EDUCATION COMMITTEE

OF THE

SOCIETY OF ACTUARIES

LONG-TERM ACTUARIAL MATHEMATICS STUDY NOTE

LONG TERM ACTUARIAL MATHEMATICS SUPPLEMENTARY NOTE

by

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Long Term Actuarial Mathematics
Supplementary Note
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Contents

Preface

Introduction
This note is provided as an accompaniment to the second edition of *Actuarial Mathematics for Life Contingent Risks* (AMLCR), by Dickson, Hardy and Waters (2013, Cambridge University Press).

AMLCR includes almost all of the material required to meet the learning objectives developed by the SOA for the Long Term Actuarial Mathematics exam which will be offered from Fall 2018. In this note we aim to provide additional material required to meet some of the newer learning objectives. This note is designed to be read in conjunction with AMLCR, and we reference section and equation numbers from that text. We expect that this material will be integrated with the text formally in a third edition.

The SUSM and SSSM used in this note refer to the standard ultimate and select models defined and used in AMLCR.

Acknowledgements
I would like to thank David Dickson and Howard Waters, my AMLCR co-authors, for innumerable hours of lively discussion about actuarial mathematics.

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Jeff Beckley, Ken Bonvallet and Steve White offered valuable information, in particular with respect to details of US practice and terminology. Jessica Ou Dang has provided invaluable and careful research and editorial assistance.

None of these brilliant and careful people bears any responsibility for any errors or omissions in this work.

Edits and Corrections
Some typos and minor edits have been incorporated in this version.

1. In Section 2.1, we have corrected a typo in equations (8) and (9), replacing '$n-\frac{k}{12}$' in the subscripts with '$n-\frac{k+1}{12}$ The numerical calculations are correct.

2. In Example 2.6 the table values for $A_x^{(12)^{03}}$ have been changed, with consequent changes in the solutions.

3. In Example 3.1 we have added a sentence clarifying that the CI diagnosis lump sum is not paid if the life is diagnosed and dies within a single month. We have also made small corrections to the $A_x^{(12)}$ functions and consequently to the solutions.

4. In Example 4.7 we have corrected the specification of $K_{2017}^{(2)} = 0.01$ in the first line. The calculations are correct.

1 Long term coverages in health insurance

1.1 Disability income insurance (DII)

Disability income insurance, also known as income protection insurance, is designed to replace income for individuals who cannot work, or cannot work to full capacity due to sickness or disability. Typically, level premiums are payable at regular intervals through the term of the policy, but are suspended during periods of disability. Benefits are paid at regular intervals during periods of disability. The benefits are usually related to the policyholder's salary, but to encourage the policyholder to return to work as soon as possible, the payments are generally capped at 50-70% of the salary that is being replaced. The policy could continue until the insured person reaches retirement age.

Common features or options of disability income insurance include the following.

- The **waiting period** or **elimination period** is the time between the beginning of a period of disability and the beginning of the benefit payments. Policyholders select a waiting period from a list offered by the insurer, with typical periods being 30, 60, 180 or 365 days.
- The payment of benefits based on **total disability** requires the policyholder to be unable to work at their usual job, and to be not working at a different job. Medical evidence of the disability is also required at intervals.
- If the policyholder can do some work, but not at the full earning capacity established before the period of sickness, they may be eligible for a lower benefit based on **partial disability**.
- The amount of disability benefits payable may be reduced if the policyholder receives disability related income from other sources, for example from workers compensation or from a government benefit program.
- The benefit payment term is selected by the policyholder from a list of options. Typical terms are two years, five years, or up to age 65. Once the disability benefit comes into payment, it will continue to the earlier of the recovery of the policyholder to full health, or the end of the selected benefit term, or the death of the policyholder. If the policyholder moves from full disability to partial disability, then the benefit payments may be decreased, but the total term of benefit payment (covering the full and partial benefit periods) could be fixed.

  For shorter benefit terms, the policy covers each separate period of sickness, so even if the full benefit term of, say, two years has expired, if the policyholder later becomes disabled again, provided sufficient time has elapsed between periods of sickness, the benefits would be payable again for another period of two years.

- Where two periods of disability occur with only a short interval between them, they may be treated as a single period of sickness for determining the benefit payment term. The **off period** determines the required interval for two periods of disability to be considered separately rather than together. It is set by the insurer.

  For example, suppose a policyholder purchases DII with a two year benefit term, monthly benefit payments, and a two-month elimination period. The insurer sets the off period at six months. The policyholder becomes sick on 1 January 2017, and remains sick until 30 June 2017. She returns to work but suffers a recurrence of the sickness on 1 September 2017.

  The first benefit payment would be made at the end of the elimination period, on 1 March 2017, and would continue through to 30 June. Since the recurrence occurs within the 6-month off period, the second period of sickness would be treated as a continuation of the first. That means that the policyholder would not have to wait another two months to receive the next payment, and it also means that on 1 September, four months of the 24-month benefit term would have expired, and the benefits would continue for another 20 months, or until earlier recovery.

- '**Own job**' or '**any job**': the definition of total disability may be based on the policyholder's inability to perform their own job, or on their ability to perform any job that is reasonable given the policyholder's qualifications and experience. Clearly the latter is a more comprehensive definition, and a policy that pays benefits only if the policyholder is unable to perform any job should be considerably cheaper than one that pays out when the policyholder is unable to do her/his own job.

- DII insurance may be purchased as a group insurance by an employer, to offset the costs of paying long term disability benefits to the employees. Group insurance rates (assuming employees cannot opt out) may be lower than the equivalent rates for individuals, because the group policies carry less risk from adverse selection. There are also economies of scale, and less risk of non-payment of premiums from group policies.

- Long term disability benefits may be increased in line with inflation.

- Policies often include additional benefits such as '**return to work assistance**' which offsets costs associated with returning to work after a period of disability; for example, the policyholder may need some re-training, or it may be appropriate for the policyholder to phase their return to work by working part-time initially. It is in the insurer's interests to ensure the return to work is as smooth and as successful as possible for the policyholder.

1.2 Long Term Care Insurance (LTC)

1.2.1 LTC in the USA and Canada

In a typical LTC contract, premiums are paid regularly while the policyholder is well. When the policyholder requires care, based on the benefit triggers defined in the policy, there is a **waiting period**, similar to the elimination period for DII; 90 days is typical. After this, the policy will pay benefits as long as the need for care continues, or until the end of the selected benefit payment period.

Common features or options associated with LTC insurance in the USA and Canada include the following.

- The trigger for the payment of benefits is usually described in terms of the Activities of Daily Living, or ADLs. There are six ADLs in common use;
  - Bathing
  - Dressing
  - Eating (does not include cooking).
  - Toileting (ability to use the toilet and manage personal hygiene).
  - Continence (ability to control bladder and bowel functions)
  - Transferring (getting in and out of a bed or chair).

  If the policyholder requires assistance to perform two or more of the ADLs, based on certification by a medical practitioner, then the LTC benefit is triggered, and the waiting period, if any, commences.

- There is often an alternative trigger based on severe cognitive impairment of the policyholder.

- Although the most common policy design uses two ADLs for the benefit trigger, some policies use three.

- At issue, the policyholder may select a definite term benefit period (typical options are between 2 years and 5 years), or may select an indefinite period, under which benefit payments continue as long as the trigger conditions apply.

- The benefit payments may be based on a reimbursement approach, under which the benefits are paid directly to the caregiving organisation, and cover the cost of providing appropriate care, up to a daily or monthly limit.

- Alternatively, the benefit may be based on a fixed annuity payable during the benefit period. The policyholder may have the flexibility to apply the benefit to whatever form of care is most suitable, but there is no guarantee that the annuity would be sufficient for the level of care required.

- The insurer may offer the option to have the payments, or payment limits, increase with inflation.
- The reimbursement type of benefit may cover different forms of care, including in-home care, delivered by visiting or live-in support workers, or residential care costs, under which the policyholder would move to a suitable residential long term care facility.
- Similarly to DII, an off-period, typically 6 months, is used to determine whether two successive periods of care are treated separately or as a single continuous period.
- Hybrid LTC and life insurance plans are becoming popular. There are different ways to combine the benefits. Under the 'return of premium' approach, if the benefits paid under the LTC insurance are less than the total of the premiums paid, the balance may be returned as part of the death benefit under the life insurance policy. Alternatively, the 'accelerated benefit' approach uses the sum insured under the life insurance policy to pay LTC benefits. If the policyholder dies before the full sum insured has been paid in LTC benefits, the balance is paid as a death benefit. The policyholder may add an *extension of benefits* option to the hybrid insurance, which would provide for the LTC benefits to continue for a pre-determined period after the original sum insured is exhausted. Typically, extension periods offered are in the range of two to five years.
- In the USA some LTC policies are tax-qualifying, which means that policyholders may deduct a portion of the premiums paid from taxable income when filing their tax returns. These policies have a trigger based on inability to perform two ADLs, or based on severe cognitive impairment, provided the disability is expected to last for at least a 90-day period.
- Premiums are designed to be level throughout the policy term, but insurers may retain the right to increase premiums for all policyholders if the experience is sufficiently adverse. Generally, insurers must obtain approval from the regulating body for such rate increases. In this circumstance, policyholders may be given the option to maintain the same premiums for a lower benefit level. Another feature that may be invoked by regulation is the *Conditional Benefit Upon Lapse*, under which policyholders who lapse their policies may use the net of all premiums paid less any paid claims as a single premium to purchase a new, paid-up LTC policy.

1.2.2 LTC in other countries

In this section we briefly describe how LTC insurance differs in some countries around the world.

LTC in France
LTC insurance is popular in France; in 2010 the market penetration (meaning the proportion of eligible adults with coverage) was higher than any other country. France provides a social security type benefit for LTC costs that is income-tested (those with high retirement income receive less benefit that those with lower income), and that is designed to cover a significant proportion of the cost of basic nursing or residential care. LTC insurance allows policyholders to supplement the government benefit.

Policies in France are much simpler and much cheaper than in the USA. Premiums in 2010 averaged around $450 per year, compared with over $2,200 in the USA[1]. Benefits are paid as a fixed or inflation indexed annuity. Policies are often purchased through group plans facilitated by employers, reducing the expenses. The policyholder may choose a policy based on 'mild or severe dependency' or one based on 'severe dependency' only. The severe dependency policy is more popular than the mild dependency type. Severe dependency is defined as *bed or chair bound, requiring assistance several times a day* or *cognitive impairment requiring constant monitoring.* Mild dependency refers to cases where the individual needs help with eating, bathing and/or some mobility, but is not bed or chair bound. Premiums in France are lower than the US partly because the average benefit (around $25 per day in 2010) is considerably less than the average payout from a US policy, where typical daily maximum reimbursement limits range from $100 to $200. Other relevant factors include the simpler contract terms, and the fact that individuals in France tend to purchase their policies at younger ages than in the USA.

LTC in Germany
In Germany basic LTC costs are covered under the government provided social health insurance. Individuals can top up the government benefit with private LTC insurance, or can opt out of the state benefit (and thereby opt out of the tax supporting the benefit) and use LTC insurance instead. The benefits are fixed annuities.

LTC in Japan
LTC insurance is provided in Japan on a stand-alone basis or as a rider on a whole life insurance policy. The benefit is payable as a lump sum or annuity, triggered when the policyholder reaches a specified level of dependency. There may be additional benefits payable when the level of dependency steps up.

LTC in the United Kingdom
In the UK regular premium LTC policies are no longer offered, as they never reached the necessary level of popularity for the business to be sustained.

In their place is a different kind of pre-funding, called an **immediate needs annuity**. This is a single premium immediate annuity that is purchased as the individual is about to move permanently into long term care, possibly funded from the proceeds of the sale of a property. The benefit is paid as a regular fixed annuity, but is paid directly to the care home, saving the policyholder from having to pay income tax on the proceeds. Because the lives are assumed to

[1] `www.ncbi.nlm.nih.gov/pmc/articles/PMC4462881/`.

be somewhat impaired, and the insurer's exposure to adverse selection with respect to longevity is reduced, the benefit amount per unit of single premium may be somewhat greater than a regular single premium life annuity.

1.3 Critical Illness (CII) Insurance

Critical illness insurance pays a lump sum benefit on diagnosis of one of a list of specified diseases and conditions. Different policies and insurers may cover slightly different illnesses, but virtually all include heart attack, stroke, major organ failure, and most forms of cancer. Policies may be whole life or for a definite term. Unlike DII or LTC insurance, once the claim arises, the benefit is paid and the policy expires. A second critical illness diagnosis would not be covered. Some policies offer a partial return of premium if the policy expires or lapses without a CI diagnosis.

Level premiums are typically paid monthly throughout the term, though they may cease at, say, 75 for a long term policy.

Critical illness cover may be added to a life insurance policy as an **accelerated benefit rider**. In this case, the critical illness diagnosis triggers the payment of some or all of the death benefit under the life insurance, with some discounting adjustment applied in some cases. Where the full benefit is accelerated, the policy expires on the CI diagnosis. If only part of the benefit is accelerated, then the remainder is paid out when the policyholder dies.

1.4 Chronic illness insurance

Chronic illness insurance pays a benefit on diagnosis of a chronic illness, defined as one from which the policyholder will not recover, although the illness does not necessarily need to be terminal. The illness must be sufficiently severe that the policyholder is no longer able to perform two or more of the ADLs listed in the LTC insurance section. The benefit under a chronic illness policy is paid as a lump sum or as an annuity.

Chronic illness insurance is typically added to a standard life insurance policy as an accelerated benefit rider, similarly to the critical illness case.

1.5 Hospital indemnity insurance (HII)

Hospital indemnity insurance pays the policyholder a lump sum each time the policyholder is admitted for hospital treatment. There may also be a daily stipend payable during a hospital stay. Other benefits may include payments for emergency room or outpatient visits that do not result in overnight admission.

The purpose of hospital indemnity insurance differs from standard health insurance, which provides reimbursement of health costs. Hospital indemnity insurance benefits are available for the policyholder to use however she wants – for example, to pay for child care or travel costs for visiting family. In the USA it can be used to offset uninsured costs associated with the hospital visit, for example, if the policyholder's health insurance cover requires the policyholder to pay some of the costs of the required treatment[2].

Premiums for HII increase each year, so the policies are essentially short term in nature. However, the insurers may guarantee renewal up to age 65, which means that the policyholder is not subject to annual medical assessment at each renewal date, and also that the premiums should be the same for a policyholder who has already made several claims under the policy as for a policyholder who has not.

1.6 Continuing Care Retirement Communities (CCRCs)

Continuing care retirement communities (CCRCs) are residential facilities for seniors, with different levels of medical and personal support designed to adapt to the residents as they age. Many CCRCs offer funding packages where the costs of future care are covered by a combination of an entry fee, and a monthly charge.

There are generally three or four categories of residence:

1. Independent Living Units (ILU) represent the first stage of residence in a CCRC. These are apartments with fairly minimal external care provided (for example, housekeeping, emergency call buttons, transport to shopping).
2. Assisted living units (ALU) allow more individual support for residents who need help with at least one, and commonly several of the activities of daily living. Most of the support at this level is non-medical – help with bathing, dressing, preparation of meals, etc.
3. The skilled nursing facility (SNF) is for residents who need ongoing medical care. The SNF often looks more like a hospital facility.
4. Memory care units (MCU) offer a separate, more secure facility for residents with severe dementia or other cognitive impairment.

The industry has developed different forms of funding for CCRCs. Not every CCRC will offer all funding options, and some will offer variants that are not described here, but these are the major forms in current use.

- Residents can choose to pay a large upfront fee, and monthly payments which are level, or which are only increasing with cost of living adjustments. The resident is guaranteed that all residential, personal assistance and health care needs will be covered without further cost. This is called a **full life care**, or life care, or **Type A contract**.

[2]Policies generally have deductibles or co-pay requirements, which mean that the full cost of health treatment is not covered by insurance.

- Under a **modified life care**, or **Type B contract** residents pay lower monthly fees, and possibly a lower entry fee, but will have to pay additional costs for some services if they need them. For example, a resident may be charged a higher monthly fee as he moves into the ALU, with further increases on entry to the SNF or the MCU. Typically, the increases would be less than the full market cost of the additional care, meaning that the costs are partially pre-funded through the entry fee and regular monthly payments.

- **Fee-for-service**, or **Type C contracts**, involve little or no pre-funding of health care. Residents pay for the health care they receive at the current market rates. Fee-for-service contracts have the lowest entry fee and monthly payments, as these only cover the accommodation costs.

- Prospective residents entering under a Type A or Type B contract must be sufficiently well to live independently when they enter the CCRC, and a medical exam is generally required. Entrants who are already sufficiently disabled to need more care are only eligible for Type C contracts.

- Under a Type A or B contract the CCRC may offer a partial refund of the entry fee on the resident's death or when the resident moves out. This may involve some options, for example, the resident can choose a higher entry fee with a partial refund, or a lower entry fee with no refund.

- There are some CCRCs that offer (partial) ownership of the ILU, in place of some or all of the entry fee. When the resident moves out of independent living permanently, or dies, the unit is sold, with the proceeds shared between the resident (or her estate) and the CCRC.

- It is common for couples to purchase CCRC membership jointly, and different payment schedules may be applied to couples as to single residents entering the CCRC.

The average age at entry to a CCRC in the USA is around 80, with Type A entrants generally being younger than Type B, who are younger than Type C, on average.

The Type A and (to a lesser extent) Type B contracts transfer the risk of increasing health care costs from the resident to the CCRC, and therefore are a form of insurance.

2 Multiple state models for long term health and disability insurance

All the long term health related insurances described in Section 1 can be modelled using the multiple state modelling framework described in AMLCR Chapter 8. In fact, several already appear in AMLCR. The disability income insurance described above is one of the examples used throughout Chapter 8; Exercises 8.7 and 8.11 in AMLCR are examples of multiple state models applied to critical illness insurance, and the permanent disability model described in AMLCR Chapter 8, is essentially the same model as we would use for the chronic illness insurance.

In this section we will consider more explicitly how we can use multiple state models for the individual long term health coverages described in Section 1. We assume that the reader has already mastered the material in AMLCR Chapter 8. What follows are additional Chapter 8 examples, using the long term health coverages for context. We also describe some different ways to value cash flows, to allow for complications such as waiting periods and discrete payment models.

2.1 Disability Income Insurance

DII is covered fairly extensively in AMLCR, specifically in Sections 8.2.4 and 8.7.1. The DII model is illustrated in Figure 8.4 in AMLCR, and we repeat it here for convenience.

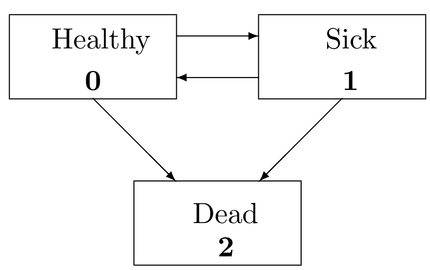


Figure 1: The disability income insurance model.

Under this model, an n-year DII policy written on a healthy life age x, with premiums of P per year payable continuously while healthy, and a benefit of B per year payable continuously while disabled, has equation of value at issue

$$P \int_0^n {}_tp_x^{00}\, e^{-\delta t} dt = B \int_0^n {}_tp_x^{01}\, e^{-\delta t} dt \tag{1}$$

$$\text{that is, } P\bar{a}^{00}_{x:\overline{n}|} = B\bar{a}^{01}_{x:\overline{n}|} \tag{2}$$

and this easily generalises to the discrete, monthly payment case,

$$\frac{P}{12}\left(1 + {}_{\frac{1}{12}}p_x^{00}\, v^{\frac{1}{12}} + {}_{\frac{2}{12}}p_x^{00}\, v^{\frac{2}{12}} + \cdots + {}_{n-\frac{1}{12}}p_x^{00}\, v^{n-\frac{1}{12}}\right)$$

$$= \frac{B}{12}\left({}_{\frac{1}{12}}p_x^{01}\, v^{\frac{1}{12}} + {}_{\frac{2}{12}}p_x^{01}\, v^{\frac{2}{12}} + \cdots + {}_{n}p_x^{01}\, v^{n}\right) \tag{3}$$

$$\text{that is, } P\ddot{a}^{(12)\,00}_{x:\overline{n}|} = Ba^{(12)\,01}_{x:\overline{n}|} \tag{4}$$

This formulation does not include an allowance for the waiting time between the onset of disability and the payment of benefits. We can adapt our results to exclude the waiting period from the benefit payment period, but we can no longer use the neat annuity formulation in equations (1) and (3) for the benefit valuation, as we need to allow for the length of time of disability. It is simplest to do this in the continuous payment case, and we will start by considering a different derivation of the annuity value $\bar{a}^{01}_{x:\overline{n}|}$.

In equation (1), we evaluate $\bar{a}^{01}_{x:\overline{n}|}$ by integrating over all the possible payment dates between time 0 and time n. We can find the same annuity value by integrating over all the possible dates of transition from State 0 (Healthy) to State 1 (Sick), and then valuing the benefit that starts at that transition and ends on the next transition out of State 1, or on the earlier expiry of the contract.

