\left[Z > \frac{19}{9}\right]$$

$$= \Pr\left[Z > 2.11\right] = 1 - \Phi(2.11) = 1 - 0.9826 = 0.0174$$

Or With the Continuity Correction

$$\Pr[N > 100] \approx \Pr\left[\frac{N-\lambda}{\sqrt{\lambda}} > \frac{100.5-81}{\sqrt{81}}\right] = \Pr\left[Z > \frac{19.5}{9}\right]$$

$$= \Pr\left[Z > 2.17\right] = 1 - \Phi(2.17) = 1 - 0.9850 = 0.0150$$

113. Let the random variable N be distributed as Poisson with an expected value of 81.

Use the normal distribution to approximate A such that $\Pr[81 - A < N < 81 + A] = 0.80.$

Solution:

$$\Pr[81 - A < N < 81 + A] = 0.80 ==> \Pr\left[\frac{81-A-81}{9} < \frac{N-\lambda}{\sqrt{\lambda}} < \frac{81+A-81}{9}\right] = 0.8$$

$$\Pr\left[\frac{-A}{9} < Z < \frac{A}{9}\right] = 0.8 ==> \Phi\left(\frac{A}{9}\right) - \Phi\left(\frac{-A}{9}\right) = \Phi\left(\frac{A}{9}\right) - \left[1 - \Phi\left(\frac{A}{9}\right)\right] = 2\Phi\left(\frac{A}{9}\right) - 1 = 0.8$$

$$\Phi\left(\frac{A}{9}\right) = 0.9 ==> \frac{A}{9} = 1.282 ==> A = 11.54$$

Or, taking into account the fact that Poison is discrete, $A = 12$.

114. An insurance company sells 500 auto insurance policies. The number of claims per year for each policy follows a Poisson distribution with a mean of 0.2 claims. The number of claims between policies is independent.

Using the normal distribution to approximate the probability that the company will have 80 claims or more.

Solution:

$$\lambda = 500(0.2) = 100$$

$$\Pr\left[N \geq 80\right] \approx \Pr\left[\frac{N-\lambda}{\sqrt{\lambda}} \geq \frac{80-100}{10}\right] = \Pr[Z \geq -2] =$$

$$1-\Phi(-2) = 1-\left[1-\Phi(2)\right] = \Phi(2) = 0.9772$$

Or with the continuity correction

$$\Pr\left[N \geq 80\right] \approx \Pr\left[\frac{N-\lambda}{\sqrt{\lambda}} \geq \frac{79.5-100}{10}\right] = \Pr[Z \geq -2.05] =$$

$$1-\Phi(-2.05) = 1-\left[1-\Phi(2.05)\right] = \Phi(2.05) = 0.9798$$

115. An insurance company issues 1250 vision care policies. The number of claims filed by a policyholder under a vision care policy during one year is Poisson with a mean of 2. The number of claims filed by each policy is independent of other policyholders.

Using the normal distribution of approximate that the total number of claims will be between 2450 and 2600 during a one year period.

Solution:

$$\lambda = (1250)(2) = 2500$$

$$\Pr\left[2450 \le N \le 2600\right] \approx \Pr\left[\frac{2450-2500}{\sqrt{2500}} \le \frac{N-\lambda}{\sqrt{\lambda}} \le \frac{2600-2500}{\sqrt{2500}}\right] = \Pr[-1 \le Z \le 2] =$$

$$\Phi(2)-\Phi(-1) = \Phi(2)-\left[1-\Phi(1)\right] = 0.9772-[1-0.8413] = 0.8185$$

Or with the continuity correction

$$\Pr\left[2450 \le N \le 2600\right] \approx \Pr\left[\frac{2449.5-2500}{\sqrt{2500}} \le \frac{N-\lambda}{\sqrt{\lambda}} \le \frac{2600.5-2500}{\sqrt{2500}}\right] = \Pr[-1.01 < Z < 2.01] =$$

$$\Phi(2.01)-\Phi(-1.01) = \Phi(2.01)-\left[1-\Phi(1.01)\right] = 0.9778-[1-0.8438] = 0.8216$$

116. You are given that the Random Variable X can have values of 1, 2, 3 and random variable Y can have values of 1, 2, 3, 4. Further, the table below lists the joint probability mass function.

	X		
Y	1	2	3
1	1/10	1/20	1/5
2	0	1/10	1/20
3	1/20	1/10	1/10
4	3/20	0	1/10

a. Calculate $\Pr[X = 2, Y = 3]$.

Solution:

$$\Pr[X = 2, Y = 3] = 1/10$$

b. Calculate the marginal pdf of X .

