STAT 472
Test 1
Fall 2019
October 1, 2019

1. (6 points) You are given that ${}_{t}q_{90} = \dfrac{t^2}{100}$ for $0 \le t \le 10$.

   Calculate $\overset{\circ}{e}_{90}$.

   Solution:

   $${}_{t}p_{90} = 1 - \frac{t^2}{100}$$

   $$\overset{\circ}{e}_{90} = \int_0^{10} {}_{t}p_x \; dt = \int_0^{10} \left[1 - \frac{t^2}{100} \right] dt = \left[t - \frac{t^3}{300} \right]_0^{10}$$

   $$= 10 - \frac{1000}{300} - 0 = 6.6667$$

2. (8 points) The curtate expectation of life, which is e_{80}, is 10.606 based on the Standard Ultimate Life Table.

   If $q_{80} = 0.015$ instead of 0.032658, but the other mortality rates are unchanged, determine e_{80} accurate to three decimal places.

   Solution:

   $$e_{80} = p_x(1+e_{81})$$

   $$10.606 = (1-0.032658)(1+e_{81}) \Rightarrow e_{81} = 9.9641$$

   $$e_{80} = (1-0.015)(1+9.9641) = 10.800$$

3. (10 points) You are given the following mortality table:

x	q_x
90	0.2
91	0.4

You are also given that deaths are uniformly distributed between ages 90 and 91 and the deaths follow a constant force of mortality between ages 91 and 92.

Calculate ${}_{0.2|0.9}q_{90.2}$.

Solution:

$$l_{90} = 1000$$
$$l_{91} = 1000(1-0.2) = 800$$
$$l_{92} = 800(1-0.4) = 480$$

$${}_{0.2|0.9}q_{90.2} = \frac{l_{90.4} - l_{91.3}}{l_{90.2}}$$

$$= \frac{[1000(0.6)+800(0.4)] - 800^{0.7}(480)^{0.3}}{1000(0.8)+800(0.2)}$$

$$= \frac{233.6662}{960} = 0.2434$$

4. (10 points) Let Z be the present value random variable for a discrete whole life policy with a death benefit of 100 sold to (101) who has just been underwritten.

You are given the following select and ultimate mortality table:

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
100	0.05	0.15	0.35	102
101	0.10	0.30	0.50	103
102	0.20	0.40	0.80	104
103	0.25	0.50	1.00	105

You are also given that $d = 0.07$.

Calculate $Var[Z]$.

Solution:

$$v = 1 - 0.07 = 0.93$$

$$Var[Z] = 100^2 \left({}^2 A_{[101]} - (A_{[101]})^2 \right)$$

$$l_{[101]} = 1000; l_{[101]+1} = 1000(0.9) = 900; l_{[101]+2} = 900(0.7) = 630$$
$$l_{[101]+3} = 630(0.5) = 315; l_{[101]+4} = 315(0.2) = 63; l_{[101]+5} = 63(0) = 0$$

$$A_{[101]} = \frac{(1000-900)(.93) + (900-630)(.93)^2 + (630-315)(.93)^3 + (315-63)(.93)^4 + (63-0)(.93)^5}{1000}$$

$$= 0.8122$$

$${}^2 A_{[101]} = \frac{(1000-900)(.93)^2 + (900-630)(.93)^4 + (630-315)(.93)^6 + (315-63)(.93)^8 + (63-0)(.93)^{10}}{1000}$$

$$= 0.6638$$

$$Var[Z] = 100^2 (0.6638 - 0.8122^2) = 40.4912$$

5. (9 points) You are given:

i. $A_{60} = 0.500$

ii. ${}^2A_{60} = 0.350$

iii. $p_{60} = 0.96$

iv. $p_{61} = 0.95$

v. $i = 0.06$

Let Z be the present value random variable for a whole life to (61) with a death benefit of 1 paid at the end of the year of death.

Calculate the $Var(Z)$.

Solution:

$$A_{60} = vq_{60} + vp_{60}A_{61}$$

$$0.5 = (1.06)^{-1}(1-0.96) + (1.06)^{-1}(0.96)A_{61}$$

$$A_{61} = 0.51042$$

$$Var[Z] = {}^2A_{61} - (A_{61})^2$$

$${}^2A_{60} = v^2q_{60} + v^2 p_{60} {}^2A_{61}$$

$$0.35 = (1.06)^{-2}(.04) + (1.06)^{-2}(0.96) \cdot {}^2A_{61}$$

$${}^2A_{61} = 0.36798$$

$$Var[Z] = 0.36798 - (0.51042)^2 = 0.10745$$

6. Jeff is (63). He wants to buy a life insurance policy from Wu Life Insurance Company. However, he is not sure which policy to buy. All calculations assume:

   i. Mortality follows the Standard Ultimate Life Table

   ii. $i = 0.05$

   iii. Deaths are uniformly distributed between integral ages.