It is helpful first to define the EPV of a **continuous sojourn annuity**. We define $\bar{a}^{\overline{ii}}_{x:\overline{n}|}$ to be the EPV of a continuous payment of 1 per year paid to a life currently age x and in state i, where the payment continues as long as the life remains in state i, or until the expiry of the n year term if earlier. The annuity ceases if the life leaves state i, even if she subsequently returns to it. That is

$$\bar{a}^{\overline{ii}}_{x:\overline{n}|} = \int_0^n {}_tp_x^{\overline{ii}}\, e^{-\delta t} dt \tag{5}$$

So, for the annuity valued by integrating over transition times we consider each infinitesimal interval $(t, t+dt)$, and take the product of the three components:

- The probability that the life transitions from healthy to sick in the interval from t to $t+dt$: ${}_tp_x^{00}\, \mu^{01}_{x+t} dt$
- The value at t of an annuity of 1 per year paid for the continuous period of sickness starting at time t and ending at the earlier of the end of the sickness period and the expiry of the remaining $n-t$ years of the contract: $\bar{a}^{\overline{11}}_{x+t:\overline{n-t}|}$.
- A discount factor to bring values back from the start of the benefit payment period, at time t, to present values: $e^{-\delta t}$

So we have

$$\bar{a}^{01}_{x:\overline{n}|} = \int_0^n {}_tp^{00}_x\, \mu^{01}_{x+t}\, \bar{a}^{\overline{11}}_{x+t:\overline{n-t}|}\, e^{-\delta t} dt \tag{6}$$

Now, the reason for doing this is it enables us to adjust the annuity value to allow for an elimination or waiting period. Suppose we are valuing a disability benefit with a waiting period of w years – that is, once the life becomes sick, she must wait for w years before receiving any benefit from that period of sickness. We can allow for this by subtracting the first w years of annuity from each period of sickness in equation (6).

That is, the EPV of a benefit of 1 paid continuously while sick to a life currently age x and healthy, with a waiting period or w years, and a policy term of $n > w$ years is

$$\int_0^{n-w} {}_tp^{00}_x\, \mu^{01}_{x+t} \left(\bar{a}^{\overline{11}}_{x+t:\overline{n-t}|} - \bar{a}^{\overline{11}}_{x+t:\overline{w}|} \right) e^{-\delta\, t} dt \tag{7}$$

Note that the term in parentheses in equation (7) is the expected present value of the w-year deferred, continuous sojourn sickness annuity starting at time t. Also, note the upper limit of integration; we do not need to consider any sickness periods that start within w years of the end of the term of the contract, because the policy will expire before the waiting period ends.

This approach can be adapted for discrete time payments. For example, if the benefit payments are monthly, and assuming all other terms are as in equation (7), then we sum over each month of possible transition from healthy to sick, recalling that the final benefit payment date is time n, as the benefit is paid at the end of each month, giving the EPV of the benefit payment of 1 per year, payable monthly, as

$$\sum_{k=0}^{12(n-w-\frac{1}{12})} {}_{\frac{k}{12}}p^{00}_x \; {}_{\frac{1}{12}}p^{01}_{x+\frac{k}{12}} \left(\ddot{a}^{(12)\overline{11}}_{x+\frac{k+1}{12}:\overline{n-\frac{k+1}{12}}|} - \ddot{a}^{(12)\overline{11}}_{x+\frac{k+1}{12}:\overline{w}|} \right) v^{\frac{k+1}{12}} \tag{8}$$

We can also use equation (7) to construct the value of a DII policy with a maximum payment term for each period of disability of, say, m years after the waiting period. We replace the term of the first continuous sojourn annuity in the equation with an $m + w$-year term annuity, unless the transition happens within $m + w$ years of the end of the contract, giving a valuation of

$$\int_0^{n-(m+w)} {}_tp^{00}_x\, \mu^{01}_{x+t} \left(\bar{a}^{\overline{11}}_{x+t:\overline{m+w}|} - \bar{a}^{\overline{11}}_{x+t:\overline{w}|} \right) e^{-\delta\, t} dt$$

$$+ \int_{n-(m+w)}^{n-w} {}_tp^{00}_x\, \mu^{01}_{x+t} \left(\bar{a}^{\overline{11}}_{x+t:\overline{n-t}|} - \bar{a}^{\overline{11}}_{x+t:\overline{w}|} \right) e^{-\delta\, t} dt$$

The sums and integrals in this section can easily be evaluated numerically, using the techniques described in AMLCR.

Example 2.1
In AMLCR Examples 8.5 and 8.6, probabilities and premiums are calculated for a 10-year DII policy with a death benefit of 50 000 payable immediately on death, and a disability income benefit of 20 000 payable monthly in arrear whilst disabled, issued to a healthy life aged 60. Consider the same policy, and assume monthly premiums and disability benefits. Calculate the revised premium assuming a waiting period of (a) 1 month (b) 3 months (c) 6 months and (d) 1 year.

Solution 2.1
From AMLCR we have that the death benefit has expected present value 8115.5.

We also have:

$$\ddot{a}^{(12)\,00}_{60:\overline{10}|} = 6.5980 \qquad a^{(12)\,01}_{60:\overline{10}|} = 0.66877$$

Using Excel, we can calculate the annuity value taking the waiting period into consideration by summing the terms in equation (8). It is simplest to use the following slightly different formulation, to avoid changing the limits of the sum for different waiting periods.

$$\sum_{k=0}^{12(n-\frac{1}{12})} {}_{\frac{k}{12}}p_x^{00}\ {}_{\frac{1}{12}}p_{x+\frac{k}{12}}^{01} \left(\ddot{a}^{(12)\,\overline{11}}_{x+\frac{k+1}{12}:\overline{n-\frac{k+1}{12}}|} - \ddot{a}^{(12)\,\overline{11}}_{x+\frac{k+1}{12}:\overline{\min(w,n-\frac{k+1}{12})}|} \right) v^{\frac{k+1}{12}} \tag{9}$$

The table below gives values for the EPV at issue of a disability benefit of 1 per year, payable monthly, for a 10-year policy issued to (60), with parameters given in AMLCR Examples 8.4 and 8.5, and also the associated premiums.

Waiting Period	EPV of benefit of 1 per year	Premium P
0 months	0.6688	3257.20
1 month	0.6539	3212.22
3 months	0.6252	3125.22
6 months	0.5839	2999.83
1 year	0.5070	2766.76

2.2 Long Term Care

The form of the multiple state model used for insurance valuation should always be adapted to the cash flows of the policy. For LTC insurance, we will use different models depending on whether the benefit reimburses the cost of care, or pays a predetermined annuity, possibly with inflation protection.

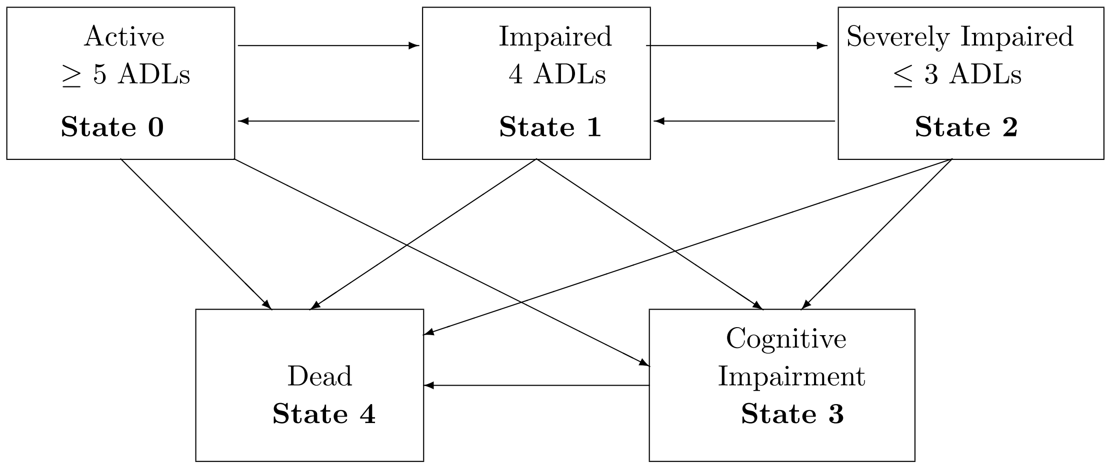


Figure 2: Example of an LTC insurance model.

For a reimbursement policy, the severity of the disability will impact the level of benefit, so there is value in using different states to model different levels of disability. For example, we could use the model illustrated in Figure 2, where the number of ADLs which a policyholder is able to manage acts as a marker for the expected amount of reimbursement, and we separately model the cognitive impairment state. The figure is more complicated than those in AMLCR Chapter 8, and could be more complicated still, for example, if we allow for recovery from cognitive impairment, or allow for simultaneous loss of more than one ADL. However, the principles from AMLCR Chapter 8 apply to this figure, and all the required probabilities and actuarial functions can be evaluated numerically.

Example 2.2
Write down the Kolmogorov forward equations for all the probabilities for a life age x, currently in State 2, for the model in Figure 2, and give boundary conditions. Assume the usual assumptions for Markov multiple state models apply.

Solution 2.2

$$\frac{d}{dt}\,{}_tp_x^{20} = {}_tp_x^{21}\,\mu_{x+t}^{10} - {}_tp_x^{20}\left(\mu_{x+t}^{01} + \mu_{x+t}^{03} + \mu_{x+t}^{04}\right)$$

$$\frac{d}{dt}\,{}_tp_x^{21} = {}_tp_x^{20}\,\mu_{x+t}^{01} + {}_tp_x^{22}\,\mu_{x+t}^{21} - {}_tp_x^{21}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14}\right)$$

$$\frac{d}{dt}\,{}_tp_x^{22} = {}_tp_x^{21}\,\mu_{x+t}^{12} - {}_tp_x^{22}\left(\mu_{x+t}^{21} + \mu_{x+t}^{23} + \mu_{x+t}^{24}\right)$$

$$\frac{d}{dt}\,{}_tp_x^{23} = {}_tp_x^{20}\,\mu_{x+t}^{03} + {}_tp_x^{21}\,\mu_{x+t}^{13} + {}_tp_x^{22}\,\mu_{x+t}^{23} - {}_tp_x^{23}\,\mu_{x+t}^{34}$$

$$\frac{d}{dt}\,{}_tp_x^{24} = {}_tp_x^{20}\,\mu_{x+t}^{04} + {}_tp_x^{21}\,\mu_{x+t}^{14} + {}_tp_x^{22}\,\mu_{x+t}^{24} + {}_tp_x^{23}\,\mu_{x+t}^{34}$$

For boundary conditions, we have ${}_0p_x^{22} = 1$ and ${}_0p_x^{2j} = 0$ for $j \neq 2$.

As a check, verify that the sum of all the terms in all the differential equations with the same starting state is 0.

Example 2.3
Write down Thiele's equation for $\bar{a}_{x+t}^{03}$.

Solution 2.3
Thiele's equation applies to policy values, but we can apply it here because we can view $\bar{a}_{x+t}^{03}$ as the policy value for a single premium annuity contract with benefit of 1 per year payable continuously in State 3, given that the life is in State 0 currently.

The general form of Thiele's equation for multiple state models is given in AMLCR Equation (8.23), which we repeat here for convenience. Recall that ${}_tV^{(i)}$ is the policy value for a generic fully continuous insurance, conditional on being in State i at time t.

For $i = 0, 1, \ldots, n$ and $0 < t < n$,

$$\boxed{\frac{d}{dt}\,{}_tV^{(i)} = \delta_t\,{}_tV^{(i)} - B_t^{(i)} - \sum_{j=0,\ j\neq i}^{n} \mu_{x+t}^{ij}\left(S_t^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)}\right).} \tag{10}$$

where δ_t is the force of interest function, $B_t^{(i)}$ is the benefit payable continuously while the life is in State i (premiums payable continuously are treated as negative benefits), and $S_t^{(ij)}$ is the lump sum paid immediately on transition from State i to State j.

In the case considered in this example, we have $i = 0$, $S_t^{(ij)} = 0$ and $B_t^{(0)} = 0$, as there are no transition benefits, and no annuity in State 0. Plugging this into Thiele's equation we have

$$\frac{d}{dt}{}_tV^{(0)} = \delta\,{}_tV^{(0)} - \mu_{x+t}^{01}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu_{x+t}^{03}\left({}_tV^{(3)} - {}_tV^{(0)}\right) - \mu_{x+t}^{04}\left(-{}_tV^{(0)}\right)$$

Now ${}_tV^{(0)} = \bar{a}_{x+t}^{03}$, and similarly ${}_tV^{(1)} = \bar{a}_{x+t}^{13}$, and ${}_tV^{(3)} = \bar{a}_{x+t}^{33}$. The policy expires when the life moves into State 4, so we have ${}_tV^{(4)} = 0$. We do not include State 2, since it is impossible to move into State 2 from State 0. Then

$$\frac{d}{dt}\bar{a}^{03}_{x+t} = \delta\,\bar{a}^{03}_{x+t} - \mu^{01}_{x+t}\left(\bar{a}^{13}_{x+t} - \bar{a}^{03}_{x+t}\right) - \mu^{03}_{x+t}\left(\bar{a}^{33}_{x+t} - \bar{a}^{03}_{x+t}\right) - \mu^{04}_{x+t}\left(-\bar{a}^{03}_{x+t}\right)$$

$$= \bar{a}^{03}_{x+t}\left(\delta + \mu^{01}_{x+t} + \mu^{03}_{x+t} + \mu^{04}_{x+t}\right) - \mu^{01}\,\bar{a}^{13}_{x+t} - \mu^{03}\,\bar{a}^{33}_{x+t}$$

Using simultaneous Thiele equations for the different state dependent annuities, we can solve numerically to determine values for all relevant $\bar{a}^{ij}_{x+t}$.

Example 2.4
Consider an LTC policy issued to (x). Premiums of P per year are payable continuously while in State 0; benefits payable continuously in States 1, 2 and 3 are assumed to increase geometrically at rate g, convertible continuously, with starting values at inception of $B^{(j)}$ in state $j = 1, 2, 3$.

Write down, and simplify as far as possible the premium equation of value for the policy.

Solution 2.4

$$P\bar{a}^{00}_{x} = \int_0^\infty B^{(1)}e^{gt}\,{}_tp^{01}_x\,e^{-\delta\,t}dt + \int_0^\infty B^{(2)}e^{gt}\,{}_tp^{02}_x\,e^{-\delta\,t}dt + \int_0^\infty B^{(3)}e^{gt}\,{}_tp^{03}_x\,e^{-\delta\,t}dt$$

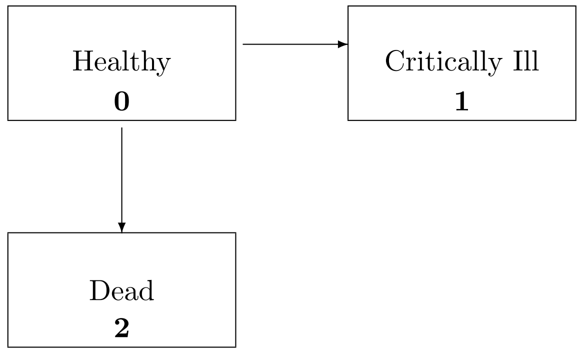
$$\Rightarrow P\bar{a}^{00}_{x} = B^{(1)}\,\bar{a}^{01}_{x}|_{\delta^*} + B^{(2)}\,\bar{a}^{02}_{x}|_{\delta^*} + B^{(3)}\,\bar{a}^{03}_{x}|_{\delta^*}$$

where the annuities on the right hand side are evaluated at a force of interest $\delta^* = \delta - g$.

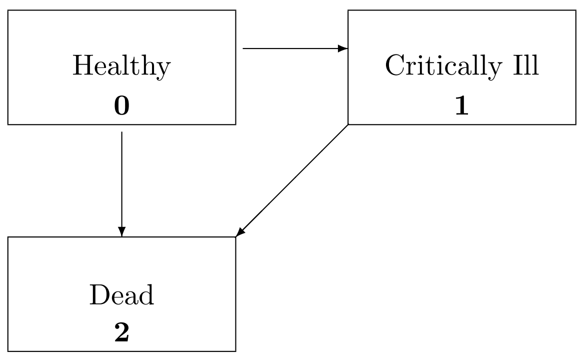
2.3 Critical Illness Insurance

We illustrate some possible models for CII insurance in Figure 3. If the CII is a stand alone policy, with a benefit on CII diagnosis, but with no death benefit, then we could use the model illustrated in Figure 3a. We can use the same model if the CII accelerates the death benefit in full. In both cases the policy expires on the earlier of the CII diagnosis or death, If there is an additional death benefit that is payable in the same amount, whether or not there is a CII diagnosis preceding, then we could use the model illustrated in Figure 3b, because in this case the policy expires on the policyholder's death, but it doesn't make a difference to the death benefit whether the life dies from State 0 or from State 1. Finally, if the CII partially accelerates the death benefit, then we could use the model in Figure 3c, which separates the case where death occurs without a preceding CII diagnosis, and the case where death occurs after a CII diagnosis.

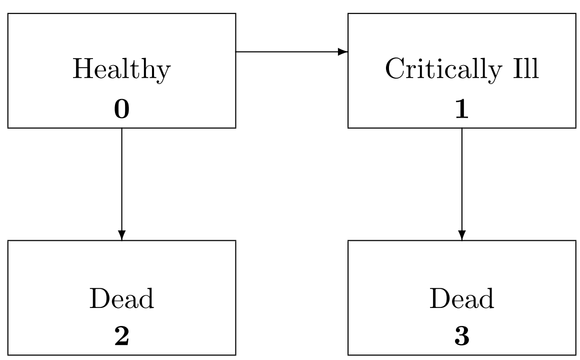
As Figure 3c is the most general form of the model, it could be used for any of the different CII forms described. The simpler models (a) and (b) in Figure 3 can't be used for the accelerated benefit case.



(a) CII1 Model 1



(b) CII Model 2



(c) CII Model 3

Figure 3: CII Models

Example 2.5

(a) Using Model 3 in Figure 3, write down the equations of value for the premiums for the following CII policies, in terms of the actuarial functions $\bar{A}^{ij}_{x:\overline{n}|}$ and $\bar{a}^{ij}_{x:\overline{n}|}$. Assume in each case that the policy is issued to a healthy life aged 50, that premiums are payable continuously while in State 0, and that all contracts are fully continuous, expiring on the policyholder's 70th birthday.

(i) A stand alone CII policy with benefit \$20,000 paid immediately on CII diagnosis.

(ii) A combined CII and life insurance policy that pays \$20,000 on CII diagnosis and \$10,000 on death.

(iii) An accelerated death benefit CII policy that pays \$20,000 immediately on the earlier of CII diagnosis and death.

(iv) A partly accelerated death benefit policy, which pays \$20,000 on CII diagnosis, and pays \$30,000 if the policyholder dies without a CII claim, or \$10,000 if the policyholder dies after a CII claim.

(b) Use the functions given in the tables below to calculate the annual rate of premium for each of the policies described in (a). The effective rate of interest is 5% per year.

(c) Use the functions in the tables below to calculate the policy value at $t = 10$ for each of the policies described in (a), assuming (i) the life is in State 0 at time 10 or (ii) the life is in State 1 at time 10.

x	$\bar{a}^{00}_x$	$\bar{A}^{01}_x$	$\bar{A}^{02}_x$	$\bar{A}^{03}_x$	$\bar{A}^{13}_x$
50	13.31267	0.22409	0.12667	0.14176	0.34988
60	10.17289	0.34249	0.16140	0.22937	0.47904
70	6.56904	0.49594	0.18317	0.36019	0.62237

t	${}_{20-t}p^{00}_{50+t}$	${}_{20-t}p^{01}_{50+t}$	${}_{20-t}p^{02}_{50+t}$	${}_{20-t}p^{03}_{50+t}$	${}_{20-t}p^{11}_{50+t}$
0	0.68222	0.15034	0.13788	0.02956	0.66485
10	0.75055	0.13135	0.09943	0.01867	0.75283

Solution 2.5

(a)

(i) $P\,\bar{a}^{00}_{50:\overline{20}|} = 20\,000\bar{A}^{01}_{50:\overline{20}|}$

(ii) $P\,\bar{a}^{00}_{50:\overline{20}|} = 20\,000\bar{A}^{01}_{50:\overline{20}|} + 10000\left(\bar{A}^{02}_{50:\overline{20}|} + \bar{A}^{03}_{50:\overline{20}|}\right)$

(iii) $P\,\bar{a}^{00}_{50:\overline{20}|} = 20\,000\left(\bar{A}^{01}_{50:\overline{20}|} + \bar{A}^{02}_{50:\overline{20}|}\right)$

(iv) $P\,\bar{a}^{00}_{50:\overline{20}|} = 20\,000\,\bar{A}^{01}_{50:\overline{20}|} + 30\,000\,\bar{A}^{02}_{50:\overline{20}|} + 10\,000\,\bar{A}^{03}_{50:\overline{20}|}$

(b) We need to calculate the 20 year actuarial functions.

$$\bar{a}^{00}_{50:\overline{20}|} = \bar{a}^{00}_{50} - {}_{20}p^{00}_{50}\,v^{20}\,\bar{a}^{00}_{70} = 11.6236$$

$$\bar{A}^{01}_{50:\overline{20}|} = \bar{A}^{01}_{50} - {}_{20}p^{00}_{50}\,v^{20}\,\bar{A}^{01}_{70} = 0.09657$$

$$\bar{A}^{02}_{50:\overline{20}|} = \bar{A}^{02}_{50} - {}_{20}p^{00}_{50}\,v^{20}\,\bar{A}^{02}_{70} = 0.07957$$

$$\bar{A}^{03}_{50:\overline{20}|} = \bar{A}^{03}_{50} - {}_{20}p^{00}_{50}\,v^{20}\,\bar{A}^{03}_{70} - {}_{20}p^{01}_{50}\,v^{20}\,\bar{A}^{13}_{70} = 0.01388$$

Note that in the last case, we need to subtract the value of the benefit on transition to State 3 paid after age 70, where the life is in State 0 at age 70, as well as the value of the benefit paid on transition to State 3 after age 70 where the life is in State 1 at age 70.

Using the equations and functions above, we have premiums for each case as follows. Since we will need the premiums for part (c), we use the superscripts to connect the premiums to the different contracts.