Solution:

$$f_X(1) = 1/10 + 0 + 1/20 + 3/20 = 6/20$$
$$f_X(2) = 1/20 + 1/10 + 1/10 + 0 = 5/20$$
$$f_X(3) = 1/5 + 1/20 + 1/10 + 1/10 = 9/20$$

c. Calculate the marginal pdf of Y .

Solution:

$$f_Y(1) = 1/10 + 1/20 + 1/5 = 7/20$$
$$f_Y(2) = 0 + 1/10 + 1/20 = 3/20$$
$$f_Y(3) = 1/20 + 1/10 + 1/10 = 5/20$$
$$f_Y(4) = 3/20 + 0 + 1/10 = 5/20$$

d. Calculate the $\Pr[X + Y > 4]$.

Solution:

$$f_{X,Y}(1,4) + f_{X,Y}(2,3) + f_{X,Y}(2,4) + f_{X,Y}(3,2) + f_{X,Y}(3,3) + f_{X,Y}(3,4)$$

$$= 3/20 + 1/10 + 0 + 1/20 + 1/10 + 1/10 = 10/20$$

e. Calculate $F_{X,Y}(1,2)$.

Solution:

$$F_{X,Y}(1,2) = \Pr[X \le 1, Y \le 2] = f_{X,Y}(1,1) + f_{X,Y}(1,2) = 1/10 + 0 = 1/10$$

117. Let X and Y be discrete random variables with joint density function:

$$f_{X,Y}(x,y)=\frac{1}{6} \quad \text{for } x=1,2,3 \text{ and } y=2,3. \quad f_{X,Y}(x,y)=0 \text{ elsewhere.}$$

Let $U=X+Y$. Calculate the probability mass function for U .

Solution:

There are six possible combinations:

$$(x,y)=(1,2),(1,3),(2,2),(2,3),(3,2),(3,3)$$

Therefore:

$$p(U)=\begin{cases}\frac{1}{6}, \text{ if } u=3\\ \frac{1}{3}, \text{ if } u=4\\ \frac{1}{3}, \text{ if } u=5\\ \frac{1}{6}, \text{ if } u=6\end{cases}$$

118. Let X and Y be discrete random variables with Joint probability function:

$$f_{X,Y}(x,y)=\begin{cases}\frac{2x+y}{12}, & \text{for } (x,y)=(0,1),(0,2),(1,2),(1,3)\\ 0, & \text{elsewhere}\end{cases}$$

Determine the marginal probability function for X.

Solution:

$$f_X(x)=\frac{2(0)+1}{12}+\frac{2(0)+2}{12}=\frac{1}{4} \quad \text{for} \quad x=0$$

$$f_X(x)=\frac{2(1)+2}{12}+\frac{2(1)+3}{12}=\frac{3}{4} \quad \text{for} \quad x=1$$

119. The continuous random variable X and Y have a joint probability density function of $f_{X,Y}(x,y)=k(x+y)$ for $0<x<2$ and $0<y<1$.

a. Determine k .

Solution:

$$\int_0^1\int_0^2 f_{X,Y}(x,y)dxdy=1$$

$$\int_0^1\int_0^2 k(x+y)dxdy=\int_0^1 k\left(\frac{x^2}{2}+yx\right)_0^2 dy=\int_0^1 k(2+2y)dy=k\left(2y+y^2\right)_0^1=3k=1$$

$$k=\frac{1}{3}$$

b. Calculate $F_{X,Y}(0.4,0.5)$.

Solution:

$$F_{X,Y}(0.4,0.5)=\int_0^{0.5}\int_0^{0.4} f_{X,Y}(x,y)dxdy=\int_0^{0.5}\int_0^{0.4}\frac{(x+y)}{3}dxdy=\frac{1}{3}\int_0^{0.5}\left(\frac{x^2}{2}+yx\right)_0^{0.4} dy$$

$$=\frac{1}{3}\int_0^{0.5}(0.08+0.4y)dy=\frac{1}{3}\left(0.08y+0.2y^2\right)_0^{0.5}=0.03$$

c. Calculate $\Pr[X+Y>2]$.

Solution:

$$\Pr[X+Y>2]=\int_0^1\int_{2-y}^2 f_{X,Y}(x,y)dxdy=\int_0^1\int_{2-y}^2\frac{(x+y)}{3}dxdy=\frac{1}{3}\int_0^1\left(\frac{x^2}{2}+yx\right)_{2-y}^2 dy$$

$$=\frac{1}{3}\int_0^1\left(2+2y-\frac{(2-y)^2}{2}-(2-y)y\right)dy=\frac{1}{3}\int_0^1\left(2y-\frac{y^2}{2}\right)dy=\frac{1}{3}\left(y^2+\frac{y^3}{6}\right)_0^1=\frac{7}{18}$$

d. Calculate $\Pr[X+Y>0.7]$.