(i) $P^{(i)} = 166.16$

(ii) $P^{(ii)} = 246.56$

(iii) $P^{(iii)} = 303.07$

(iv) $P^{(iv)} = 383.47$

(c) We will need the following actuarial functions:

$$\bar{a}^{00}_{60:\overline{10}|} = \bar{a}^{00}_{60} - {}_{10}p^{00}_{60}\,v^{10}\,\bar{a}^{00}_{70} = 7.14606$$

$$\bar{A}^{01}_{60:\overline{10}|} = \bar{A}^{01}_{60} - {}_{10}p^{00}_{60}\,v^{10}\,\bar{A}^{01}_{70} = 0.11397$$

$$\bar{A}^{02}_{60:\overline{10}|} = \bar{A}^{02}_{60} - {}_{10}p^{00}_{60}\,v^{10}\,\bar{A}^{02}_{70} = 0.07700$$

$$\bar{A}^{03}_{60:\overline{10}|} = \bar{A}^{03}_{60} - {}_{10}p^{00}_{60}\,v^{10}\,\bar{A}^{03}_{70} - {}_{10}p^{01}_{60}\,v^{10}\,\bar{A}^{13}_{70} = 0.01322$$

$$\bar{A}^{13}_{60:\overline{10}|} = \bar{A}^{13}_{60} - {}_{10}p^{11}_{60}\,v^{10}\,\bar{A}^{13}_{70} = 0.19140$$

The policy values are

(i) $${}_{10}V^{(0)} = 20\,000\bar{A}^{01}_{60:\overline{10}|} - P^{(i)}\,\bar{a}^{00}_{60:\overline{10}|} = 1092.01$$

$${}_{10}V^{(1)} = 0 \text{ (or undefined, as the policy has expired)}$$

(ii) $${}_{10}V^{(0)} = 20\,000\bar{A}^{01}_{60:\overline{10}|} + 10\,000\left(\bar{A}^{02}_{60:\overline{10}|} + \bar{A}^{03}_{60:\overline{10}|}\right) - P^{(ii)}\,\bar{a}^{00}_{60:\overline{10}|} = 1419.67$$

$${}_{10}V^{(1)} = 10\,000\bar{A}^{13}_{60:\overline{10}|} = 1914.00$$

(iii) $${}_{10}V^{(0)} = 20\,000\left(\bar{A}^{01}_{60:\overline{10}|} + \bar{A}^{02}_{60:\overline{10}|}\right) - P^{(iii)}\,\bar{a}^{00}_{60:\overline{10}|} = 1653.64$$

$${}_{10}V^{(1)} = 0 \text{ (or undefined, as the policy has expired)}$$

(iv) $${}_{10}V^{(0)} = 20\,000\bar{A}^{01}_{60:\overline{10}|} + 30\,000\,\bar{A}^{02}_{60:\overline{10}|} + 10\,000\bar{A}^{03}_{60:\overline{10}|} - P^{(iv)}\,\bar{a}^{00}_{60:\overline{10}|} = 1981.30$$

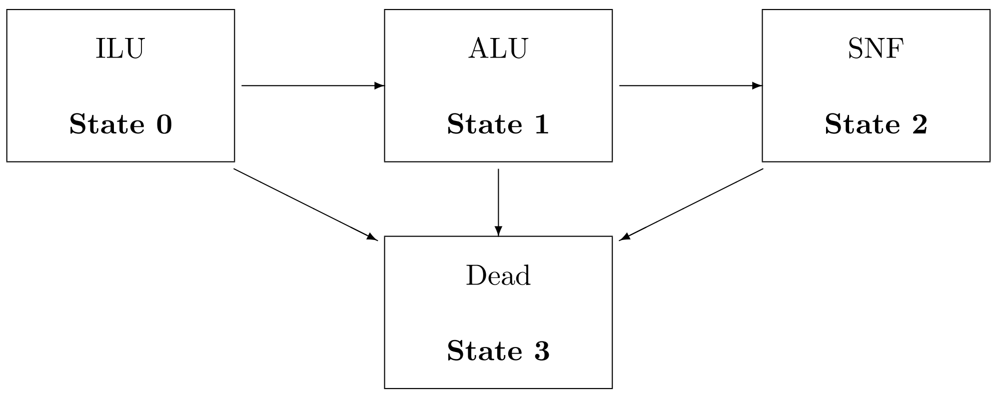
$${}_{10}V^{(1)} = 10\,000\bar{A}^{13}_{60:\overline{10}|} = 1914.00$$

2.4 Continuing Care Retirement Communities (CCRC)

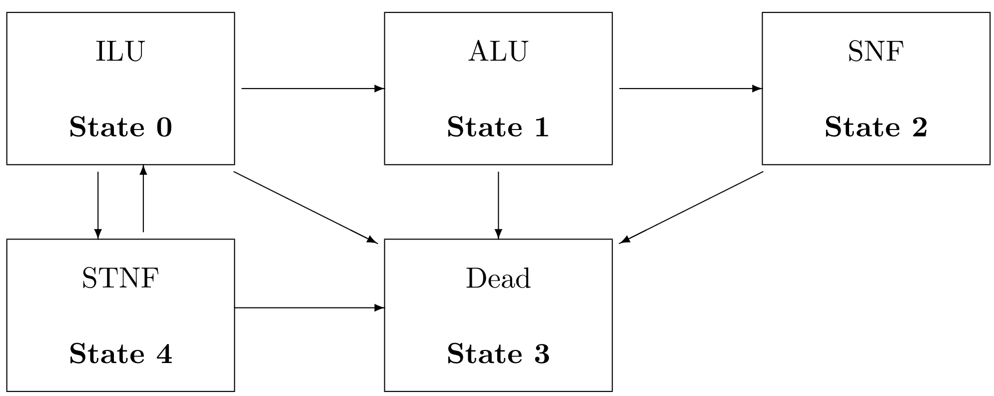
The model for a CCRC (without a memory care facility) would look something like the examples in Figure 4. In Fig 4a, the model allows for a simple forward transition from the Independent Living Unit (ILU) through the Assisted Living Unit (ALU) to the Skilled Nursing Facility (SNF). In Fig. 4b the model allows explicitly for short term stays in the skilled nursing facility (STNF) while in ILU. This would cover periods of temporary ill-health of residents who will recover sufficiently to return to independent living. Of course, the model could be made more complex by allowing periods of temporary disability from the ALU state, or by allowing for direct transitions from STNF to ALU or to SNF.

Another possible complication is a joint life version of the model, which would allow for each partner of a couple to move separately through the stages.

Some widely used CCRC contract types are described in Section 1.6. For the full lifecare (Type A) and modified lifecare (Type B) contracts, the price is expressed as a combination of entry fee and monthly fees that for Type A increase with inflation, but do not change when the resident moves between different residence categories. Type B monthly fees increase with inflation and also increase as residents move through the different categories, but the increases are less than the actual difference in cost, so there is some prefunding of the costs of the more expensive ALU and SNF facilities. Type C contracts are pay as you go, and so do not involve pre-funding, and therefore do not need actuarial modelling for costing purposes.



(a) The simplified CCRC model; ILU is Independent Living Unit; ALU is Assisted Living Unit; SNF is Specialized Nursing Facility.



(b) The extended CCRC model, adding a Short Term Nursing Facility (STNF) state.

Figure 4: CCRC Models

The allocation of costs between the entry fee and the monthly charge is mainly determined by market forces. Often, residents fund the entry fee by selling their home, and so the CCRC may set the entry fee to be close to the average home price in the area, and then allocate the remaining costs to the monthly fee.

Example 2.6
A CCRC wishes to charge an entry fee of 200,000 under a full lifecare (Type A) contract, for lives entering the independent living unit. Subsequently, residents will pay a level monthly fee regardless of the level of care provided. Fees are payable at the start of each month.

The actual monthly costs incurred by the CCRC, including medical care, provision of services, maintenance of buildings and all other expenses and loadings, are as follows:

Independent Living Unit:	3 500
Assisted Living Unit	6 000
Specialized Nursing Facility:	12 000

(a) The actuarial functions given in Table 1 have been calculated using the model in Figure 4a, and an interest rate of 5%. Use these functions to calculate the level monthly fee for entrants age 65, 70, 80 and 90, assuming a 200 000 entry fee, which is not refunded.

(b) Calculate the revised monthly fees from (a), assuming 70% of the entry fee is refunded at the end of the month of death.

(c) The CCRC wants to charge a level monthly fee for all residents using the full life care contract, regardless of age at entry. Assume that all residents enter at one of the four ages in (a), and the proportions of entrants at each age are

Entry age	Proportion of entrants
65	5%
70	30%
80	55%
90	10%

Calculate a suitable monthly fee which is not age-dependent, assuming (i) no refund of entry fee and (ii) 70% refund of entry fee on death.

(d) What are the advantages and disadvantages of offering the refund, compared with the no refund contract?

x	$\ddot{a}_x^{(12)\,00}$	$\ddot{a}_x^{(12)\,01}$	$\ddot{a}_x^{(12)\,02}$	$A_x^{(12)\,03}$
65	11.6416	0.75373	0.24118	0.38472
66	11.3412	0.75157	0.25330	0.39885
67	11.0323	0.74994	0.26614	0.41335
68	10.7149	0.74877	0.27974	0.42820
69	10.3892	0.74790	0.29415	0.44340
70	10.0554	0.74720	0.30944	0.45894
71	9.7139	0.74648	0.32566	0.47482
72	9.3650	0.74554	0.34285	0.49101
73	9.0092	0.74415	0.36108	0.50752
74	8.6471	0.74204	0.38040	0.52431
75	8.2792	0.73897	0.40085	0.54138
76	7.9064	0.73463	0.42248	0.55869
77	7.5295	0.72875	0.44531	0.57621
78	7.1495	0.72105	0.46937	0.59392
79	6.7675	0.71127	0.49465	0.61177
80	6.3846	0.69917	0.52113	0.62971
81	6.0023	0.68455	0.54875	0.64769
82	5.6218	0.66729	0.57745	0.66566
83	5.2449	0.64731	0.60710	0.68354
84	4.8730	0.62461	0.63755	0.70127
85	4.5078	0.59929	0.66861	0.71877
86	4.1511	0.57151	0.70002	0.73597
87	3.8044	0.54153	0.73150	0.75277
88	3.4696	0.50968	0.76274	0.76911
89	3.1481	0.47637	0.79335	0.78489
90	2.8414	0.44205	0.82295	0.80005

Table 1: CCRC actuarial functions at 5% per year interest

Solution 2.6

(a) The expected present value at entry of the future costs, for an entrant age x, is

$$\mathrm{EPV}_x = 12\left(3500\ddot{a}_x^{(12)\,00} + 6000\ddot{a}_x^{(12)\,01} + 12000\ddot{a}_x^{(12)\,03}\right)$$

which gives EPV of future costs at entry for the different ages of

Entry age 65	577,944
Entry age 70	520,686
Entry age 80	393,535
Entry age 90	269,671

To find the annual fee rate, we subtract the entry fee, and divide by the value of an annuity of 1 per year, payable monthly, while the life is in State 0 or State 1 or State 2. To get the monthly fee, we divide this by 12, giving the fee equation:

$$\mathrm{Fee}_x = \frac{\mathrm{EPV}_x - 200\,000}{12\left(\ddot{a}_x^{(12)\,00} + \ddot{a}_x^{(12)\,01} + \ddot{a}_x^{(12)\,03}\right)}$$

The resulting monthly fees are

Entry age 65:	2492.4
Entry age 70:	2404.9
Entry age 80:	2120.7
Entry age 90:	1413.9

(b) We need to add an extra term to the EPV in (a) to allow for the value of the refund. The revised EPV for entry age x is

$$\mathrm{EPV}_x^r = 12\left(3500\ddot{a}_x^{(12)\,00} + 6000\ddot{a}_x^{(12)\,01} + 12000\ddot{a}_x^{(12)\,03}\right) + 0.7(200\,000)A_x^{(12)\,03}$$

To get the revised monthly fee, we proceed as in (a); subtract the entry fee, and divide by the annuity sum, to give:

Entry age 65:	2847.6
Entry age 70:	2886.8
Entry age 80:	3086.8
Entry age 90:	3686.9

(c) We can take a weighted average of the fees to get a single fee for all ages, that is, without refund,

$$\mathrm{Fee} = 0.05(2492.4) + 0.30(2404.9) + 0.55(2120.7) + 0.10(1413.9) = 2153.9$$

and with refund

$$\text{Fee} = 0.05(2847.6) + 0.30(2886.8) + 0.55(3086.8) + 0.10(3686.9) = 3074.8$$

(d) Adding the refund feature would be popular with residents who are concerned about losing much of their capital if the resident dies soon after entering the facility. It ensures a bequest is available for the resident's family.

The refund also has an advantage when charging a non-entry age-dependent fee, because the range of fees across entry ages with the refund feature is smaller than without, so that the CCRC would be less exposed to the risk that the entry age distribution changes.

For example, suppose the CCRC fixes the monthly fees at 2153.9, with no refund, or 3074.8 with refund, as calculated above, but the entry age distribution shifts to

Entry age	Proportion of entrants
65	5%
70	35%
80	55%
90	5%

Then without the refund, the fee should be 2203.4, giving a deficit of 49.5 per person per month. With the refund, the fee should be 3034.8, giving a smaller surplus of 40.0 per person per month.

A disadvantage of introducing the refund is that the monthly fees are higher, which might discourage potential residents who are more constrained by their ability to meet the monthly payments than they are concerned about their bequest. Those who have spare income could replace some or all bequest using separate life insurance.

3 Recursions for policy values with multiple states

3.1 Review of policy value recursions for traditional life insurance

In this section we will give examples of policy value recursions for discrete cash flows in a multiple state model context. We assume the reader has already covered Chapters 7 and 8 of AMLCR.

We start by recalling the policy value recursion for a traditional, 2-state model, from Chapter 7 of AMLCR. Consider a whole life policy issued to (x), with sum insured S paid at the end of the year of death, and with premium P paid annually. Ignoring expenses, we have policy values at integer durations $t = 0, 1, ...$ related recursively as follows:

$$({}_tV + P)\,(1+i) = q_{x+t}\,S + p_{x+t}\,({}_{t+1}V)$$

This is a simplified version of Equation (7.6) in AMLCR.

The intuitive explanation of the recursion is that the left hand side represents the available funds at the end of the year, if the policy value at the start of the year is held as a reserve, and if interest is earned at the assumed rate i per year.

The right hand side represents the expected costs at the end of the year; if the policyholder dies, with probability q_{x+t}, then the funds must support the payment of the sum insured, S; if the policyholder survives, then the new policy value (or reserve) must be carried forward to the next year. The policy value is determined such that the expected income and outgo in each year are balanced, which gives us the equation of left hand side (funds available) and right hand side (funds required). AMLCR gives a much more rigorous proof, of course, of the recursion, but the intuition is useful in generalising the result to the multiple state case.

First, we generalise to the case where payments are made at intervals of h years – for example, $h = 1/12$ for monthly policies. In this case, we have a recursion from t to $t + h$. The premium of P per year is paid in level instalments of hP, giving the following recursion for the whole life example, with premiums paid at the start of each h-year period, and benefits paid at the end of each h year period, for $t = 0, h, 2h, ...$

$$({}_tV + hP)\,(1+i)^h = {}_hq_{x+t}\,S + {}_hp_{x+t}\,({}_{t+h}V) \tag{11}$$

One way to derive Thiele's equation for $\frac{d}{dt}\,{}_tV$ for continuous policies is to use equation (11), divide by h, and take the limit as $h \to 0$.

3.2 Recursion for DII with discrete time and benefits

In this section, we will use the disability income insurance model, Figure 1, to illustrate the policy value recursion in a multiple state model setting. We will construct the recursion using the principle of equating expected income and outgo each period, as we did in the previous section, noting that the policy value at the start of each period represents the funds available from the previous period, and is treated as income, while the policy value at the end of each period represents the cost of continuing the policy, and is treated as outgo.

Suppose an insurer issues a DII to (x), with level premiums of hP payable every h years, at the start of each interval t to $t+h$, provided the policyholder is in State 0 at the payment date. A benefit of hB is paid at the end of every h years provided the policyholder is in state 1 at the payment date. We assume here that there is no waiting or off period[3].

From equation (4) we see the premium equation in this case is

$$hP\left(1 + {}_{h}p_x^{00}\, v^h + {}_{2h}p_x^{00}\, v^{2h} + \cdots + {}_{n-h}p_x^{00}\, v^{n-h}\right) = hB\left({}_{h}p_x^{01}\, v^h + {}_{2h}p_x^{01}\, v^{2h} + \cdots + {}_{n}p_x^{01}\, v^n\right)$$

that is, $P\ddot{a}_{x:\overline{n}|}^{(\frac{1}{h})00} = Ba_{x:\overline{n}|}^{(\frac{1}{h})01}$

For cashflow dates, $t = 0, h, 2h, ..., n-h$, let ${}_tV^{(0)}$ denote the policy value given that the policyholder is in State 0, and ${}_tV^{(1)}$ denote the policy value if the policyholder is in State 1. The policy value at cash flow dates, considered as a prospective value of future outgo minus income, includes the premium paid at t, but does not include the benefit paid at t, if any. So

$${}_tV^{(0)} = Ba_{x+t:\overline{n-t}|}^{(\frac{1}{h})01} - P\ddot{a}_{x+t:\overline{n-t}|}^{(\frac{1}{h})00} \tag{12}$$

$${}_tV^{(1)} = Ba_{x+t:\overline{n-t}|}^{(\frac{1}{h})11} - P\ddot{a}_{x+t:\overline{n-t}|}^{(\frac{1}{h})10} \tag{13}$$

Now we will construct the recursions, one for each of ${}_tV^{(0)}$ and ${}_tV^{(1)}$, from first principles. For simplicity, we assume net premium policy values (no expenses in premium or policy value), but it is straightforward to incorporate expenses in a gross premium approach.

Suppose the policyholder is in State 0 at t. In this case, she pays her premium of hP at t. This is added to the policy value brought forward, and accumulated to $t+h$, giving the left hand side of the equation

$$\left({}_tV^{(0)} + hP\right)(1+i)^h$$

If the policyholder is in State 1 at t, then there is no premium, and the left hand side of the recursion is

$$\left({}_tV^{(1)}\right)(1+i)^h$$

At $t+h$, if the policyholder is in State 0 the insurer will need a policy value of ${}_{t+h}V^{(0)}$ to carry forward to the next time period; if she is in State 1, the insurer will need to pay the benefit,

[3]Waiting and off periods can be incorporated in a recursion, but it makes the formulas messy, and harder to interpret. For clarity, we will consider only the simpler case.

hB, and will also need a policy value of ${}_{t+h}V^{(1)}$ to carry forward. If the policyholder has moved to State 3, there is no payment, and no policy value required. Applying the appropriate probabilities to the two relevant cases for the end of period states, we have the recursions

$$\left({}_tV^{(0)} + hP\right)(1+i)^h = {}_hp^{00}_{x+t}\,({}_{t+h}V^{(0)}) + {}_hp^{01}_{x+t}\,\left(hB + {}_{t+h}V^{(1)}\right) \tag{14}$$

$$\left({}_tV^{(1)}\right)(1+i)^h = {}_hp^{10}_{x+t}\,({}_{t+h}V^{(0)}) + {}_hp^{11}_{x+t}\,\left(hB + {}_{t+h}V^{(1)}\right) \tag{15}$$

Deriving the DII policy value recursions

We can prove the recursion equations (14) and (15) more formally, starting from equations (12) and (13).

First, we note that for the DII model, for any $k > h$

$${}_kp^{00}_x = {}_hp^{00}_x\;{}_{k-h}p^{00}_{x+h} + {}_hp^{01}_x\;{}_{k-h}p^{10}_{x+h} \tag{16}$$

Next, we decompose the state dependent annuity functions as follows

$$\ddot{a}^{(\frac{1}{h})00}_{x:\overline{n}|} = h\left(1 + v^h\,{}_hp^{00}_x + v^{2h}\,{}_{2h}p^{00}_x + ... + v^{n-h}\,{}_{n-h}p^{00}_x\right) \tag{17}$$

$$= h + v^h\,{}_hp^{00}_x\,h\left(1 + v^h\,{}_hp^{00}_{x+h} + v^{2h}\,{}_{2h}p^{00}_{x+h} + ... + v^{n-2h}\,{}_{n-2h}p^{00}_{x+h}\right) \tag{18}$$

$$+ v^h\,{}_hp^{01}_x\,h\left(v^h\,{}_hp^{10}_{x+h} + v^{2h}\,{}_{2h}p^{10}_{x+h} + ... + v^{n-2h}\,{}_{n-2h}p^{10}_{x+h}\right) \tag{19}$$

$$= h + v^h\,{}_hp^{00}_x\,\ddot{a}^{(\frac{1}{h})00}_{x+h:\overline{n-h}|} + v^h\,{}_hp^{01}_x\,\ddot{a}^{(\frac{1}{h})10}_{x+h:\overline{n-h}|} \tag{20}$$

$$a^{(\frac{1}{h})01}_{x:\overline{n}|} = h\left(v^h\,{}_hp^{01}_x + v^{2h}\,{}_{2h}p^{01}_x + ... + v^n\,{}_np^{01}_x\right) \tag{21}$$

$$= hv^h\,{}_hp^{01}_x + v^h\,{}_hp^{01}_x\,h\left(v^h\,{}_hp^{11}_{x+h} + v^{2h}\,{}_{2h}p^{11}_{x+h} + ... + v^{n-h}\,{}_{n-h}p^{11}_{x+h}\right) \tag{22}$$

$$+ v^h\,{}_hp^{00}_x\,h\left(v^h\,{}_hp^{01}_{x+h} + v^{2h}\,{}_{2h}p^{01}_{x+h} + ... + v^{n-h}\,{}_{n-h}p^{01}_{x+h}\right) \tag{23}$$

$$= hv^h\,{}_hp^{01}_x + v^h\,{}_hp^{01}_x\,a^{(\frac{1}{h})11}_{x+h:\overline{n-h}|} + v^h\,{}_hp^{00}_x\,a^{(\frac{1}{h})01}_{x+h:\overline{n-h}|} \tag{24}$$

Similarly

$$\ddot{a}^{(\frac{1}{h})10}_{x:\overline{n}|} = hv^h\,{}_hp^{10}_x + v^h\,{}_hp^{10}_x\,\ddot{a}^{(\frac{1}{h})00}_{x+h:\overline{n-h}|} + v^h\,{}_hp^{11}_x\,\ddot{a}^{(\frac{1}{h})10}_{x+h:\overline{n-h}|} \tag{25}$$

$$a^{(\frac{1}{h})11}_{x:\overline{n}|} = hv^h\,{}_hp^{11}_x + v^h\,{}_hp^{11}_x\,a^{(\frac{1}{h})11}_{x+h:\overline{n-h}|} + v^h\,{}_hp^{10}_x\,a^{(\frac{1}{h})01}_{x+h:\overline{n-h}|} \tag{26}$$

So we have

$$
\begin{aligned}
{}_tV^{(0)} &= B a^{(\frac{1}{h})01}_{x+t:\overline{n-t|}} - P \ddot{a}^{(\frac{1}{h})00}_{x+t:\overline{n-t|}} \\
&= B \left(h v^h \, {}_hp^{01}_{x+t} + v^h \, {}_hp^{01}_{x+t} \, a^{(\frac{1}{h})11}_{x+t+h:\overline{n-t-h|}} + v^h \, {}_hp^{00}_{x+t} \, a^{(\frac{1}{h})01}_{x+t+h:\overline{n-t-h|}} \right) \\
&\qquad - P \left(h + v^h \, {}_hp^{00}_{x+t} \ddot{a}^{(\frac{1}{h})00}_{x+t+h:\overline{n-t-h|}} + v^h \, {}_hp^{01}_{x+t} \ddot{a}^{(\frac{1}{h})10}_{x+t+h:\overline{n-t-h|}} \right)
\end{aligned}
$$

Multiply both sides by $(1+i)^h$, and collect together terms on the left hand side in ${}_hp^{01}_{x+t}$ and ${}_hp^{00}_{x+t}$, to give

$$
\begin{aligned}
\left({}_tV^{(0)} + hP\right)(1+i)^h &= {}_hp^{01}_{x+t} \left(hB + B a^{(\frac{1}{h})11}_{x+t+h:\overline{n-t-h|}} - P \ddot{a}^{(\frac{1}{h})10}_{x+t+h:\overline{n-t-h|}} \right) \\
&\qquad + {}_hp^{00}_{x+t} \left(B a^{(\frac{1}{h})01}_{x+t+h:\overline{n-t-h|}} - P \ddot{a}^{(\frac{1}{h})00}_{x+t+h:\overline{n-t-h|}} \right) \\
&= {}_hp^{01}_{x+t} \left(hB + {}_{t+h}V^{(1)} \right) + {}_hp^{00}_{x+t} \left({}_{t+h}V^{(0)} \right) \qquad \text{as required.}
\end{aligned}
$$

The recursion in equation (15) can be derived similarly.

Deriving Thiele's equations from the h-yearly recursion

For very small h, we might use the continuous time Thiele's equation, which for ${}_tV^{(0)}$ in this example is

$$
\frac{d}{dt}\, {}_tV^{(0)} = \delta\, {}_tV^{(0)} + P - \mu^{01}_{x+t} \left({}_tV^{(1)} - {}_tV^{(0)} \right) - \mu^{02}_{x+t} \left(-{}_tV^{(0)} \right)
$$

We can derive Thiele's equation for this policy using equations (14) and (15) by letting $h \to 0$, as follows:

$$
\left({}_tV^{(0)} + hP\right)(1+i)^h = {}_hp^{01}_{x+t} \left(hB + {}_{t+h}V^{(1)} \right) + {}_hp^{00}_{x+t} \left({}_{t+h}V^{(0)} \right)
$$

$$
\Rightarrow \left({}_tV^{(0)} + hP\right) e^{\delta h} - {}_hp^{01}_{x+t} \left(hB + {}_{t+h}V^{(1)} \right) = {}_{t+h}V^{(0)} (1 - {}_hp^{01}_{x+t} - {}_hp^{02}_{x+t})
$$

subtract ${}_tV^{(0)}$ from both sides, and rearrange

$$
\begin{aligned}
{}_{t+h}V^{(0)} - {}_tV^{(0)} = {}_tV^{(0)}(e^{\delta h} - 1) + hPe^{\delta h} - {}_hp^{01}_{x+t} hB - {}_hp^{01}_{x+t} \left({}_{t+h}V^{(1)}\right) \\
+ ({}_hp^{01}_{x+t} + {}_hp^{02}_{x+t})({}_{t+h}V^{(0)})
\end{aligned}
$$

Divide by h

$$\frac{{}_{t+h}V^{(0)} - {}_tV^{(0)}}{h} = {}_tV^{(0)}\frac{e^{\delta h}-1}{h} + Pe^{\delta h} - {}_hp^{01}_{x+t}B - \frac{{}_hp^{01}_{x+t}}{h}{}_{t+h}V^{(1)} + \frac{{}_hp^{01}_{x+t} + {}_hp^{02}_{x+t}}{h}{}_tV^{(0)}$$

Now we take limits as $h \to 0$, recalling that

$$\lim_{h\to 0}\frac{e^{\delta h}-1}{h} = \delta; \quad \lim_{h\to 0} e^{\delta h} = 1; \quad \lim_{h\to 0} {}_hp^{01}_{x+t} = 0; \quad \lim_{h\to 0}\frac{{}_hp^{01}_{x+t} + {}_hp^{02}_{x+t}}{h} = \mu^{01}_{x+t} + \mu^{02}_{x+t}$$

So that

$$\begin{aligned}\frac{d}{dt}{}_tV^{(0)} &= \delta\,{}_tV^{(0)} + P - \mu^{01}_{x+t}\left({}_tV^{(1)}\right) + \mu^{01}_{x+t}\left({}_tV^{(0)}\right) + \mu^{02}_{x+t}\left({}_tV^{(0)}\right) \\ &= \delta\,{}_tV^{(0)} + P - \mu^{01}_{x+}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{x+}\left(-{}_tV^{(0)}\right)\end{aligned}$$

as required.

3.3 General recursion for h-yearly cash flows

In this section we generalise the recursion in Section 3.2 for a general multiple state dependent insurance policy.

Note that in the recursions in Section 3.2, the cashflows depend only on the state at the payment date. Discrete recursions for multiple state dependent cashflows will not work if the payments at the end of the time period depend on intermediate transitions. For example, consider a model with a death benefit, payable at the end of the month of death, where the amount of benefit depends on whether the life became sick and then died, or died directly from the healthy state. In this model the cash flow at $t+h$ depends on the intermediate states between the states at t and $t+h$, not solely on the starting and end states. The discrete recursion approach will not give accurate answers, as intermediate states are not accommodated. The inaccuracy will tend to zero as $h \to 0$, as the probability of intermediate transfers will also tend to zero.

For our general recursion, we will assume that payments depend at most on the state at the start and end of the period between cashflows[4]. We also assume, as in the previous section, that cash flows are h-yearly.

For the general recursion, we will use the following notation; these are consistent with those used in the general Thiele equation (8.23) in AMLCR.

- $P^{(j)}_t$ denotes the annual rate of premium paid at the start of the interval t to $t+h$, conditional on the policyholder being in State j at that time.

[4] A stronger assumption, that is consistent with the recursions, but is stronger than we require, is that at most one transition may occur between cashflow dates.

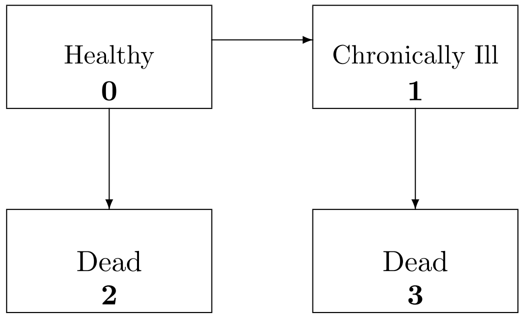


Figure 5: Life and Chronic Illness Insurance Model

- $B_{t+h}^{(j)}$ denotes the annual rate of benefit paid at the end of the interval t to $t+h$, conditional on the policyholder being in State j at that time.
- $S_{t+h}^{(jk)}$ denotes a lump sum benefit paid at the end of the interval t to $t+h$, conditional on the policyholder being in state j at the start of the interval and State k at the end.
- There are $m+1$ states, labelled State 0 to State m.

Then the general net premium policy value recursion for a policy issued to (x), with h-yearly cash flows, and where the policy is in state j at time t, is

$$\left({}_tV^{(j)} + hP_t^{(j)}\right)(1+i)^h = \sum_{k=0}^{m} {}_hp_{x+t}^{jk}\left(hB_{t+h}^{(k)} + S_{t+h}^{(jk)} + {}_{t+h}V^{(k)}\right) \tag{27}$$

Example 3.1
A whole life insurance policy with a chronic illness rider is sold to a healthy life age 50. If the policyholder contracts a chronic illness, the policy pays a lump sum of 10,000 at the end of the month of diagnosis (if they are still alive), plus an additional 1000 at the end of each subsequent month while the life survives. A benefit of 40 000 is paid at the end of the month of death if the life dies after a chronic illness diagnosis. The policy pays 50,000 at the end of the month of death if the life dies without suffering a chronic illness.

Premiums are payable monthly while the life is healthy.

The company uses the model in Figure 5 to evaluate premiums and policy values, with an interest rate of 5% per year effective.

You are given the following actuarial functions, at 5% per year interest.

x	$\ddot{a}_x^{(12)00}$	$a_x^{(12)01}$	$a_x^{(12)11}$
50	16.08747	0.45320	13.22971
70	11.01946	0.60398	6.14524

x	$A_x^{(12)01}$	$A_x^{(12)02}$	$A_x^{(12)03}$	$A_x^{(12)13}$
50	0.109701	0.106972	0.087644	0.351773
70	0.246405	0.218444	0.215597	0.696724

(a) Calculate the monthly premium for the policy.

(b) Calculate the net premium policy values at $t = 20$ for the policy.

(c) You are given the following probabilities for a policyholder aged 70:

$${}_{\frac{1}{12}}p_{70}^{00} = 0.998866 \quad {}_{\frac{1}{12}}p_{70}^{01} = 0.000552 \quad {}_{\frac{1}{12}}p_{70}^{02} = 0.000582 \quad {}_{\frac{1}{12}}p_{70}^{03} = 0.0$$

$${}_{\frac{1}{12}}p_{70}^{11} = 0.995489 \quad {}_{\frac{1}{12}}p_{70}^{13} = 0.004511$$

Use the recursion equation to calculate the policy values at $t = 20\frac{1}{12}$.

Solution 3.1

(a) Let P denote the monthly premium. Then the premium equation is

$$12P\,\ddot{a}_{50}^{(12)\,00} = 12000a_{50}^{(12)\,01} + 10000A_{50}^{(12)\,01} + 40000A_{50}^{(12)\,03} + 50000A_{50}^{(12)\,02}$$

$$\Rightarrow 12P = 956.63 \Rightarrow P = 79.72 \text{ per month}$$

(b)

$$\begin{aligned} {}_{20}V^{(0)} &= 12000a_{70}^{(12)\,01} + 10000A_{70}^{(12)\,01} + 40000A_{70}^{(12)\,03} + 50000A_{70}^{(12)\,02} - 12P\,\ddot{a}_{70}^{(12)\,00} \\ &= 18716.35 \end{aligned}$$

$$\begin{aligned} {}_{20}V^{(1)} &= 12000\,a_{70}^{(12)\,11} + 40000A_{70}^{(12)\,13} \\ &= 101611.8 \end{aligned}$$

(c) The recursions in this case are as follows, where $h = \frac{1}{12}$.

$$\left({}_{20}V^{(0)} + P\right)(1+i)^h = {}_hp_{70}^{00}\left({}_{20+h}V^{(0)}\right) + {}_hp_{70}^{01}\left(1000 + 10000 + {}_{20+h}V^{(1)}\right) + {}_hp_{70}^{02}\,(50000)$$

$$\left({}_{20}V^{(1)}\right)(1+i)^h = {}_hp_{70}^{11}\left(1000 + {}_{20+h}V^{(1)}\right) + {}_hp_{70}^{13}\,(40000)$$

We can solve the recursion for ${}_tV^{(1)}$ first:

$${}_{20\frac{1}{12}}V^{(1)} = \frac{(101611.8)(1.05)^{\frac{1}{12}} - 0.004511(40000) - 0.995489(1000)}{0.995489} = 101306.9$$

We use this in the recursion for ${}_tV^{(0)}$:

$$\begin{aligned}{}_{20\frac{1}{12}}V^{(0)} &= \frac{(18716.35 + 79.73)(1.05)^{\frac{1}{12}} - 0.000552(11000 + 101306.9) - 0.000582(50000)}{0.998866} \\ &= 18802.95\end{aligned}$$

3.4 Approximating continuous payments in discrete recursions

For cases where the within-period transitions impact the cash flows the continuous approach, using Thiele's formula, will always work, and the values determined using Thiele can be adjusted to allow for discrete payments. As we have demonstrated above, Thiele's equation is the same as the recursion with infinitesimal h. However, if the time step h is sufficiently small, then the assumption that only one transition can occur in each time period will not significantly impact results, and the discrete recursion is used in practice.

Where the cash flows are a mixture of continuous and discrete, we may still use the discrete recursion approach, but adjust for the continuous payments. Usually, annuity benefits are discrete, but lump sum transition benefits, which are represented by $S^{(jk)}$ in the general recursion, may be paid immediately on transition. Assuming payment at the end of the period of transition will marginally under-value the benefit (for small h). A practical adjustment is to apply the claims acceleration approach, described in AMLCR Section 4.5.2. We approximate the continuous payment by assuming that the benefits are paid in the middle of the interval in which the transition occurs, rather than at the end. The general recursion formula is then slightly adjusted as

$$\left({}_tV^{(j)} + hP_t^{(j)}\right)(1+i)^h = \sum_{k=0}^{m} {}_hp_{x+t}^{jk}\left(hB_{t+h}^{(k)} + S_{t+h}^{(jk)}(1+i)^{h/2} + {}_{t+h}V^{(k)}\right) \tag{28}$$

4 Mortality improvement modelling

4.1 Introduction

In this section we will present some models and methods for integrating mortality improvement into actuarial analysis for life contingent risks.

We will expand on the short descriptive coverage in Section 3.11 of AMLCR, and will consider how mortality trends can be incorporated in actuarial valuations of annuity benefits in pensions or insurance.

First, it might be valuable to demonstrate what we mean by mortality or longevity improvement. In Figure 6 we show raw (that is, with no smoothing) mortality rates for US Males aged 30-44 from 1960-2015, and for US females aged 50-69 for the same period[5].

In each figure the higher lines are for the oldest ages, and the lower lines for the youngest ages.

The data in Figure 6 are rates derived from taking the number of deaths registered at each age, divided by an approximate count of the number of people at that age in the population, which we call the Exposed to Risk. Overall, we see that for each age, mortality rates are generally declining over time, although there are exceptions.

We also note that the rates are not very smooth. There appears to be some random variation around the general trends.

When modelling mortality we generally smooth the raw data to reduce the impact of sampling variability. It is also common in longevity modelling to use heatmaps of mortality improvement to illustrate the two dimensional data, rather than the age curves of mortality rates in Figure 6.

In Figure 7 we show a plot of smoothed mortality improvement factors for US data, for 1951-2007. The mortality improvement factor is the percentage reduction in the mortality rate for each age over each successive calendar year. That is, if the smoothed mortality rate for age x in year y is $\tilde{q}(x,y)$ then the smoothed improvement factor at age x and year y is

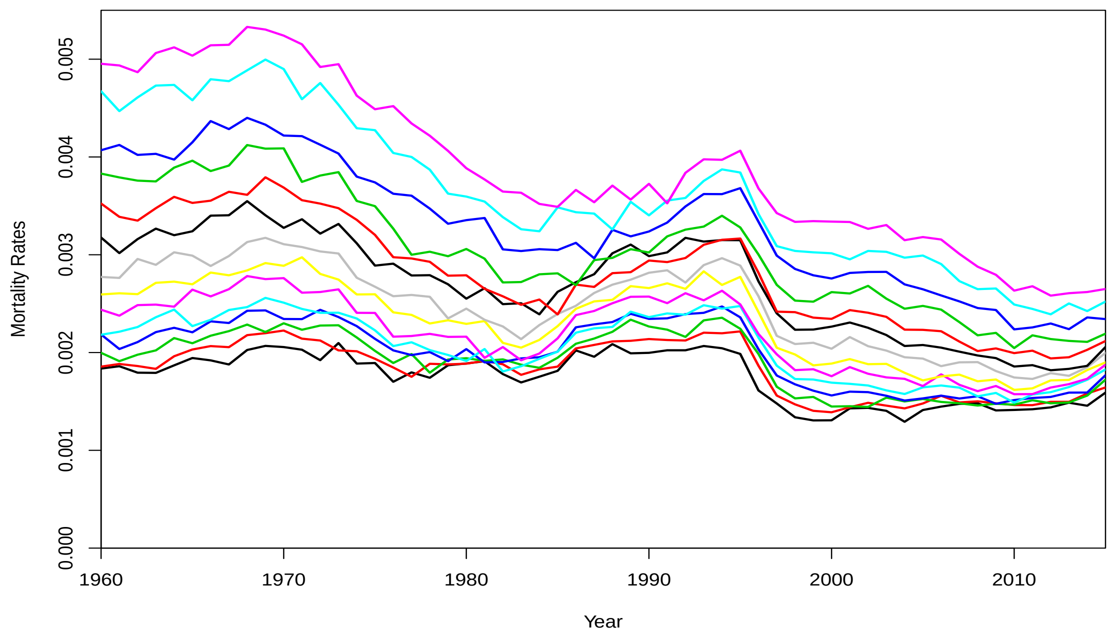
$$\varphi(x,y) = 1 - \frac{\tilde{q}(x,y)}{\tilde{q}(x,y-1)}$$

The heatmaps illustrate the following three effects.

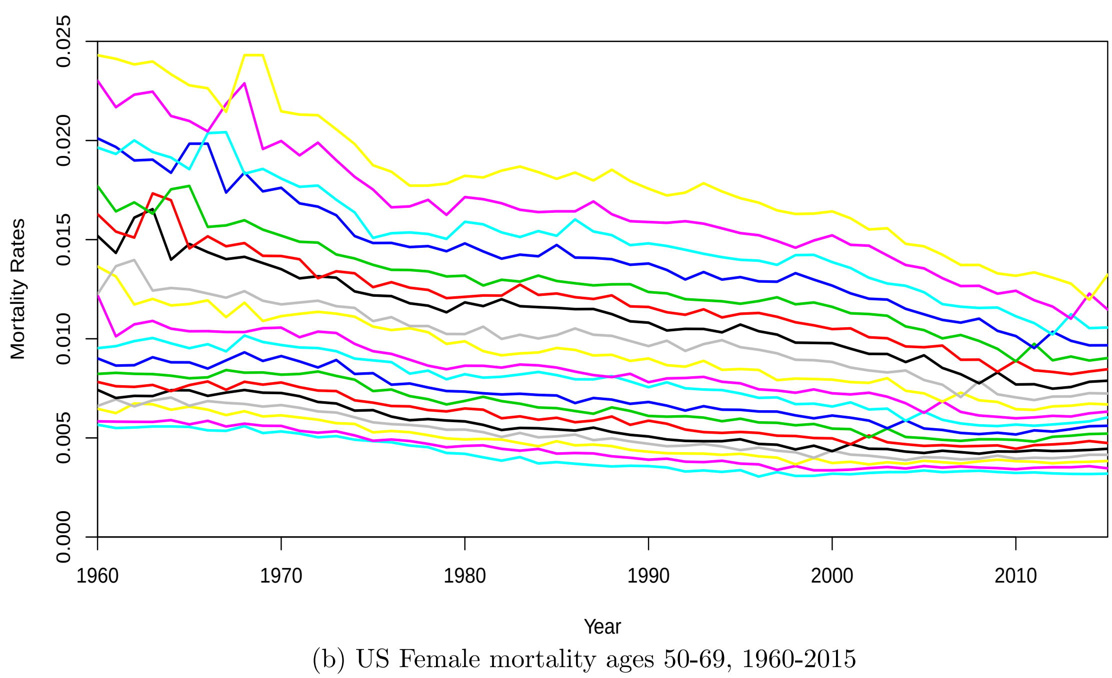
Year effects

Calendar year effects are identified in the heatmaps with vertical patterns. For example, consider the years 1958-1970 in Figure 7a. The vertical column of light blue for those

[5] Data from the National Center for Health Statistics. *Vital Statistics of the United States, Volume II: Mortality, Part A.* Washington, D.C.: Government Printing Office. Data obtained through the Human Mortality Database, www.mortality.org.