Solution:

$$\Pr[X+Y>0.7]=1-\int_0^{0.7}\int_0^{0.7-y} f_{X,Y}(x,y)dxdy=1-\int_0^{0.7}\int_0^{0.7-y}\frac{(x+y)}{3}dxdy$$

$$=1-\frac{1}{3}\int_0^{0.7}\left(\frac{x^2}{2}+yx\right)_0^{0.7-y}dy=1-\frac{1}{3}\int_0^{0.7}\left(\frac{(0.7-y)^2}{2}+(0.7-y)y\right)dy$$

$$=1-\frac{1}{3}\int_0^{0.7}\left(0.245-0.5y^2\right)dy=1-\frac{1}{3}\left(0.245y-\frac{y^3}{6}\right)_0^{0.7}=0.96189$$

e. Determine the margin pdf of x .

Solution:

$$f_X(x)=\int_0^1\frac{(x+y)}{3}dy=\frac{1}{3}\left(xy+\frac{y^2}{2}\right)_0^1=\frac{1}{3}\left(x+\frac{1}{2}\right)$$

120. Let X and Y continuous random variables with joint density function:

$$f_{X,Y}(x,y)=\begin{cases}\frac{12}{25}\left(x+y^2\right), & \text{for } 1<x<y<2\\ 0, & \text{elsewhere}\end{cases}$$

Determine the marginal density function of Y.

Solution:

$$f_Y(y)=\int_1^y f_{X,Y}(x,y)dx=\int_1^y \frac{12}{25}\left(x+y^2\right)dx=\frac{12}{25}\left(\frac{x^2}{2}+y^2x\right)_1^y=\frac{12}{25}\left(\frac{y^2}{2}+y^2y-\frac{1^2}{2}-y^2(1)\right)$$

$$=\frac{12}{25}\left(\frac{y^2}{2}+y^3-\frac{1}{2}-y^2\right)=\frac{12}{25}y^3-\frac{6}{25}y^2-\frac{6}{25} \quad \text{for} \quad 1<y<2 \quad \text{and } 0 \text{ elsewhere.}$$

121. Let X and Y be continuous random variables with a joint density function of $f_{X,Y}(x,y)=e^{-(x+y)},\ x>0,\ y>0$.

Calculate the $\Pr[X+Y<1]$.

Solution:

$$\Pr[X+Y<1]=\int_0^1\int_0^{1-x} f_{X,Y}(x,y)dydx=\int_0^1\int_0^{1-x} e^{-(x+y)}dydx$$

$$=\int_0^1\left(\frac{e^{-(x+y)}}{-1}\right)_0^{1-x}dx=\int_0^1\left(e^{-x}-e^{-1}\right)dx=\left(\frac{e^{-x}}{-1}-xe^{-1}\right)_0^1=1-2e^{-1}$$

122. A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f_{X,Y}(x,y)=\frac{x+y}{8},\quad \text{for } 0<x<2,\ 0<y<2\ .$$

Calculate the probability that the device fails in the first hour of operation.

Solution:

$$\Pr[\{X\le 1\}\cup\{Y\le 1\}]=1-\Pr[X>1,Y>1]=1-\int_1^2\int_1^2 f_{X,Y}(x,y)dydx$$

$$1-\int_1^2\int_1^2\frac{x+y}{8}dydx=1-\int_1^2\frac{x+1.5}{8}dx=1-\frac{3}{8}=\frac{5}{8}$$

123. Let X and Y be continuous random variables with joint density function:

$$f_{X,Y}(x,y)=\begin{cases} xy, & \text{for } 0\le x\le 2 \text{ and } 0\le y\le 1\\ 0, & \text{elsewhere}\end{cases}$$

Calculate the $\Pr[X/2\le Y\le X]$.

Solution:

$$\Pr[X/2\le Y\le X]=\int_0^1\int_{x/2}^{x} xydydx+\int_1^2\int_{x/2}^{1} xydydx=\int_0^1 x\left[\frac{y^2}{2}\right]_{x/2}^{x}dx+\int_1^2 x\left[\frac{y^2}{2}\right]_{x/2}^{1}dx$$

$$=\int_0^1\frac{3x^3}{8}dx+\int_1^2\left[\frac{x}{2}-\frac{x^3}{8}\right]dx=\frac{3}{32}+\frac{9}{32}=\frac{3}{8}$$

124. Let X and Y continuous random variables be independent with the following density functions:

$$f_X(x) = 1, \quad \text{for } 0 < x < 1 \quad \text{and} \quad f_Y(y) = 2y, \quad \text{for } 0 < y < 1 \ .$$

Calculate the $\Pr[X < Y]$.