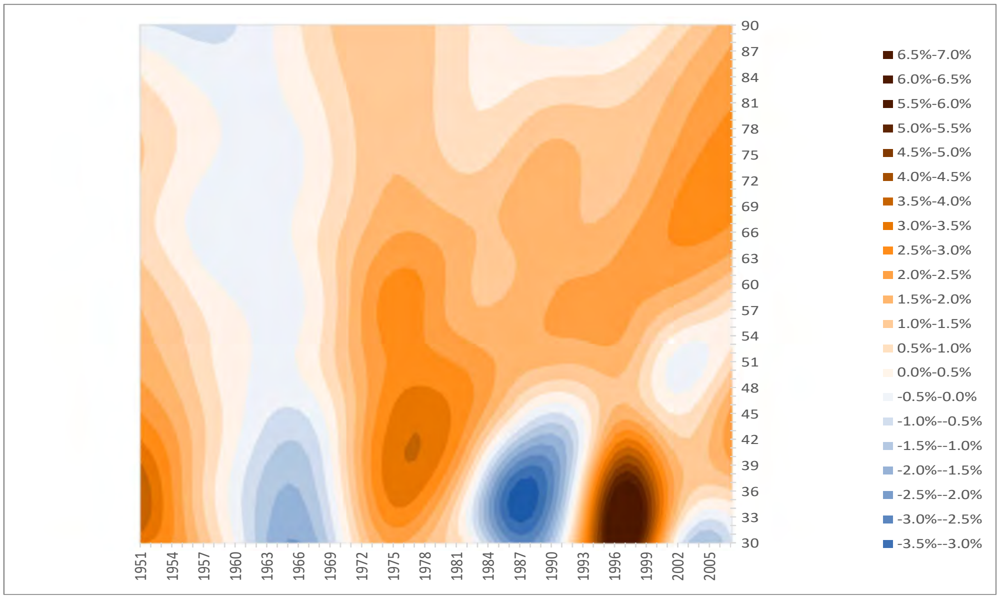


(a) US Male mortality ages 30-44, 1960-2015

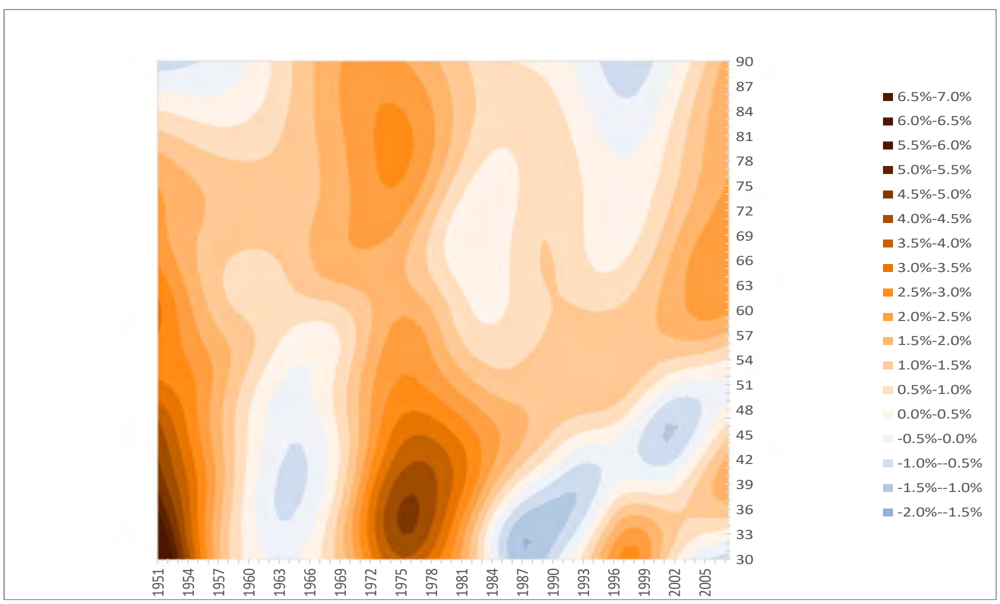


(b) US Female mortality ages 50-69, 1960-2015

Figure 6: US mortality experience 1960-2015 (from HMD).



(a) US smoothed mortality improvement, males, 1951-2007



(b) US Female smoothed mortality improvement, females, 1951-2007

Figure 7: US smoothed mortality improvement heatmap

years indicates that longevity improvement was paused or reversed for all ages in those years, though the impact was different for different age groups (the same phenomenon is apparent in the raw data in Figure 6a). The next vertical section of the graph shows longevity picking up again, with improvement more marked for younger lives than for older.

In Figure 7a we also see a very clear and severe deterioration in mortality between 1984 and 1991 affecting younger males. This area illustrates the impact of the HIV/AIDS epidemic on younger male mortality in the US. In the following period, from around 1993-2000, mortality in the same age range showed very strong improvement, as medical and social management of HIV/AIDS produced an extraordinary turnaround in mortality from the disease.

Age effects

Age effects in the heatmaps are evident from horizontal patterns; in Figure 7 there is little evidence of pure age effects that are protracted across the whole period. The most obvious impact of age in the heatmaps is in the way that different age groups are impacted differently by the calendar year effects. For example, the mortality improvement experienced by US females in the 1970's was more significant for people below 45 years old than for older lives.

In both the heat maps, we see less intense patterns of improvement or decline at older ages. It is common to assume that we will not see any significant mortality improvement at the very oldest ages, say, beyond age 95. The idea is that, although more people are living to older ages, there is not much evidence that the oldest attainable age is increasing. This phenomenon is referred to as the *rectangularization* of mortality, from the fact that the trend in longevity is generating more rectangular-looking survival curves, (i.e. curves of ${}_tp_0$ for values of t from age 0 to, say, age 120 years) without significantly shifting the right tail of the survival curve.

Cohort Effects

Cohort effects refer to patterns of mortality that are consistent for lives born in the same year. Cohort effects can be seen in the individual age rates in Figure 6 as spikes or troughs that move up diagonally across the curves, as the lives who are, say, age 40 in 1951, if they survive, become the lives who are age 41 in 1952, and so on. Cohort effects are more clearly seen in the heatmaps as diagonal patterns from lower left to upper right. In Figure 7a there is a diagonal band of higher improvement applying to lives born around 1935-1942, and a similar band in 7b, but for lives born a few years later, in the period from around 1940-1945.

Cohort mortality effects are not observed in all populations, and there is still substantial uncertainty as to why they occur.

If mortality rates are generally declining over time, it may not be suitable to assume the same rate of mortality in actuarial calculations regardless of how far ahead we are looking. There is a general trend across the globe of decreasing underlying mortality, as medical science advances, and from improving the other social determinants of longevity such as nutrition and access to health care.

Allowing for mortality improvement means that we model mortality as a function of both age and time, so we replace the age based mortality rate q_x with a rate based on the attained age x and on the calendar year that the age is attained, t. We let $q(x,t)$ denote the mortality rate applying to lives who attain age x in year t, and $p(x,t) = 1 - q(x,t)$. We may measure t relative to some base year, or it may be used to indicate the full calendar year.

There are two approaches to modelling how mortality changes over time. The first is a deterministic approach, where we model $q(x,t)$ as a fixed, known function, using a deterministic **Mortality Improvement Scale** function. The second is a stochastic approach, where we treat future values of $q(x,t)$ as a series of random variables.

4.2 Mortality Improvement Scales

The deterministic approach to mortality improvement models at the highest level uses a two step process:

Step 1 Choose a base year and construct tables of mortality rates for lives attaining each integer age in the base year. This gives the values $q(x,0)$.

Step 2 Construct a scale function that can be applied to the base mortality rates to generate appropriate rates for future years.

4.2.1 Single factor mortality improvement models

The simplest scale functions depend only on age. If we denote the improvement factor for age x as φ_x, then for $t = 1, 2, ...,$

$$q(x,t) = q(x,0)(1 - \varphi_x)^t$$

For example, the Scale AA factors were published in 1994 by the Society of Actuaries. They proposed the age based improvement factors illustrated in Figure 8.

Example 4.1
In Table 2 we show base mortality rates for males in the year 2000, and we show the Scale AA mortality improvement factors, denoted φ_x, for the same age range.

Calculate (a) the 10-year survival probability and (b) the 10-year term life annuity due, with $i = 5\%$, for a life age 60, with and without the mortality improvement scale.

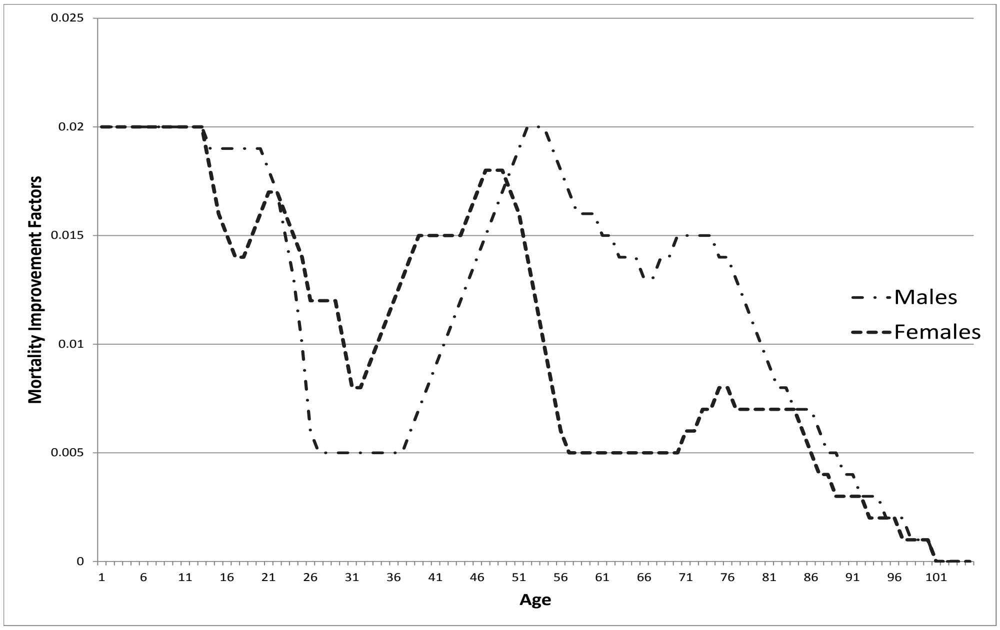


Figure 8: Scale AA mortality improvement factors

Age x	$q(x, 0)$	φ_x
60	0.008196	0.016
61	0.009001	0.015
62	0.009915	0.015
63	0.010951	0.014
64	0.012117	0.014
65	0.013419	0.014
66	0.014868	0.013
67	0.016460	0.013
68	0.018200	0.014
69	0.020105	0.014
70	0.022206	0.015

Table 2: RP2000 Male healthy annuitant mortality rates, with Scale AA improvement factors.

Solution 4.1
(a) Without mortality improvement we have

$$ {}_{10}p_{60} = \prod_{t=0}^{9}(1 - q(60+t, 0)) = 0.87441 $$

With mortality improvement we have

$$ \begin{aligned} {}_{10}p_{60} &= \prod_{t=0}^{9}(1 - q(60+t, t)) \\ &= \prod_{t=0}^{9}\left(1 - q(60+t, 0)(1-\varphi_{60+t})^{t}\right) = 0.88277 \end{aligned} $$

(b) Without mortality improvement we have

$$ \ddot{a}_{60:\overline{10|}} = 1 + \sum_{t=1}^{9} v^t \prod_{k=0}^{t-1}(1 - q(60+k, 0)) = 7.7606 $$

With mortality improvement we have

$$ \begin{aligned} \ddot{a}_{60:\overline{10|}} &= 1 + \sum_{t=1}^{9} v^t \prod_{k=0}^{t-1}(1 - q(60+k, k)) \\ &= 1 + \sum_{t=1}^{9} v^t \prod_{k=0}^{t-1}(1 - q(60+k, 0)(1-\varphi_{60+k})^{k}) \\ &= 7.7744 \end{aligned} $$

Notice that the impact on the annuity in this case is small, both because the term is short and because the mortality is fairly light even without the improvement factors.

Using mortality rates that allow for improvement leads to an increase in the value of annuities, as lives are expected to live longer and therefore collect more annuity. We use the term **longevity risk** for the risk that we underestimate the cost of life contingent benefits through unanticipated changes in the mortality rates experienced by annuitants or policyholders. In recent years, this risk has been associated with higher longevity than expected, which leads to underestimation of the cost of annuity type payments. For life insurance policies, longer life reduces the costs, as it allows for more premium collection, and a longer period to the payment of the sum insured, and so life insurance portfolios are not exposed to significant longevity risk, although, this would change if trends started to show increasing mortality rates over time.

The one factor mortality improvement scales have proven too simplistic. The AA scale predicted that mortality at age 70 would improve by 1.5% per year indefinitely, but the heat map shows improvement rates of around 2.75% in the mid 2000's. On the other hand, the heat map shows that the higher values of the improvement factors might not persist for later cohorts.

A more robust approach to deterministic mortality improvement scales uses improvement factors that are a function of both age and calendar year. This approach is used in the MP2014 tables of the Society of Actuaries as well as in the CPM scales of the Canadian Institute of Actuaries. The method used for both of these was first proposed by the Continuous Mortality Investigation Bureau (CMIB), which is a standing committee of the Institute and Faculty of Actuaries in the UK.

The improvement scales are determined in three steps

- Determine **short term improvement factors**, using regression or other smoothing techniques applied to recent experience.
- Determine **long term improvement factors**, and the time at which the long term rates will be reached. After this time, the rates are assumed to be constant. This step is usually based on subjective judgment.
- Determine **intermediate improvement factors** using smooth functions that will connect the short and long term factors.

For the MP2014 tables, the Society of Actuaries used the following three steps to generate past and future improvement factors, $\varphi(x,t)$, where x is the age (integer values from 15 to 95) and t is the calendar year from 1950 forwards.

- Improvement factors for calendar years 1950-2007 are determined by taking the raw mortality experience from the US Social Security Administration (SSA) database. A two dimensional smoothing method is applied to the logarithm of the raw mortality rates, generating smooth log-rates denoted $s(x,t)$. The two dimensional smoothing ensures that $s(x,t)$ is smooth across ages x and across calendar years y. The smoothed historical rates up to 2007 are then

  $$\tilde{q}(x,t) = e^{s(x,t)}$$

  and the historical improvement factors for 1950 -2007 are

  $$\varphi(x,t) = 1 - \frac{\tilde{q}(x,t)}{\tilde{q}(x,t-1)} = 1 - e^{s(x,t)-s(x,t-1)}$$

- Long term improvement factors were set at 1% at all ages up to age 65, decreasing linearly to 0% at age 115, for both males and females. These rates are assumed to apply from 2027.

- Intermediate factors covering calendar years 2008-2026 are determined using a blend of age-based cubic splines and cohort-based cubic splines.

A spline is a smooth function that can be used to interpolate between two other functions. In our context, we have the historical improvement factors up to 2007, and we have the assumed long term improvement factors applying from 2027, which are assumed to be constant. A cubic spline is a cubic function of time (in years) measured from 2007 which matches the improvement function values at 2007 and 2027, and also matches the gradient of the improvement function at 2007 and at 2027. These four constraints will give us four simultaneous equations for the four parameters of the cubic function. The two end points joined by the spline are called knots.

The age-based cubic spline uses the same age for the knots at 2007 and 2027. For a given age x it is a function of time t measured in years from 2007, joining the functions $\varphi(x, 2007 + t)$, for $t = 0, -1, -2, ...$ and $\varphi(x, 2007 + t)$, for $t = 20, 21, 22, ...$ We denote the function $C_a(x, t)$ so that

$$C_a(x, t) = at^3 + bt^2 + ct + d$$

$$C'_a(x, t) = 3at^2 + 2bt + c$$

And we have four equations for the four parameters:

(1) $C_a(x, 0) = d = \varphi(x, 2007)$
(Set the starting value of the spline)

(2) $C'_a(x, 0) = c = \varphi(x, 2007) - \varphi(x, 2006)$
(Set the starting derivative of the spline equal to the 2007 gradient)

(3) $C_a(x, 20) = 8000a + 400b + 20c + d = \varphi(x, 2027)$
(Set the end value of the spline)

(4) $C'_a(x, 20) = 1200a + 40b + c = 0$
(Set the end value of the spline equal to the 2027 gradient)

And we can solve the four equations to determine the coefficients a, b, c, d of the polynomial.

The cohort-based spline is similar, but it smooths the improvement factors for a cohort age x in 2007, which means it interpolates between the functions $\varphi(x + t, 2007 + t)$, for $t = 0, -1, -2...$ and $\varphi(x + t, 2007 + t)$ for $t = 20, 21, 22, ...$ We denote the cohort-based spline for a life age x at $2007 + t$ as $C_c(x, t)$, where

$$C_c(x, t) = a^*t^3 + b^*t^2 + c^*t + d^*$$

$$C'_c(x, t) = 3a^*t^2 + 2b^*t + c^*$$

and the four equations are

(1) $\quad C_c(x-t,0) = d^* = \varphi(x-t,2007)$

(2) $\quad C_c'(x-t,0) = c^* = \varphi(x-t,2007) - \varphi(x-t-1,2006)$

(3) $\quad C_c(x-t+20,20) = 8000a^* + 400b^* + 20c^* + d^* = \varphi(x-t+20,2027)$

(4) $\quad C_c'(x-t+20,20) = 1200a^* + 40b^* + c^* = \varphi(x-t+21,2028) - \varphi(x-t+20,2027)$

The improvement factor for age x in year t is then taken as the average of the two splines

$$\varphi(x,2007+t) = 0.5C_a(x,t) + 0.5C_c(x,t) \qquad t = 1,2,...,19$$

Example 4.2
Calculate the MP2014 1-year improvement factor for a female life age 40 in 2020, given the following values for short and long term improvement factors:

$$\varphi(40,2006) = 0.0162;\ \varphi(40,2007) = 0.0192;\ \varphi(40,2027) = 0.01;\ \varphi(40,2028) = 0.01$$

$$\varphi(26,2006) = -0.0088;\ \varphi(27,2007) = -0.0088;\ \varphi(47,2027) = 0.01;\ \varphi(48,2028) = 0.01$$

Solution 4.2
The age-based improvement spline applicable to a life age 40 in 2020 is the cubic joining $\varphi(40,2007)$ and $\varphi(40,2027)$. The value for 2020 is $C_a(40,13)$. Solving for the parameters of the $C_a(40,t)$ spline, we have

$$\varphi(40,2007) = C_a(40,0) = d \implies d = 0.0192$$

$$\varphi(40,2007) - \varphi(40,2006) = C_a'(40,0) = c \implies c = 0.003$$

$$\left.\begin{array}{l} \varphi(40,2027) = 8000a + 400b + 20c + d = 0.01 \\ \varphi(40,2028) - \varphi(40,2027) = 1200a + 40b + c = 0 \end{array}\right\} \implies \begin{array}{l} a = 9.8 * 10^{-6} \\ b = -3.69 * 10^{-4} \end{array}$$

So we have

$$C_a(40,13) = a \times 13^3 + b \times 13^2 + c \times 13 + d = 0.01737$$

The cohort-based spline applicable to a life age 40 in 2020 is the cubic joining $\varphi(27,2007)$ and

x	$q(x,2010)$	$\varphi(x,2010+t)$, $t=1$	2	3	4	5	6	7	8	9	10
50	0.002768	0.0206	0.0227	0.0238	0.0243	0.0241	0.0233	0.0221	0.0205	0.0188	0.0170
51	0.002905	0.0180	0.0205	0.0221	0.0229	0.0230	0.0226	0.0216	0.0203	0.0188	0.0171
52	0.003057	0.0156	0.0181	0.0201	0.0213	0.0218	0.0217	0.0210	0.0200	0.0186	0.0171
53	0.003225	0.0124	0.0148	0.0168	0.0184	0.0193	0.0195	0.0192	0.0185	0.0175	0.0162
54	0.003412	0.0093	0.0115	0.0134	0.0150	0.0164	0.0170	0.0171	0.0167	0.0160	0.0151
55	0.003622	0.0066	0.0085	0.0104	0.0120	0.0134	0.0145	0.0150	0.0150	0.0146	0.0140
56	0.003858	0.0045	0.0061	0.0078	0.0094	0.0109	0.0121	0.0130	0.0134	0.0134	0.0131
57	0.004128	0.0033	0.0045	0.0060	0.0075	0.0090	0.0103	0.0113	0.0121	0.0125	0.0124
58	0.004436	0.0031	0.0037	0.0049	0.0063	0.0078	0.0091	0.0102	0.0111	0.0117	0.0120
59	0.004789	0.0039	0.0039	0.0046	0.0057	0.0071	0.0084	0.0096	0.0105	0.0112	0.0117
60	0.005191	0.0055	0.0049	0.0050	0.0058	0.0069	0.0082	0.0094	0.0103	0.0110	0.0115

Table 3: Mortality rates and improvement factors, $\varphi(x, 2010+t)$ for Example 4.3

$\varphi(47, 2027)$. Solving for the parameters of the $C_c(27+t, t)$ spline we have

$$\varphi(27, 2007) = C_c(27, 0) = d^* \implies d^* = -0.0088$$

$$\varphi(27, 2007) - \varphi(26, 2006) = C_c'(27, 0) = c^* \implies c^* = 0.0$$

$$\left.\begin{array}{l} \varphi(47, 2027) = 8000a^* + 400b^* + 20c^* + d^* = 0.01 \\ \varphi(48, 2028) - \varphi(47, 2027) = 1200a^* + 40b^* + c^* = 0 \end{array}\right\} \implies \begin{array}{l} a^* = -4.7 * 10^{-6} \\ b^* = 1.41 * 10^{-4} \end{array}$$

So we have

$$C_c(40, 13) = a^* \times 13^3 + b^* \times 13^2 + c^* \times 13 + d^* = 0.00470$$

Hence, the improvement factor for age 40 in 2020 is

$$\varphi(40, 2020) = 0.5C_a(40, 13) + 0.5C_c(40, 13) = 0.011035$$

Example 4.3
In Table 3 you are given the mortality rates for lives age 50-60 in 2010, together with improvement factors for 2011 to 2020.

Calculate ${}_{10}p_{50}$ and $\ddot{a}_{50:\overline{10|}}$ for a life age 50 in 2010 assuming (i) no mortality improvement and (ii) mortality improvement using the two-way factors in Table 3. Use $i = 5\%$.