Solution:

Since X and Y are independent, $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = (1)(2y) = 2y$, for $0 < x < 1$ and for $0 < y < 1$.

$$P[Y < X] = \int_0^1 \int_x^1 2y \cdot dydx = \int_0^1 \left[y^2\right]_x^1 \cdot dx = \int_0^1 1 - x^2 \cdot dx = \left[x - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

125. Let X and Y continuous random variables with joint density function:

$$f_{X,Y}(x,y)=\begin{cases}\frac{12}{25}\left(x+y^2\right), & \text{for } 1<x<y<2\\ 0, & \text{elsewhere}\end{cases}$$

Calculate $E[X+Y]$.

Solution:

$$E[X+Y]=\int_1^2\int_1^y (x+y)\frac{12}{25}\left(x+y^2\right)dxdy=\int_1^2\int_1^y \frac{12}{25}\left(x^2+y^2x+xy+y^3\right)dxdy$$

$$=\frac{12}{25}\int_1^2\left(\frac{x^3}{3}+\frac{y^2x^2}{2}+\frac{x^2y}{2}+y^3x\right)_1^y dy=\frac{12}{25}\int_1^2\left(\frac{y^3}{3}+\frac{y^2y^2}{2}+\frac{y^2y}{2}+y^3y-\left[\frac{1}{3}+\frac{y^2}{2}+\frac{y}{2}+y^3\right]\right)dy=$$

$$\frac{12}{25}\int_1^2\left(\frac{y^3}{3}+\frac{y^4}{2}+\frac{y^3}{2}+y^4-\frac{1}{3}-\frac{y^2}{2}-\frac{y}{2}-y^3\right)dy=\frac{12}{25}\int_1^2\left(\frac{-1y^3}{6}+\frac{3y^4}{2}-\frac{1}{3}-\frac{y^2}{2}-\frac{y}{2}\right)dy$$

$$=\frac{12}{25}\left[\frac{-1y^4}{24}+\frac{3y^5}{10}-\frac{1y}{3}-\frac{y^3}{6}-\frac{y^2}{4}\right]_1^2$$

$$=\frac{12}{25}\left[\frac{-1(2)^4}{24}+\frac{3(2)^5}{10}-\frac{2}{3}-\frac{(2)^3}{6}-\frac{(2)^2}{4}-\left\{\frac{-1}{24}+\frac{3}{10}-\frac{1}{3}-\frac{1}{6}-\frac{1}{4}\right\}\right]=3.084$$

126. Let X and Y be continuous random variables with a joint density function of:

$$f_{X,Y}(x,y) = 0.25, \quad \text{for } 0 \le x \le 2, \;\; x-2 \le y \le x \qquad \text{and 0 elsewhere}$$

Calculate $E[X^3Y]$.

Solution:

$$E[X^3Y] = \int_0^2 \int_{x-2}^{x} 0.25x^3 y \cdot dydx = \int_0^2 \left[0.25x^3 \frac{y^2}{2} \right]_{x-2}^{x} dx = \int_0^2 \left[0.25x^3 \frac{x^2}{2} - 0.25x^3 \frac{(x-2)^2}{2} \right] dx$$

$$= \int_0^2 \left[\frac{x^5}{8} - \frac{x^3}{4} \frac{(x^2-4x+4)}{2} \right] dx = \int_0^2 \left[\frac{(x^4-x^3)}{2} \right] dx = \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^2 = \frac{6}{5}$$

127. You are given that the Random Variable X can have values of 1, 2, 3 and random variable Y can have values of 1, 2, 3, 4. Further, the table below lists the joint probability mass function.

	X		
Y	1	2	3
1	1/10	1/20	1/5
2	0	1/10	1/20
3	1/20	1/10	1/10
4	3/20	0	1/10

a. Calculate $E[X+Y]$.

Solution:

$$E[X+Y] = \sum\sum (x+y)p(x,y)$$

$$= (1+1)(0.1) + (2+1)(0.05) + (3+1)(0.2) + (2+2)(0.1) + (3+2)(0.05) + (1+3)(0.05)$$
$$+(2+3)(0.1) + (3+3)(0.1) + (1+4)(0.15) + (3+4)(0.1) = 4.55$$

b. Calculate $E[Min(X,Y)]$.