Solution 4.3
With no mortality improvement we have

$${}_{10}p_{50} = \prod_{t=0}^{9}(1 - q(50+t, 2010)) = 0.96438$$

The annuity value is

$$\sum_{t=0}^{9} p((50+t,2010)\, v^t = 8.0026$$

With mortality improvement we have the following age-year mortality rates

$$q(50,2010) = 0.002768$$

$$q(51,2011) = q(51,2010)(1-\varphi(51,2011)) = 0.002905(1-0.018) = 0.002853$$

$$q(52,2012) = q(52,2010)(1-\varphi(52,2011))(1-\varphi(52,2012)) = 0.002955$$

$$\vdots$$

$$q(59,2019) = q(59,0)(1-\varphi(59,2011))(1-\varphi(59,2012))...(1-\varphi(59,2019)) = 0.004487$$

We can construct a life table excerpt from the mortality rates applicable to the life age 50 in 2010, using an arbitrary radix of 100,000:

$50+t$	$l(50+t,2010+t)$	$50+t$	$l(50+t,2010+t)$
50	100 000.0	56	98 179.0
51	99 723.2	57	97 819.1
52	99 438.7	58	97 435.8
53	99 144.9	59	97 027.3
54	98 839.0	60	96 592.0
55	98 518.2		

Using this table we have ${}_{10}p_{50} = 0.96592$, and $\ddot{a}_{50:\overline{10|}} = 8.0059$

4.3 Stochastic mortality models

The deterministic approach to modelling mortality improvement assumes that rates are determined, known and smooth. This may be reasonable for calculating expected present values, or median present values, but it does not allow us to analyse or measure the risk from random variation in the underlying mortality rates. When we look at the curves in Figure 6 we see that the rates are quite variable from year to year. To analyse the potential impact of random variation in the year to year mortality experience we may use stochastic processes to model future mortality experience.

In AMLCR Chapter 8 we define a stochastic process as a collection of random variables indexed by a time variable. A stochastic mortality model is a stochastic process for the rate of mortality experienced by lives at different ages for different future dates. We generally use discrete time stochastic processes, so that we generate stochastic mortality rates suitable for each age in each future calendar year.

In the following sections we will describe some current stochastic mortality models that have been suggested for analysing long term annuity and insurance risks.

4.4 The Lee Carter Model

The Lee Carter model is probably the most famous stochastic mortality model of the past 30 years. It was first presented in Lee and Carter in 1992, and has been widely analysed and extended since then.

The Lee Carter model works with central death rates, so we first define what these are.

Given a mortality model, defined (say) by the force of mortality function μ_x, the central death rate is m_x where

$$m_x = \frac{q_x}{\int_0^1 {}_r p_x \, dr} = \frac{\int_0^1 {}_r p_x \, \mu_{x+r} \, dr}{\int_0^1 {}_r p_x \, dr} \tag{29}$$

so that m_x is a weighted average of the force of mortality experienced by lives between age x and $x+1$. If the force of mortality is reasonably flat over each year of age, then m_x will be close to μ_x, and

$$q_x \approx 1 - e^{-m_x}$$

In the Lee Carter model, the central death rate varies for age x and for calendar year t, so we write it as $m(x,t)$. For each integer age x the natural log of the central death rate is assumed to follow a discrete time stochastic process as follows:

$$\log m(x,t) = \alpha_x + \beta_x \, K_t + \epsilon_{x,t} \tag{30}$$

where

- α_x and β_x are parameters depending only on the attained age x.
- Past values of K_t are parameters found by fitting data to the model. Future values of K_t are modelled as a time series that is fitted to the estimated historical values. The K_t series is not age dependent.

  The fitted values for K_t often appear to be fairly linear, and the usual forecasting model is a random walk with drift, which we also assume in this note. This means that we assume

  $$K_{t+1} = K_t + c + \sigma_k \, Z_t \tag{31}$$

where c is a constant drift term, σ_k is the standard deviation of the annual change in K_t, and the Z_t are assumed to be i.i.d. random variables, with standard $N(0,1)$ distribution.

- $\epsilon_{x,t}$ is a random error term which is assumed to be sufficiently small to be negligible, and is often ignored in the definition and analysis of the Lee Carter model. In this note we follow this custom, which means we assume all of the uncertainty in the model is generated by the stochastic process K_t.

In this note we let $lm(x,t) = \log(m(x,t))$ so that

$$m(x,t) = e^{lm(x,t)} \tag{32}$$

Then, with the assumptions listed above, we can write the Lee Carter model as

$$lm(x,t) = \alpha_x + \beta_x\, K_t \tag{33}$$

$$= \alpha_x + \beta_x\, (K_{t-1} + c + \sigma_k\, Z_t) \tag{34}$$

The key to the model is the separation of age effects and year effects. The time series for K_t introduces random year effects, and the factor β_x allows for the year effects to have a different impact on different ages.

The model has an identifiability problem in the form presented. We could, for example, multiply all the β_x by 2 and divide all the K_t by 2 and end up with the same values. To solve this we add two constraints which depend on the data used to fit the model parameters.

Suppose the data covers ages x_0 to x_w, and calendar years t_0 to t_n, then the constraints are

$$\sum_{x=x_0}^{x_w} \beta_x = 1.0 \qquad \sum_{t=t_0}^{t_n} K_t = 0.0 \tag{35}$$

Applying these constraints we can see how the α_x parameters can be interpreted, as, for a given age x

$$\sum_{t=t_0}^{t_n} lm(x,t) = (t_n - t_0)\alpha_x + \beta_x \sum_{t=t_0}^{t_n} K_t \implies \alpha_x = \frac{\sum_{t=t_0}^{t_n} lm(x,t)}{t_n - t_0}$$

so that the α_x parameters represent the average of the log-central death rates for age x over the period of the data.

The estimation process for the model is quite complex, and is beyond the scope of this note.

In Figure 9 we show typical plots for the fitted parameters for the Lee Carter model for a data set of retirement age lives [6].

[6] This figure was generously provided by Professor Johnny S-H Li.

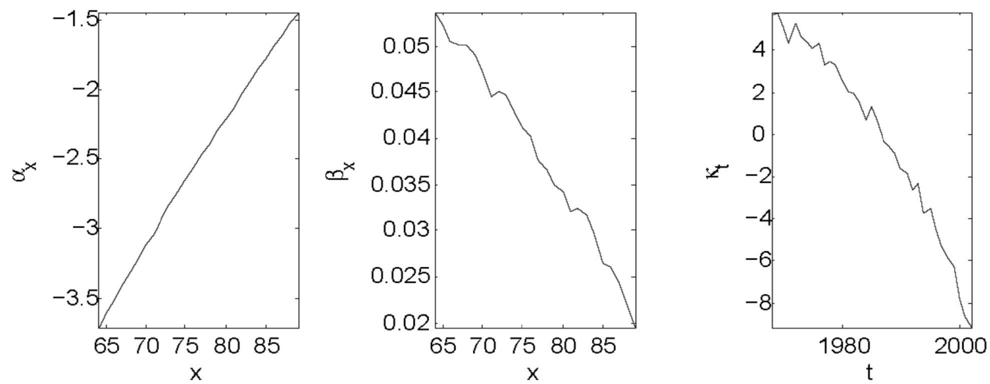


Figure 9: Example of fitted parameters for the Lee Carter model

For this data set we see a decreasing mortality trend from the negative slope of the K_t, and also that younger ages in the group experienced more benefit from the decreasing mortality trend than older ages, evident from the negative slope for β_x.

Although the Lee Carter model has been widely applied since its first publication, there are some problems, particularly for actuarial applications. The most important is that the fit to data tends not to be very good. This is partly because it does not allow for any cohort effect, yet we often see cohort effects in the actuarial and population data. Also, the model assumes (implictly) perfect correlation between mortality improvements at different ages, as we show in Example 4.4, but the mortality data shows that improvements may be far from perfectly correlated.

Example 4.4
Let $R(x,t) = lm(x,t) - lm(x,t-1)$. Show that $R(x,t)$ and $R(y,t)$ are perfectly correlated for $y \neq x$.

Solution 4.4
The correlation is

$$\rho = \frac{E[R(x,t)R(y,t)] - E[R(x,t)]E[R(y,t)]}{SD[R(x,t)]SD[R(y,t)]}$$

where $R(x,t) = (\alpha_x + \beta_x K_t) - (\alpha_x + \beta_x K_{t-1}) = \beta_x (K_t - K_{t-1})$

and $K_t - K_{t-1} = c + \sigma_k Z_t \Rightarrow E[K_t - K_{t-1}] = c$

and $SD[K_t - K_{t-1}] = \sigma_k$

so $E[R(x,t)] = \beta_x c; \qquad SD[R(x,t)] = \beta_x \sigma_k$

$E[R(x,t)R(y,t)] = E[\beta_x\beta_y (K_t - K_{t-1})^2] = \beta_x\beta_y \left(c^2 + \sigma_k^2\right)$

$$\implies \rho = \frac{\beta_x\beta_y(c^2+\sigma_k^2) - \beta_x\beta_y c^2}{(\beta_x\sigma_k)(\beta_y\sigma_k)} = 1$$

Example 4.5
You are given the following parameters for the Lee Carter model.

$\alpha_{70} = -2.684 \quad \beta_{70} = 0.04 \quad K_{2017} = -10.0 \quad c = -0.4 \quad \sigma_k = 0.7$

(a) (i) Calculate the mean and standard deviation of $m(70, 2018)$.

(ii) Calculate the median and the 5% quantile of $m(70, 2018)$

(b) (i) Calculate $p(70, 2018)$ assuming that the central death rate takes the mean value from (a), and assuming constant force of mortality between integer ages. Explain why this is not the mean value of $p(70, 2018)$.

(ii) Calculate the 50% and 95% quantiles of $p(70, 2018)$ assuming constant force of mortality between integer ages.

(c) (i) Calculate $p(70, 2018)$ assuming that the central death rate takes the mean value from (a), and assuming UDD between integer ages.

(ii) Calculate the 50% and 95% quantiles of $p(70, 2018)$ assuming UDD.

Solution 4.5

(a)

$$\begin{aligned} lm(70,2018) &= \alpha_{70} + \beta_{70} K_{2018} \\ &= \alpha_{70} + \beta_{70} (K_{2017} + c + \sigma_k Z_t) \\ &= -2.684 + 0.040(-10.0 - 0.4 + 0.7Z_t) = -3.100 + 0.028Z_t \qquad (36) \end{aligned}$$

$$\implies lm(70,2018) \sim N(-3.1,\, 0.028^2) \quad \text{because } Z_t \sim N(0,1) \qquad (37)$$

Now, for any normally distributed random variable $X \sim N(\mu, \sigma^2)$, the function $Y = e^X$ has a lognormal distribution. We write $Y \sim \text{logN}(\mu, \sigma)$. The mean and variance of a lognormal random variable with parameters μ and σ are

$$\mathrm{E}[Y] = e^{\mu+\sigma^2/2} \qquad \mathrm{V}[Y] = (\mathrm{E}[Y])^2 \left(e^{\sigma^2} - 1\right) \tag{38}$$

$$\text{So} \quad m(70, 2018) = e^{lm(70,2018)} \sim \text{logN}(-3.1, 0.028)$$

$$\implies \mathrm{E}[m(70, 2018)] = e^{-3.1+0.028^2/2} = 0.04507$$

$$\text{and } \mathrm{V}[m(70, 2018)] = (0.04507)^2(e^{0.028^2} - 1) = 0.0013^2$$

For the quantiles, because lm is an increasing function of Z_t, we can find the q-quantile of lm by replacing Z_t in equation (36) with its q-quantile. Then, because m is an increasing function of lm, we can find the q-quantile of m by replacing lm with its q-quantile in equation (32).

Let $Q_q(X)$ denote the q quantile of the random variable X. Then the median corresponds to $q = 50\%$,

$$Q_{50\%}(Z_t) = 0 \implies Q_{50\%}(lm(70, 2018)) = -3.1$$

$$\implies Q_{50\%}(m(70, 2018)) = e^{-3.1} = 0.04505$$

$$\text{and} \quad Q_{5\%}(Z_t) = -1.645$$

$$\implies Q_{5\%}(lm(70, 2018)) = -3.1 + (0.028)(-1.645) = -3.146$$

$$\implies Q_{5\%}(m(70, 2018)) = e^{-3.146} = 0.04302$$

(b) With the constant force assumption, the central death rate is equal to the constant force, so a central death rate of $m_{70} = e^{-3.1+0.028^2/2}$ (from the expected value in (a)) corresponds to a survival probability of $p_{70} = e^{-m_{70}} = 0.9559336$.

This is not the expected value of $p(70, 2018)$; for this we need

$$\mathrm{E}[p(70, 2018)] = \mathrm{E}\left[e^{-m(70,2018)}\right] = \mathrm{E}\left[e^{-(e^{-3.1+0.028Z_t})}\right] = 0.9559338$$

The survival probability is a decreasing function of the central death rate, so the q-quantile for $p(70, 2018)$ corresponds to the $(1-q)$-quantile for $m(70, 2018)$:

$$Q_{50\%}(p(70, 2018)) = e^{-Q_{50\%}(m(70,2018))} = 0.95595$$

$$Q_{95\%}(p(70, 2018)) = e^{-Q_{5\%}(m(70,2018))} = 0.95789$$

(c) Under UDD we have ${}_r q_x = r({}_1 q_x)$ for $0 \le r \le 1$. Then

$$m_x = \frac{q_x}{\int_0^1 {}_r p_x \, dr} = \frac{q_x}{\int_0^1 (1 - r(q_x))dr}$$

$$= \frac{q_x}{1 - q_x/2}$$

$$\implies q_x = \frac{m_x}{1 + m_x/2} \implies p_x = \frac{1 - m_x/2}{1 + m_x/2}$$

So assuming $m_x = 0.04507$ we have $p_x = 0.95592$

For (c)(ii), following the same process as in (b), we have

$$Q_{50\%}(p(70, 2018)) = \frac{1 - Q_{50\%}(m(70, 2018))/2}{1 + Q_{50\%}(m(70, 2018))/2} = 0.95594$$

$$Q_{95\%}(p(70, 2018)) = \frac{1 - Q_{5\%}(m(70, 2018))/2}{1 + Q_{5\%}(m(70, 2018))/2} = 0.95789$$

Example 4.6

Define the central death rate improvement factor as the random variable

$$\varphi^m(x, t) = 1 - \frac{m(x, t)}{m(x, t-1)}$$

(a) Show that the distribution of $\varphi^m(x, t)$ does not depend on t.

(b) Calculate the mean, standard deviation, median and the 95% quantile of $\varphi^m(70, t)$, using the parameters given in Example 4.5 above.

Solution 4.6

(a)

$$\log \frac{m(x, t)}{m(x, t-1)} = lm(x, t) - lm(x, t-1) = \beta_x(K_t - K_{t-1})$$

$$= \beta_x(c + \sigma_k \, Z_t) \quad \text{from eqn (31)}$$

$$\implies \log(1 - \varphi^m(x, t)) \sim N\left(\beta_x \, c, \, (\beta_x \, \sigma_k)^2\right)$$

$$\implies (1 - \varphi^m(x, t)) \sim \text{logN}\left(\beta_x \, c, \, (\beta_x \, \sigma_k)\right)$$

which demonstrates that the distribution of $\varphi^m(x,t)$ does not depend on t.

(b)

$$(1-\varphi^m(70,t)) \sim \text{logN}\left(-0.016, 0.028\right)$$

$$\text{E}\left[\varphi^m(70,t)\right] = 1 - e^{-0.016+0.028^2/2} = 0.015486$$

$$\text{V}\left[\varphi^m(70,t)\right] = \text{V}\left[1-\varphi^m(70,t)\right] = \left(e^{-0.016+0.028^2/2}\right)^2\left(e^{0.028^2}-1\right) = 0.02757^2$$

$$Q_{50\%}(\varphi^m(70,t)) = 1 - e^{-0.016} = 0.01587$$

$$Q_{95\%}(\varphi^m(70,t)) = 1 - Q_{5\%}(\varphi^m(70,t)) = 1 - e^{-0.016-1.645(0.028)} = 0.06017$$

The quantiles calculated in the examples above can be used to give some measure of longevity risk, but it should be noted that there is significant uncertainty in the parameters that is not captured in these calculations. We have also ignored the error term $\epsilon_{x,t}$ from equation (30), and there is additional uncertainty arising from model risk, given that the model fit to data is often not that compelling.

4.5 The Cairns-Blake-Dowd Models

The Cairns-Blake-Dowd (CBD) family of models has become very popular for actuarial applications. The original model fits a two factor time series to the logit of the mortality rate; the model has been extended to four or more terms.

In this section we describe the original model, and briefly discuss a popular extension of the model.

The model works with the logit function of mortality rates. The logit function is defined as $\text{logit}(x) = \log(x/(1-x))$. In this section we let

$$lq(x,t) = \log\frac{q(x,t)}{1-q(x,t)}$$

4.5.1 The original CBD model

The original CBD model is defined as

$$lq(x,t) = K_t^{(1)} + K_t^{(2)}\left(x-\bar{x}\right) \tag{39}$$

where

- $\bar{x}$ is the average age in the data set.

- $K_t^{(1)}$ and $K_t^{(2)}$ are correlated time series. Usually, each is assumed to follow a random walk with drift, so that

  $$K_t^{(1)} = K_{t-1}^{(1)} + c^{(1)} + \sigma_{k_1} Z_t^{(1)}$$

  $$K_t^{(2)} = K_{t-1}^{(2)} + c^{(2)} + \sigma_{k_2} Z_t^{(2)}$$

- $Z_t^{(1)}$ and $Z_t^{(2)}$ are standard $\mathrm{N}(0,1)$ random variables, which are correlated with each other in each given year, but are independent from year to year. That means that

  $$\mathrm{E}[Z_t^{(1)}] = \mathrm{E}[Z_t^{(2)}] = 0 \quad \mathrm{V}[Z_t^{(1)}] = \mathrm{V}[Z_t^{(2)}] = 1 \quad \mathrm{E}[Z_t^{(1)} Z_t^{(2)}] = \rho \quad -1 \le \rho \le 1$$

  $$\mathrm{E}[Z_t^{(i)} Z_u^{(j)}] = 0 \text{ for } t \neq u,\ i = 1,2,\ j = 1,2$$

Example 4.7

Suppose $K_{2017}^{(1)} = -3.2,\ K_{2017}^{(2)} = 0.01,\ c^{(1)} = -0.02,\ c^{(2)} = 0.0006,\ \bar{x} = 70$

$\sigma_{k_1} = 0.03,\ \sigma_{k_2} = 0.005,\ \rho = 0.2$

(a) Calculate the mean and variance of the $lq(65, 2018)$.

(b) Calculate the median and 95% quantile of $p(65, 2018)$.

Solution 4.7
(a)

$$lq(65, 2018) = K_{2018}^{(1)} + K_{2018}^{(2)}(65 - 70)$$

$$K_{2018}^{(1)} = -3.2 - 0.02 + 0.03 Z_{2018}^{(1)}$$

$$K_{2018}^{(2)} = 0.01 + 0.0006 + 0.005 Z_{2018}^{(2)}$$

$$\implies lq(65, 2018) = -3.273 + \left(0.03 Z_{2018}^{(1)} - 0.025 Z_{2018}^{(2)}\right)$$

On the right hand side we have a linear function of correlated Gaussian random variables, which means that $lq(65,2018)$ is also normally distributed, with mean -3.273 and variance

$$\mathrm{V}[lq(65,2018)] = 0.03^2 + 0.025^2 - 2(0.03)(0.025)(0.2) = 0.001225 = 0.035^2$$

$$\implies lq(65,2018) \sim N(-3.273,\ 0.035^2)$$

$$\implies \frac{q(65,2018)}{1-q(65,2018)} \sim \mathrm{logN}(-3.273, 0.035)$$

$$\implies \mathrm{E}\left[\frac{q(65,2018)}{1-q(65,2018)}\right] = e^{-3.273+0.035^2/2} = 0.0379$$

$$\text{and} \quad \mathrm{V}\left[\frac{q(65,2018)}{1-q(65,2018)}\right] = \left(e^{-3.273+0.035^2/2}\right)^2\left(e^{0.035^2}-1\right) = 0.00133^2$$

(b) $lq(x,t)$ is an increasing function of $q(x,t)$, and is therefore a decreasing function of $p(x,t)$

$$Q_{50\%}(lq(65,2018)) = -3.273$$

$$\implies Q_{50\%}\left(\frac{q(65,2018)}{1-q(65,2018)}\right) = e^{-3.273} = 0.03789$$

$$\implies Q_{50\%}(q(65,2018)) = \frac{0.03789}{1+0.03789} = 0.03651$$

$$\implies Q_{50\%}(p(65,2018)) = 0.96349$$

$$Q_{5\%}(lq(65,2018)) = -3.273 - 1.645(0.035) = -3.33057$$

$$\implies Q_{5\%}\left(\frac{q(65,2018)}{1-q(65,2018)}\right) = e^{-3.33057} = 0.03577$$

$$\implies Q_{5\%}(q(65,2018)) = \frac{0.03577}{1+0.03577} = 0.03454$$

$$\implies Q_{95\%}(p(65,2018)) = 0.96546$$

4.5.2 The CBD M7 Model

The CBD model has some advantages over the Lee Carter model, with fewer parameters, and less parameter uncertainty in practice, but the fit to population mortality data of the original model is not significantly better than Lee Carter, and in some cases is worse.

By adding one or two terms to the CBD model the fit can be significantly improved, while the advantages of the model are mostly retained. A popular extension is the CBD M7 model[7] which is defined as

$$lq(x,t) = K_t^{(1)} + K_t^{(2)}\,(x-\bar{x}) + K_t^{(3)}\left((x-\bar{x})^2 - s_x^2\right) + G_{t-x}$$

There are two terms that are not in the original model.