Solution:

$$E[Min(x,y)] = \sum\sum Min(x,y)\, p(x,y)$$

$$= (1)(0.1) + (1)(0.05) + (1)(0.2) + (2)(0.1) + (2)(0.05) + (1)(0.05)$$
$$+ (2)(0.1) + (3)(0.1) + (1)(0.15) + (3)(0.1) = 1.65$$

128. The profit on a new product is given by $Z = 3X - Y - 5$. Furthermore, you are given that X and Y are independent with $Var[X] = 1$ and $Var[Y] = 2$.

Calculate the $Var[Z]$.

Solutions:

$$Var[Z] = Var[3X - Y - 5] = 3^2 Var[X] + (-1)^2 Var[Y] = (9)(1) + (1)(2) = 11$$

129. The profit on a new product is given by $Z = 5X - 2Y - 3$. Furthermore, you are given that $Var[X] = 1$, $Var[Y] = 2$, and $Cov[X,Y] = -1$.

a. Calculate the $Var[Z]$.

Solutions:

Because of dependence,

$$Var[Z] = Var[5X - 2Y - 3] = 5^2 Var[X] + (-2)^2 Var[Y] + (2)(5)(-2) Cov[X,Y]$$

$$= (25)(1) + (4)(2) - 20(-1) = 53$$

b. Calculate $\rho(X,Y)$.

$$\rho(X,Y) = \frac{Cov[X,Y]}{\sigma_X \cdot \sigma_Y} = \frac{-1}{\sqrt{1} \cdot \sqrt{2}} = \frac{-1}{\sqrt{2}}$$

130. The continuous random variables X and Y have a joint density function of

$$f_{X,Y}(x,y) = 6x, \quad \text{for } 0 < x < y < 1 \quad \text{and} \quad 0 \text{ elsewhere.}$$

You are also given that $E[X]=1/2$ and $E[Y]=3/4$.

Calculate $Cov[X,Y]$.

Solution:

$$Cov[X,Y] = E[XY] - E[X]E[Y] = E[XY] - \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)$$

$$E[XY] = \int_0^1\int_0^y xyf_{X,Y}(x,y)dxdy = \int_0^1\int_0^y xy(6x)dxdy = \int_0^1\int_0^y 6x^2 ydxdy =$$

$$\int_0^1 \left[2x^3 y\right]_0^y dy = \int_0^1 \left[2y^4\right]dy = \left[\frac{2y^5}{5}\right]_0^1 = \frac{2}{5}$$

$$Cov[X,Y] = \frac{2}{5} - \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \frac{1}{40}$$

131. The continuous random variables X and Y have a joint density function of

$$f_{X,Y}(x,y) = \frac{8xy}{3}, \quad \text{for } 0 \le x \le 1 \text{ and } x < y < 2x \quad \text{and} \quad 0 \text{ elsewhere.}$$

Calculate $Cov[X,Y]$.

Solution:

$$E[X] = \int_0^1 \int_x^{2x} x f_{X,Y}(x,y)dydx = \int_0^1 \int_x^{2x} \frac{8}{3}x^2 ydydx = \int_0^1 \left[\frac{4}{3}x^2 y^2\right]_x^{2x} dx =$$

$$\int_0^1 \left[\frac{4}{3}x^2(4x^2 - x^2)\right]dx = \int_0^1 4x^4 dx = \left[\frac{4x^5}{5}\right]_0^1 = \frac{4}{5}$$

$$E[Y] = \int_0^1 \int_x^{2x} y f_{X,Y}(x,y)dydx = \int_0^1 \int_x^{2x} \frac{8}{3}xy^2 dydx = \int_0^1 \left[\frac{8}{9}xy^3\right]_x^{2x} dx =$$

$$\int_0^1 \left[\frac{8}{9}x(8x^3 - x^3)\right]_x^{2x} dx = \int_0^1 \left[\frac{56}{9}x^4\right]dx = \left[\frac{56}{45}x^5\right]_0^1 = \frac{56}{45}$$

$$E[XY] = \int_0^1 \int_x^{2x} xy f_{X,Y}(x,y)dydx = \int_0^1 \int_x^{2x} \frac{8}{3}x^2 y^2 dydx = \int_0^1 \left[\frac{8}{9}x^2 y^3\right]_x^{2x} dx =$$

$$\int_0^1 \left[\frac{8}{9}x^2(8x^3 - x^3)\right]_x^{2x} dx = \int_0^1 \left[\frac{56}{9}x^5\right]dx = \left[\frac{56}{54}x^6\right]_0^1 = \frac{56}{54}$$

$$Cov[X,Y] = E[XY] - E[X]\cdot E[Y] = \frac{56}{54} - \left(\frac{4}{5}\right)\left(\frac{56}{45}\right) = 0.0415$$