- The first is an extra year-effect time series, $K_t^{(3)}$, which has a quadratic impact across the age groups. The s_x term is just the standard deviation of the age range used. So, if the age range used to fit the model runs from age 50 to age 90, then

$$\bar{x} = \frac{1}{41}\sum_{x=50}^{90} x = 70 \quad \text{and} \quad s_x^2 = \frac{1}{41}\sum_{x=50}^{90}(x-70)^2 = 140$$

- The second additional term is G_{t-x} which introduces a cohort effect time series. For a life age, say, 65 in 2018, the lq function would use G_{1953}, where 1953 is the birth year of the cohort, and that same G_{1953} term would appear in the subsequent lq functions for the group (that is $lq(66,2019)$, $lq(67,2020)$, $\cdots$). For G_t where t lies beyond the range of the data, we fit a time series, but it is more cyclical than the K_t, typically, and would be fitted to an ARIMA type model.

4.6 Actuarial applications of stochastic mortality models

In the examples in Section 4.3 we calculated some measures of risk or variability relating to one-period ahead mortality. The impact of stochastic mortality on the cash flows of a portfolio of annuities projected much farther into the future is not so analytically tractable, especially for more complex models such as CBD M7.

Commonly, actuaries use Monte Carlo simulation to assess the potential impact of longevity risk. Monte Carlo simulation is described in Chapter 11 of AMLCR. The method can be used to generate a large number of random[8] paths for $p(x,t)$ into the future. We can use these paths to estimate distributions of cash flows and present values.

For example, suppose an insurer uses Monte Carlo simulation to generate 10,000 different paths for $p(x, 2017+k)$, for $x = 60, ..., 110$, and for $k = 1, ..., 50$, where we assume 110 is the ultimate age attainable. That means, for each path we are simulating survival probabilities for all ages, for each of the 50 years, which is a total of 2550 values for each path. We repeat this 10,000 times,

[7] The M7 comes from the numbering in Cairns et al (2009), where it was the seventh of eight models analysed.

[8] Technically, the paths are not random, but are sufficiently indistinguishable from a true random process that they can be treated as random.

using random number generators that create independent paths from the underlying process that are equally likely, and that can be treated as a random sample from the distribution.

For example, let $p_j(x, 2017+k)$ denote the simulated value for $p(x, 2017+k)$ from the jth simulated path.

Given a single path of survival probabilities starting at age 60, for example, that is, given $p_j(60, 2017)$, $p_j(61, 2018)$, etc, we can calculate the actuarial value of an annuity issued to (60) in 2017, conditional on that path. Let $\ddot{a}(60, 2017)_j$ denote the expected present value given the jth path, then

$$\ddot{a}_j(60, 2017) = \sum_{k=0}^{50} p_j(60+k, 2017+k)v^k$$

Repeating this for all of the 10,000 paths for the survival probabilities gives us a sample of 10,000 values for the present value of an annuity-due issued to age 60, taking longevity risk into consideration. From this sample, we can calculate moments such as the mean and variance, and we can assess the exposure to longevity risk by considering the impact if the annuity value takes an adverse value, such as the 95% quantile of the distribution.

Since longevity changes tend to impact the whole portfolio, not just a single age range, the insurer would not look separately at values for each age group, but would generate valuations for the whole portfolio of annuities at different ages, valued along each separate path for $p(x, 2017+k)$ for all ages x. The results could be used to assess the adequacy of reserves and of pricing, taking longevity risk into consideration. The ability of the insurer to survive extreme scenarios could be investigated by considering, for example, the worst case quantiles of the simulated distribution of the portfolio value.

4.7 Notes on stochastic mortality models

1. Different populations can display very different combinations of age, year and cohort effects. In some populations cohort effects are not at all strong, while in others they are crucial to model fit. We also need to consider sub-populations. We often use data from national census or social security records to fit stochastic mortality models, as a large amount of data is needed, due to the large number of parameters to be fitted, and because at much older ages we have less reliable parameter estimates as we may have very few lives in the database.

   There is a problem though if the population statistics are then used to model improvement factors for insured lives and annuitants. Typically, people who buy annuities and insurance are healthier and wealthier than the population as a whole, and they tend to be the first to benefit from the medical and social advances that improve longevity. The whole-population models may underestimate the longevity risk from current annuitants, if population and annuitant longevity continues to improve. On the other hand, we may overestimate the risk if the annuitant population mortality trends slow down before the population as a whole. For example, suppose we use the annuitant mortality from 2015 as our base mortality table, and then use estimated population mortality improvement factors to project the annuitant mortality rates out to the future. Suppose further that population longevity improvement is likely to be generated by improved population access to new drugs such as statins. If the annuitant population already benefited from early access to the drugs, based on their relatively privileged social position, then applying the population improvement factors to the base annuitant mortality will double count the impact of statins, and will overestimate the longevity of the group.

2. Stochastic mortality improvement models can be used to determine deterministic improvement scales. Typically, we would use the median or mean improvement factors from the stochastic model to generate scales to apply to current tables.

3. In the descriptions of the Lee Carter and CBD models we assumed the year effects time series followed random walk with drift. However, other time series models can be used.

   The modelling process for the K_t series starts with estimating the values for all the years in the data, since we don't directly observe these values. Then the estimated values are analysed using standard time series methods. For the Lee Carter and original CBD models, applied to population data from the US or the UK, the estimated values appear to follow a reasonably straight line, but for more complex models such as the CBD M7, and for some other populations, a better fit might be obtained using an autoregressive or ARIMA type model.

4. We have not given much indication of whether or why one stochastic mortality model is better than another. We can do statistical analysis of goodness of fit, but the model and parameter uncertainty is typically very large, and it is not unusual for all the models to fail standard tests of fit. We need to find ways of choosing the least bad of the models, but selecting a model is never an automatic process. Some models do well on relative fit, but demonstrate extreme parameter uncertainty – for example, generating very different values when we use slightly different historic periods to estimate the parameters.

   Ultimately, there is still a very large amount of parameter and model uncertainty in stochastic longevity models. Nevertheless, they are becoming essential tools for actuarial risk management of annuity portfolios and pension plans, because they are so much better than no model at all. At least we can generate some indication of the range of possible outcomes for an annuity portfolio's cashflows. Still, it is essential for actuaries using these models to understand the significant limitations of the models.

5. Given the sometimes conflicting information from the statistical metrics for model selection, we might want to assess the reasonableness of the models intuitively. The CBD models generate smooth survival rates across ages within each year, because of the $(x - \bar{x})$

and $((x-\bar{x})^2 - s_x^2)$ terms. The Lee Carter model may generate rather less smoothness across the ages depending on the parameters.

4.8 References and further reading

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5 Structured Settlements

5.1 Introduction and background

When a person is injured because of a negligent or criminal act committed by another person, or by an institution, legal processes will determine a suitable amount of compensation paid to the injured party (IP) by the person or institution who caused the injury (responsible person, RP)[9].

The compensation may be paid as a lump sum, but in some jurisdictions it is more common for the payment to be paid as an annuity, or as a combination of a lump sum and an annuity. The annuity may be a term annuity, a whole life annuity, or an annuity that ceases when the IP recovers from their injuries. There may be additional future lump sum benefits payable under the settlement to cover predictable future costs. The annuity payments may increase from time to time to offset the effects of inflation.

A **structured settlement** is the payment schedule agreed between the IP and the RP, usually through their lawyers, or through an insurer where the RP's liability is covered by an insurance policy. The annuity part may be funded with a single premium immediate annuity purchased from an insurer or from a firm that specialises in structured settlements.

Structured settlements are often used for payments under **Workers Compensation** insurance. Workers compensation (also known as Workers Comp, or Employer's Liability) is a type of insurance purchased by employers to fund the costs of compensating employees who are injured at work. In many jurisdictions, employers are required to purchase workers compensation insurance. In some areas the insurance is provided through a government agency.

Structured settlements are also commonly used in medical malpractice cases, and for other personal injury claims, such as from motor vehicle accidents.

The reason for using an annuity format rather than a lump sum is that the annuity better replicates the losses of the IP, in the form of lost wages, and/or ongoing expenses associated with medical care or additional needs arising from the injury. Rehabilitation costs, and any expenses associated with re-training for the workplace would also be covered through the settlement.

If the injury is very serious, such as paralysis, loss of limbs, or permanent brain damage, the settlement will be a whole life annuity. Less severe injuries may be compensated with a term life annuity, extending to the point where the individual is expected to be recovered.

The injuries involved in structured settlement compensation cases are often complex, and assessing appropriate mortality and morbidity rates is challenging. A serious injury would increase the IP's mortality, which would make the annuity cheaper compared with the equivalent amount payable to a life subject to standard table mortality. However, there is not much data on the

[9]If the issue is settled through a court case, the IP might be referred to as the plaintiff, and the RP as the defendant, but often cases are solved outside of the formal court system.

relationship between different types of injury and mortality, so the adjustments to standard annuitant mortality tables may be quite arbitrary. It is common for underwriters to apply a simple age rating, but it is likely that a better adjustment would be an addition to the force of mortality, which models an exponentially increasing mortality impact[10]. Impaired lives mortality is covered in Chapter 6 of AMLCR, in Section 6.9.

Example 5.1
Darren, who is 45 years old, has been awarded 1 000 000 in damages for injuries from a car crash. The court decides the benefit should be paid as a structured settlement with the following terms:

- An immediate payment of 100 000
- A lump sum of 50 000 on reaching age 65, or payable immediately on earlier death.
- An annuity payable annually in advance from age 45. The first payment under the annuity is X; and subsequent payments increase at 2% each year. At age 65 the annuity reduces by 20% (so the amount paid at age 65 is $0.8X(1.02)^{20}$).

Calculate X, assuming interest at 4% per year and that mortality follows the SUSM with an addition of 1% to the force of mortality.

Solution 5.1
Let the $*$ superscript denote mortality functions allowing for the extra mortality, where no $*$ denotes functions from the unadjusted SUSM mortality. Then the annuity value is

$$X\Big(1 + (1.02)v_{4\%}\;{}_1p^*_{45} + (1.02)^2\,v^2_{4\%}\;{}_2p^*_{45} + ... + (1.02)^{19}\,v^{19}_{4\%}\;{}_{19}p^*_{45}$$

$$+\,0.8\left((1.02)^{20}\,v^{20}_{4\%}\;{}_{20}p^*_{45} + (1.02)^{21}v^{21}_{4\%}\;{}_{21}p^*_{45} + ...\right)\Big)$$

Now $\;{}_tp^*_{45} = e^{-0.01t}\;{}_tp_{45}$

$$\implies (1.02)^t\,v^t_{4\%}\;{}_tp^*_{45} = (1.02)^t\,v^t_{4\%}\,e^{-0.01t}\;{}_tp_{45} = v^t_j\;{}_tp_{45},$$

where $(1+j) = (1.04)e^{0.01}/(1.02) \implies j = 2.9855\%$

So the annuity value is

$$X\left(\ddot{a}_{45:\overline{20}|} + 0.8\;{}_{20}E_{45}\,\ddot{a}_{65}\right) \quad \text{at} \quad j = 2.9855\%$$

[10] See Singer and Schmidt(2000) for further discussion and analysis.

Using an Excel calculator for the SUSM we find that at 2.9855%

$$\ddot{a}_{45:\overline{20}|} = 15.15268 \qquad {}_{20}E_{45} = 0.53026 \qquad \ddot{a}_{65} = 16.46437$$

$$\Longrightarrow \text{ Annuity Value } = 22.13704X$$

For the endowment benefit, let $\delta = \log(1.04)$, and $\delta^* = \delta + 0.01$, then the value of the 50 000 benefit is

$$50\,000\bar{A}^*_{45:\overline{20}|} = 50\,000\left(1 - \delta\bar{a}^*_{45:\overline{20}|}\right)$$

$$= 50\,000\left(1 - \delta_{4\%}\ \bar{a}_{45:\overline{20}|\,i^*}\right) = 24\,711.38$$

Note that the endowment insurance benefit cannot be valued by simply changing the interest rate in $\bar{A}_{45:\overline{20}|}$; this only works for benefits which are dependent on survival, not for benefits which are dependent on death.

The equation of value for the structured settlement is

$$1\,000\,000 = 100\,000 + 24\,711 + X(22.13704) \implies X = 39\,540$$

5.2 Notes on structured settlements

1. Replacement of income will normally be at less than 100% of pre-injury earnings. There are two reasons cited.

    - In some countries (including the US and the UK) income from a structured settlement annuity is not taxed. Hence, less annuity is required to support the IP's pre-injury lifestyle.
    - The insurer wants to ensure that the IP has a strong incentive to return to work.
    - The amount of compensation may be reduced if the IP is determined to be partially at fault in the incident.

2. The annuity will typically include some allowance for inflation. This may be a fixed annual increase, such as 2% per year as in the example above, or the annuity may be fully indexed to inflation.

3. In cases of potentially severe injury, there is often a period of uncertainty as to the extent of damage and long term prognosis for the IP. For example, it may take a year of treatment and rehabilitation to determine the level of permanent damage from a spinal cord injury. In these cases there may be an interim arrangement of benefit until the time of **maximum medical improvement** (MMI), at which point the final structured settlement will be determined.

4. Structured settlements evolved from a system where the entire compensation was in a lump sum form. Paying compensation as a lump sum requires the IP to manage a potentially very large amount of money. There is a strong temptation to overspend; research indicates that 80%-90% of recipients spend their entire lump sum compensation within 5 years[11]. Even a fairly prudent individual who invests the award in stocks and bonds could lose 30% of their funds in a stock market crash. An annuity relieves the IP from investment risk and from dissipation risk (the risk of spending the funds too early). The move from lump sum to annuities in structured settlements has led to two different approaches to determining the payments.

   The **top-down approach** starts with determination of an appropriate lump sum compensation, and then converts that to an annuity.

   The **bottom-up approach** starts with a suitable income stream, and then converts that to a capital value.

   Because the purpose of the settlement is to restore the IP to her former financial position, as far as possible, the bottom-up method seems most appropriate.

5. In some areas of the US the IP may transfer her annuity to a specialist firm in exchange for a lump sum, under a 'structured settlement buy-out'. After concerns that the buy-out firms were making excessive profits on these transactions, the market has become more regulated, with buy-outs in many areas prohibited or at least requiring court approval. Structured settlement buy-outs are not permitted in Canada, where the structured settlement provider must ensure that the payments are going directly to the IP.

6. The structured settlement determined through a court case is generally final and non-reviewable. This means that if the IP makes an unexpectedly strong recovery, the insurer cannot reduce or stop the payments under the settlement. It also means that if the settlement is insufficient, for example because of inflation or because the IP's condition worsens, the IP may not apply for an increase in the payments. However, reviewable settlements are common in workers comp where the employee may not have the right to sue for a fixed structured settlement, and where the insurer believes that the reviewable approach will be more cost effective.

5.3 Reviewable settlements under workers comp

Workers compensation awards are not always in the form of fixed, structured settlements.

If the injury is less severe then the compensation may take the form of a disability annuity, ceasing when the IP recovers sufficiently to return to work. If that level of recovery does not occur, then the annuity may just cease at some point, or it may continue indefinitely, depending on the

[11] See, for example, Charles et al, (2000).

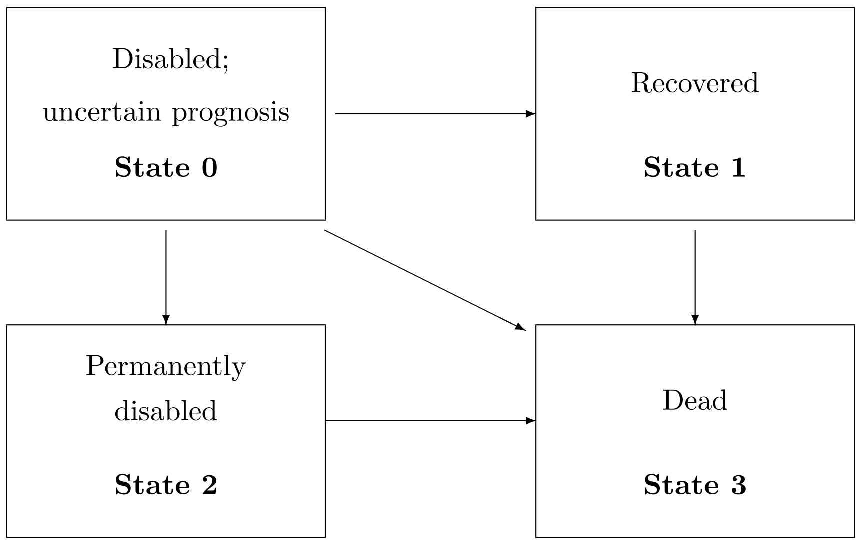

Figure 10: Model for workers compensation benefits.

agreement and on the legal requirements governing the award. For example, some jurisdictions set maximum payments for workers compensation awards to protect employers.

The payments would be similar to the disability income annuity described in Figure 1. We propose a slightly different form in Fig 10.

An individual in State 0 has had an injury, but it is not yet known whether the disability will be permanent or temporary. If the individual recovers, they move to state 1. This may be associated with a lump sum back-to-work benefit. If the injury proves to cause permanent impairment to income, then the individual moves to State 2, and receives annuity for their lifetime, or for the maximum term under the agreement, if shorter. When the individual is in State 2, the insurer may agree to a structured settlement.

Example 5.2

A life aged 50 (the IP) has recently suffered a workplace injury which is covered by a workers compensation insurance policy. The insurer uses the model illustrated in Figure 10 to value the benefits. You are given the following information.

- The IP is currently in State 0.
- μ^{13}_{x+t} follows the SUSM, i.e.

  $\mu^{13}_{x+t} = A + Bc^{x+t}$ where $A = 2.2 * 10^{-4}$; $B = 2.7 * 10^{-6}$; $c = 1.124$

  $\mu^{01}_{x+t} = 0.5$ $\quad \mu^{02}_{x+t} = 1.2$ $\quad \mu^{03}_{x+t} = \mu^{23}_{x+t} = \mu^{13}_{x+t} + 0.05$
- $i = 0.04$
- The insurer calculates the following probabilities and annuity values (at 4%).

x	$\bar{a}^{00}_x$	$\bar{a}^{01}_x$	$\bar{a}^{02}_x$	$\bar{a}^{11}_x$	$\bar{a}^{22}_x$	${}_1p^{00}_x$	${}_1p^{01}_x$	${}_1p^{02}_x$
50	0.5585	5.2706	7.0657	18.8441	10.4785	0.17356	0.23785	0.55308
51	0.5585	5.2006	7.0268	18.6011	10.4238	0.17354	0.23782	0.55301

(a) The insurer considers a reviewable annuity of 150 000 per year, payable continuously, which would cease when the IP dies or recovers. Alternatively, the insurer may pay a non-reviewable whole life annuity of 100 000 per year, payable continuously.

Calculate the value of the two annuity options.

(b) (i) Assume the settlement uses the reviewable annuity option. Calculate the expected value at time 0 of the total policy value at time 1.

(ii) Assume the settlement uses the non-reviewable annuity option. Calculate the expected value at time 0 of the total policy value at time 1.

Solution 5.2

(a) The value of the non-reviewable annuity is

$$100\,000\left(\bar{a}^{00}_{50} + \bar{a}^{01}_{50} + \bar{a}^{02}_{50}\right)$$

$$= 100\,000\,(0.5585 + 5.2706 + 7.0657) = 1\,289\,480$$

The value of the reviewable annuity is

$$150\,000\left(\bar{a}^{00}_{50} + \bar{a}^{02}_{50}\right)$$

$$= 150\,000(0.5585 + 7.0657) = 1\,143\,630$$

(b) (i) First, determine the state dependent policy values:

$$ {}_1V^{(0)} = 150\,000\left(\bar{a}_{51}^{00} + \bar{a}_{51}^{02}\right) = 1\,137\,790 $$

$$ {}_1V^{(1)} = 0 $$

$$ {}_1V^{(2)} = 150\,000\left(\bar{a}_{51}^{22}\right) = 1\,563\,570 $$

Then the EPV of the time 1 policy value at time 0 is

$$ v\left({}_1p_{50}^{00}\ {}_1V^{(0)} + {}_1p_{50}^{02}\ {}_1V^{(2)}\right) = 1\,021\,400 $$

(ii) The policy values are now

$$ {}_1V^{(0)} = 100\,000\left(\bar{a}_{51}^{00} + \bar{a}_{51}^{01} + \bar{a}_{51}^{02}\right) = 1\,278\,580 $$

$$ {}_1V^{(1)} = 100\,000\left(\bar{a}_{51}^{11}\right) = 1\,860\,110 $$

$$ {}_1V^{(2)} = 100\,000\left(\bar{a}_{51}^{22}\right) = 1\,042\,380 $$

Hence, the EPV of the time 1 policy value at time 0 is

$$ v\left({}_1p_{50}^{00}\ {}_1V^{(0)} + {}_1p_{50}^{01}\ {}_1V^{(1)} + {}_1p_{50}^{02}\ {}_1V^{(2)}\right) = 1\,193\,140 $$

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6 Retiree Health Benefits

6.1 Introduction

Retiree health benefits are provided by some employers through supplementary health insurance cover. The supplementary insurance reimburses a portion of the costs of health care which are not covered by the relevant government funded system. For example, seniors in the USA are eligible for socialized health cover through the Medicare program, but there are significant deductible and co-pay requirements. This means, for example, that individuals may have to pay out-of-pocket for each doctors visit and hospital stay. Supplementary health insurance may also provide cover for care that is not included in the Medicare program, such as dental treatment.

In this section we discuss some methods for valuing and funding the future costs to the employer of retirees' health insurance premiums. This section will reference principles and terminology from Chapter 10 of AMLCR.

Retiree health benefits are typically offered only to those who retire from the firm, so that those who leave before they are eligible to retire would not receive retirement health benefits, even if they leave a deferred pension benefit in the pension plan. In addition to reaching a retirement eligible age, there are typically minimum service requirements, such as at least 10 years of service with the firm.

Retiree health benefits may be pre-funded, similarly to defined benefit pensions, or may be funded on a pay-as-you-go basis, meaning that the annual premiums are met by the firm from year to year earnings. However, even when pay-as-you-go is used, there may be accounting regulations requiring benefits to be valued and declared in the financial statements. Unlike accrued pension benefits, post-retirement health benefits do not represent a legal obligation for the employer. The benefits could be withdrawn at any point with no ongoing liability for the employer. This means that there is no need to hold separate funds to meet the future liability.

6.2 Valuing retiree health benefits

In this section we will consider the valuation and funding of a simplified retiree health care benefit plan. We assume that the health care benefits are provided through a health insurance company. The employer pays premiums to the insurer to secure the post-retirement health benefits. So the cost of the benefits to the sponsoring employer is the cost of the premiums that it pays to the health insurance company.

Premiums for health insurance increase with age, as costs are clearly age-dependent. They also increase with time, as health care costs tend to increase faster than normal price inflation.

Let $B(x,t)$ denote the annual premium payable for health insurance under an employer sponsored plan, for a life who is age x at time t.

For a life retiring at age xr in t years, the value at retirement of the supplementary health insurance is

$$B(xr,t) + v\, p_{xr} B(xr+1,t+1) + v^2\ {}_2p_{xr} B(xr+2,t+2) + +v^3\ {}_3p_{xr} B(xr+3,t+3) + ...$$

$$= B(xr,t)\left(1 + v\ p_{xr}\frac{B(xr+1,t+1)}{B(xr,t)} + v^2\ {}_2p_{xr}\frac{B(xr+2,t+2)}{B(xr,t)} + v^3\ {}_3p_{xr}\frac{B(xr+3,t+3)}{B(xr,t)} + ...\right)$$

Let $\ddot{a}_B(xr,t) = \left(1 + v\ p_{xr}\frac{B(xr+1,t+1)}{B(xr,t)} + v^2\ {}_2p_{xr}\frac{B(xr+2,t+2)}{B(xr,t)} + ...\right)$

then the value of the benefit at retirement is:

$$B(xr,t)\ddot{a}_B(xr,t)$$

We call $\ddot{a}_B(xr,t)$ the benefit premium annuity at t for a life then aged xr.

Now, suppose that the annual rate of inflation for the healthcare premiums is $100j\%$, and that in any year the premiums increase exponentially with age, so that $B(x+1,t)/B(x,t) = c$, say. Then

$$B(xr+k,t+k) = c^k(1+j)^k B(xr,t)$$

$$\implies \ddot{a}_B(xr,t) = 1 + v\, p_{xr} c\,(1+j) + v^2{}_2p_{xr} c^2(1+j)^2 + ...$$

$$\implies \ddot{a}_B(xr,t) = \ddot{a}_{xr|\, i^*} \qquad \text{where } (1+i^*) = \frac{1+i}{c\,(1+j)}$$

For example, if $c = 1.02$ and $j = 5\%$, then the value of the benefit premium annuity for a life age 65 at an interest rate of 6%, and assuming SUSM mortality is

$$\ddot{a}_B(65,t) = 26.6403$$

The adjusted interest rate $i^* = -1.027\%$ is negative because the premiums are increasing faster than the interest rate which discounts them.

Now consider a life age x, who may retire at any age up to 65 with full eligibility for the healthcare insurance. We assume first that all lives retire at exact ages. We use the service table functions defined in Chapter 10 in AMLCR.

Then the actuarial value of the total retiree health benefit at age x is $AVTHB$ where

$$AVTHB = \sum_{t=0}^{65-x} \frac{r_{x+t}}{l_x} v^t\, B(x+t,t)\, \ddot{a}_B(x+t,t)$$

If we assume the same structure for the premium as above, we can replace $B(x+t,t)$ with $B(x,0)c^t(1+j)^t$, giving

$$AVTHB = B(x,0)\sum_{t=0}^{65-x}\frac{r_{x+t}}{l_x}v_{i^*}^t\ddot{a}_{x+t|\,i^*}$$

where i^*, as above, is $(1+i)/(c(1+j))-1$.

We may refine this to allow for lives retiring between integer ages. We follow the examples in AMLCR Chapter 10, and assume that such exits occur half-way through the year of age. So, for example, if the exits at all ages up to age 64 are assumed to be half-way through the year, and the age 65 exits are exact age retirements, we have

$$AVTHB = B(x,0)\left(\sum_{t=0}^{64-x}\frac{r_{x+t}}{l_x}v_{i^*}^{t+\frac{1}{2}}\ddot{a}_{x+t+\frac{1}{2}|\,i^*} + \frac{r_{65}}{l_x}v_{i^*}^{65-x}\ddot{a}_{65|\,i^*}\right)$$

Usually the valuation will include one or two exact age exit terms, and the rest will be mid-year exits. This is demonstrated in the following example.

Example 6.1
You are given the following information:

- The annual benefit premium for retiree healthcare cover, for a life age 60 at the valuation date is $B(60,0) = 5000$.
- $c = 1.02$, $j = 5\%$, $i = 6\%$.
- Retirements follow the Service Table from AMLCR (Appendix D). This table allows age retirement from age 60 to 65. It also separates exact age retirements at age 60 from mid-year retirements between age 60 and 61.
- Mortality after age retirement follows the SUSM.

(a) Calculate the $AVTHB$ for a life age 60 at the valuation date.

(b) Calculate the $AVTHB$ for a life age 50 at the valuation date.

Solution 6.1

(a) The service table identifies age retirement decrements at exact ages 60 and 65; all other retirements are assumed to occur half-way through the year of age. All lives are assumed to retire by age 65.

Let r_{60^e} and r_{65} denote the exact age retirement decrements at age 60 and 65, while r_{60^+}, r_{61} ..., r_{64}, represent the mid-year retirement decrements.

Then the AVTHB is

$$AVTHB = B(60,0)\frac{r_{60^e}}{l_{60}}\,\ddot{a}_{60|\,i^*} + B(60.5,0.5)\,v_i^{0.5}\,\frac{r_{60^+}}{l_{60}}\,\ddot{a}_{60.5|\,i^*} + B(61.5,1.5)\,v_i^{1.5}\,\frac{r_{61}}{l_{60}}\,\ddot{a}_{61.5|\,i^*}$$

$$+ \quad \cdots \quad + B(65,5)\,v_i^{5}\,\frac{r_{65}}{l_{60}}\,\ddot{a}_{65|\,i^*}$$

Substituting for $B(60+t,t)$, using $B(60+t,t) = B(60,0)(1+j)^t\,c^t$, we have

$$AVTHB = \frac{B(60,0)}{l_{60}}\Bigg(r_{60^e}\,\ddot{a}_{60|\,i^*} + (1+j)^{0.5}\,c^{0.5}\,v_i^{0.5}\,r_{60^+}\,\ddot{a}_{60.5|\,i^*} + (1+j)^{1.5}\,c^{1.5}\,v_i^{1.5}\,r_{61}\,\ddot{a}_{61.5|\,i^*}$$

$$+ \quad \cdots \quad + (1+j)^5\,c^5\,v_i^5\,r_{65}\,\ddot{a}_{65|\,i^*}\Bigg)$$

$$= \frac{B(60,0)}{l_{60}}\left(r_{60^e}\ddot{a}_{60|\,i^*} + v_{i^*}^{0.5}r_{60^+}\ddot{a}_{60.5|\,i^*} + v_{i^*}^{1.5}r_{61}\ddot{a}_{61.5|\,i^*} + \cdots + v_{i^*}^{5}r_{65}\ddot{a}_{65|\,i^*}\right)$$

(Change the discount rate from i to i^*)

We have: $i^* = (1+i)/(c(1+j)) - 1 = -1.027\%$, (as above), and we can calculate the required annuity values using an SUSM excel calculator:

$x+t$	$\ddot{a}_{x+t|\,i^*}$
60.0	32.5209
60.5	31.9097
61.5	30.7024
62.5	29.5156
63.5	28.3496
64.5	27.2047
65.0	26.6403

So

$$AVTHB = \frac{5000}{93085}\left(27926\,(32.5209) + v_{i^*}^{0.5}\,6188\,(31.9097) + \cdots + v_{i^*}^{5}\,38488\,(26.6403)\right)$$

$$= \frac{5000}{93085}(2\,760\,503) = 148\,279$$

(b) At age 50 we need

$$\begin{aligned}
AVTHB =& B(60,10)\,v_i^{10}\,\frac{r_{60^e}}{l_{50}}\,\ddot{a}_{60|\,i^*} + B(60.5,10.5)\,v_i^{10.5}\,\frac{r_{60+}}{l_{50}}\,\ddot{a}_{60.5|\,i^*} + B(61.5,11.5)\,v_i^{11.5}\,\frac{r_{61}}{l_{50}}\,\ddot{a}_{61.5|\,i^*} \\
& + \quad \cdots \quad + B(65,5)\,v_i^{15}\,\frac{r_{65}}{l_{60}}\,\ddot{a}_{65|\,i^*} \\
&= \frac{B(60,10)}{l_{50}}\,v_i^{10}\left(r_{60^e}\,\ddot{a}_{60|\,i^*} + v_{i^*}^{0.5}\,r_{60+}\,\ddot{a}_{60.5|\,i^*} + v_{i^*}^{1.5}\,r_{61}\,\ddot{a}_{61.5|\,i^*} + \;\cdots\; + v_{i^*}^{5}\,r_{65}\,\ddot{a}_{65|\,i^*}\right) \\
&= \frac{B(60,0)(1+j)^{10}}{l_{50}}\,v_i^{10}\left(r_{60^e}\ddot{a}_{60|\,i^*} + v_{i^*}^{0.5}r_{60+}\ddot{a}_{60.5|\,i^*} + v_{i^*}^{1.5}r_{61}\ddot{a}_{61.5|\,i^*} + \cdots + v_{i^*}^{5}r_{65}\ddot{a}_{65|\,i^*}\right)
\end{aligned}$$

(Allowing for 10 years premium inflation)

$$= \frac{B(60,0)}{l_{50}}\,v_{i^\dagger}^{10}\left(r_{60^e}\ddot{a}_{60|\,i^*} + v_{i^*}^{0.5}r_{60+}\ddot{a}_{60.5|\,i^*} + v_{i^*}^{1.5}r_{61}\ddot{a}_{61.5|\,i^*} + \cdots + v_{i^*}^{5}r_{65}\ddot{a}_{65|\,i^*}\right)$$

$$\text{where } i^\dagger = \frac{1+i}{1+j} - 1 = 0.952\%$$

$$= \frac{5000}{117145}\,v_{0.952\%}^{10}\,(2\,760\,503)$$

$$= 107\,169$$

Notice that the net rate of interest in the first 10 years is different to the subsequent period, because of the netting off of the premium inflation rate from age 50 to 60, then the premium inflation rate and the increase in premiums from aging after age 60.

We see from this example, that it is relatively straightforward to derive the AVTHB from first principles, and this is usually preferable to applying memorized valuation formulas, which may need adapting, as in the case, depending on the information given, and on the specific details of each case.

6.3 Funding retiree health benefits

Although employers may not be required to pre-fund retiree health benefits, they may choose to do so. Even if they choose the pay-as-you-go route, it may be necessary to determine the value of the benefits and a (nominal) normal contribution for accounting purposes.

In AMLCR Chapter 10 we discussed two methods of funding a final salary pension, the traditional unit credit and the projected unit credit. Both of these methods are accruals based methods, meaning that we assume the actuarial liability at each valuation date is based on the pension earned from past service; we do not include future service benefits in the liability valuation, as these are assumed to be funded from future contributions.

The notion of accrual for retiree health benefits is less natural than for a final salary pension, as the pension benefit is a linear function of the number of years of service. This is not true for the health insurance benefits described in this section. In the simplest retiree health benefit plan, two employees retiring on the same day, at the same age will both be entitled to the same benefit even if one has 20 years of service and the other has 40 years. The benefit is also independent of salary, so paying contributions as % of payroll is less natural than for salary related benefits.

We can adapt the accruals principle to retiree health benefits by assuming that the benefits accrue linearly over each employee's period of employment. It would be prudent to assume that the employee retires at the earliest possible date for this purpose. Alternatively, we may assume a linear accrual to retirement for each possible retirement date.

Once we define the accrued benefit and the accrued liability (which is the actuarial value of the accrued benefit), then the normal cost is found using the same principles as in AMLCR Chapter 10. That is, if ${}_tV^h$ represents the actuarial liability at time t for retiree health insurance for an employee, and C_t^h represents the normal contribution for the year, payable at t, for the same employee, then

$$ {}_tV^h + C_t^h = \text{EPV of benefits for mid-year exits } + v\ {}_1p_x^{(\tau)}\ {}_{t+1}V^h, \qquad (40) $$

that is

$$ C_t^h = v\ {}_1p_x^{(\tau)}\ {}_{t+1}V^h + \text{EPV of benefits for mid-year exits} -\ {}_tV^h. $$

By *EPV of benefits for mid-year exits* we mean the EPV at the start of the year of benefits that would be payable given that the life exits during the year, multiplied by the probability of exit during the year. In other words, the normal cost is the expected present value of one additional year of benefit accrual, except in respect of exits in the valuation year, for which the normal cost is the expected present value of $\frac{1}{2}$-year of additional accrual.

Example 6.2

(a) Suppose the retiree health benefits in Example 6.1 are funded using a pro-rata accruals method, assuming retirement at age 60 for the accruals period.

(i) Calculate the accrued liability and normal contribution for an employee age 50 with 15 years service.

(ii) Calculate the accrued liability and normal contribution for an employee age 60 with 25 years service.

(b) Repeat (a), but assume linear accrual to each retirement age, so that for an employee who entered service at age 40, benefits on retirement at age 60 would be accrued over 20 years, benefits on retirement at age 60.5 would be accrued over 20.5 years, and so on.

Solution 6.2

(a)(i) We assume the benefits are accrued over 25 years, from entry until age 60. So the actuarial liability is

$$ {}_0V = AVTHB\frac{15}{25} = 64301 $$

Since there are no mid year exits from age retirement between ages 50 and 51, the normal cost is

$$ NC = \frac{{}_0V}{15} = \frac{AVTHB}{25} = 4286.76 $$

The normal cost is exactly sufficient to pay for one more year of accrual, and under this approach, each year's accrual costs ($APVTHB/25$), up to age 60.

(a)(ii) Under this approach, by age 60 the benefit is fully funded, so the actuarial liability is $AVTHB = 148\,279$, and the normal cost is 0. There is no additional accrual to fund from the normal cost.

(b)(i) We cannot use the AVTHB calculation here, because each term in the valuation is assumed to accrue at a different rate, depending on the retirement age.

Instead we consider each possible retirement date separately. For an employee age 50 with 15 years past service, the benefit payable on retirement at age 60 is assumed to accrue over 25 years; the benefit payable if the employee retires at age 60.5 is assumed to accrue over 25.5 years, and so on to the age 65 retirement benefit, which accrues over the 30 years of total service. So, for each possible retirement age xr, say, the accrued benefit at age 50 is $(\frac{15}{xr-35})$ years of the total cost given exit at age xr.

In symbols we have

$$
\begin{aligned}
{}_0V &= \frac{B(60,0)v_{i^\dagger}^{10}}{l_{50}}\left(r_{60^e}\ddot{a}_{60|\,i^*}\left(\frac{15}{25}\right)+v_{i^*}^{0.5}r_{60^+}\ddot{a}_{60.5|\,i^*}\left(\frac{15}{25.5}\right)\right.\\
&\qquad\left.+v_{i^*}^{1.5}r_{61}\ddot{a}_{61.5|\,i^*}\left(\frac{15}{26.5}\right)+\cdots+v_{i^*}^{5}r_{65}\ddot{a}_{65|\,i^*}\left(\frac{15}{30}\right)\right)\\
&= \frac{(15)B(60,0)v_{i^\dagger}^{10}}{l_{50}}\left(\frac{r_{60^e}\,\ddot{a}_{60|\,i^*}}{25}+\frac{v_{i^*}^{0.5}\,r_{60^+}\,\ddot{a}_{60.5|\,i^*}}{25.5}+\frac{v_{i^*}^{1.5}\,r_{61}\,\ddot{a}_{61.5|\,i^*}}{26.5}\right.\\
&\qquad\left.+\quad\cdots\quad+\frac{v_{i^*}^{5}\,r_{65}\,\ddot{a}_{65|\,i^*}}{30}\right)\\
&= 58677.46
\end{aligned}
$$

The normal cost is the expected present value of one additional year of accrual, as there are no mid-year retirements between age 50 and 51. Hence the normal cost is

$$NC = 58677.46/15 = 3911.83$$

(b)(ii) The actuarial liability is very similar to the age 50 case:

$$
\begin{aligned}
{}_0V &= \frac{B(60,0)}{l_{60}}\left(r_{60^e}\ddot{a}_{60|\,i^*}\left(\frac{25}{25}\right)+v_{i^*}^{0.5}r_{60^+}\ddot{a}_{60.5|\,i^*}\left(\frac{25}{25.5}\right)\right.\\
&\qquad\left.+v_{i^*}^{1.5}r_{61}\ddot{a}_{61.5|\,i^*}\left(\frac{25}{26.5}\right)+\cdots+v_{i^*}^{5}r_{65}\ddot{a}_{65|\,i^*}\left(\frac{25}{30}\right)\right)\\
&= \frac{(25)B(60,0)}{l_{60}}\left(\frac{r_{60^e}\ddot{a}_{60|\,i^*}}{25}+\frac{v_{i^*}^{0.5}r_{60^+}\ddot{a}_{60.5|\,i^*}}{25.5}+\frac{v_{i^*}^{1.5}r_{61}\ddot{a}_{61.5|\,i^*}}{26.5}\right.\\
&\qquad\left.+\cdots+\frac{v_{i^*}^{5}r_{65}\ddot{a}_{65|\,i^*}}{30}\right)\\
&= 135310.0
\end{aligned}
$$

For the normal contribution, consider each of the terms in the actuarial liability calculation:

- The age 60 term is fully funded at the valuation; no additional contribution is required.
- The age 60.5 term is 25/25.5 =98.04% funded in the actuarial liability, and by the end of the year must be fully funded, so the normal contribution must fund the additional

(1/2)-year of accrual, at a cost of

$$\frac{(0.5)B(60,0)\,v_{i^*}^{0.5}\,r_{60^+}\,\ddot{a}_{60.5|i^*}}{l_{60}\,(25.5)} = 209.04$$

- The age 61.5 term is $25/26.5 = 94.3\%$ funded in the actuarial liability and must be 26/26.5 funded by the end of the year. So the normal cost for age 61.5 retirements is the cost of an additional 1-year of accrual

$$\frac{B(60,0)\,v_{i^*}^{1.5}\,r_{61}\,\ddot{a}_{61.5|i^*}}{l_{60}\,(26.5)} = 352.23$$

- Similarly, for retirement ages, $xr = 62.5$, 63.5, 64.5, and 65 the NC must fund an extra 1-year of accrual as follows:

Age 62.5: $$\frac{B(60,0)\,v_{i^*}^{2.5}\,r_{62}\,\ddot{a}_{62.5|i^*}}{l_{60}\,(27.5)} = 296.86$$

Age 63.5: $$\frac{B(60,0)\,v_{i^*}^{3.5}\,r_{63}\,\ddot{a}_{63.5|i^*}}{l_{60}\,(28.5)} = 250.12$$

Age 64.5: $$\frac{B(60,0)\,v_{i^*}^{4.5}\,r_{64}\,\ddot{a}_{64.5|i^*}}{l_{60}\,(29.5)} = 210.73$$

Age 65: $$\frac{B(60,0)\,v_{i^*}^{5}\,r_{65}\,\ddot{a}_{65|i^*}}{l_{60}\,(30)} = 1933.09$$

Combining these costs of accrual, we have

$$NC = 3252.07$$

We can generalise the valuation where we allow different accrual periods for the different retirement dates. Let xe denote the entry age for an employee, and let $x \geq xe$ denote the valuation age, so that $x - xe$ is the past service at the valuation date, and $xr - xe$ is the total service for an employee who retires at age xe. We will separate the exact age retirements, represented by decrement r_{y^e} and the mid-year decrements, represented by r_y. We let xw denote the maximum retirement age. The actuarial liability is

$$_0V = \frac{B(x,0)\,(x-xe)}{l_x}\left(\underbrace{\sum_{y=x}^{xw}\frac{r_{y^e}\,v_{i^*}^{y-x}\,\ddot{a}_{y|i^*}}{y-xe}}_{\text{exact age exits}} + \underbrace{\sum_{y=x}^{xw}\frac{r_y\,v_{i^*}^{x-y+\frac{1}{2}}\,\ddot{a}_{y+\frac{1}{2}|i^*}}{y+\frac{1}{2}-xe}}_{\text{mid year exits}}\right)$$

For the NC we have three terms; the first allows for the 1/2 year accrual for exits in the valuation year. The second sums all the exact age exit terms, and the third sums all the mid-year exit terms:

$$NC = \frac{B(x,0)}{l_x}\left(\frac{0.5\, r_x\, v^{0.5}\, \ddot{a}_{x+\frac{1}{2}|i^*}}{x+\frac{1}{2}-xe} + \sum_{y=x+1}^{xw} \frac{r_{y^e}\, v_{i^*}^{y-x}\, \ddot{a}_{y|i^*}}{y-xe} + \sum_{y=x+1}^{xw} \frac{r_y\, v_{i^*}^{y-x+\frac{1}{2}}\, \ddot{a}_{y+\frac{1}{2}|i^*}}{y+\frac{1}{2}-xe}\right)$$

and if $r_x = 0$, the NC is simply ${}_0V/(x-xe)$.

6.4 References and further reading

Fundamentals of Retiree Group Benefits Second Edition Dale H. Yamamoto ACTEX Publications, Inc. Winsted, CT ActexMadRiver.com