   First, he decides to consider a whole life insurance policy that pays a death benefit of 100,000 at the end of the year of death. Jeff asks Shina who is the chief actuary at Wu Life to do several things for him.

   a. (4 points) Write the present value random variable Z for this policy.

$$Z = 100,000v^{k_x+1} = 100,000\left(\frac{1}{1.05}\right)^{k_x+1}$$

   b. (4 points) Jeff estimates that the expected present value of this whole life policy is 33,000 to the nearest 1000. Calculate it to the nearest 1.

$$100,000A_{63} = 100,000(0.32785) = 32,785$$

   c. (4 points) What would be the expected present value of the whole life policy if it paid a death benefit at the moment of death instead of at the end of the year of death.

$$100,000\bar{A}_{63} = 100,000A_{63}\left(\frac{i}{\delta}\right) = 32,785\left(\frac{0.05}{\ln(1.05)}\right) = 33,597.96$$

d. (12 points) Jeff decides to purchase the whole life insurance policy in Part b. With this sale, Wu Life now has 400 identical whole life policies sold to 400 independent lives. Wu decides to hold 13,500,000 to cover the future death benefit payments on these policies. Using the normal distribution, calculate the probability that the present value of the death benefits will be greater than the 13,500,000.

Solution:

$$E[Port] = 400(32,785) = 13,114,000$$

$$Var[Z] = 100,000^2[{}^2A_{63} - (A_{63})^2] = 100,000^2[0.13421 - (0.32785)^2] = 267,243,775$$

$$Var[Port] = 400(267,243,775)$$

$$13,114,000 + Z\left(\sqrt{400(267,243,775)}\right) = 13,500,000$$

$$Z = 1.18$$

$$P(Z > 1.18) = 1 - 0.8810 = 0.1190$$

e. (8 points) Jeff asks Shina to calculate his median future life time. The median future life time is the point at which ${}_{n+s}p_{63} = 0.5$ where $0 \le s \le 1$. Jeff knows that $n = 25$.

Determine s accurate to three decimal places.

Solution:

$${}_{25+s}p_{63} = 0.5 = \frac{l_{88+s}}{l_{63}} = \frac{l_{88+s}}{95,534.40} ==> l_{88+s} = 47,767.20$$

$$l_{88}(1-s) + l_{89}(s) = 47,767.20$$

$$50,038.60(1-s) + 45,995.6(s) = 47,767.20$$

$$50,038.6 - 4043s = 47,767,20$$

$$s = 0.562$$

f. (10 points) Jeff decides to also purchase a term insurance policy that pays a death benefit of 100,000 payable at the moment of death if he dies in the next 26 years. Determine the expected present value of this term insurance.

Solution:

$$100,000\bar{A}^{1}_{63:\overline{26}|} = 100,000\left[A_{63}\left(\frac{i}{\delta}\right) - {}_{26}E_{63}A_{89}\left(\frac{i}{\delta}\right)\right]$$

$$\frac{i}{\delta} = \frac{0.05}{\ln(1.05)} = 1.0248$$

$${}_{26}E_{63} = (1.05)^{-26}\left(\frac{l_{89}}{l_{63}}\right)$$

$$100,000\left[0.32785(1.0248) - (1.05)^{-26}\left(\frac{45,995.6}{95,534.4}\right)(0.73853)(1.0248)\right] = 23,349.93$$

g. (5 points) Explain why the expected present value of the term insurance is less than the expected present value of the whole life insurance.

Solution:

The expected present value of term insurance is lower because the death benefit only gets paid to the beneficiary if Jeff dies within the 26-year term; with whole life insurance the death benefit is paid no matter when Jeff dies therefore the present value of whole life is greater than that of term.

7. (10 points) Alisa (20) buys a special whole life policy with a non-level death benefit. The death benefits are paid at the end of the year of death and are listed in the following table:

Years	Death Benefit
1-30	125,000
31-50	300,000
51+	50,000

Using the Standard Ultimate Life Table with $i = 5\%$, calculate the expected present value of this insurance.

Solution:

$$125,000A_{20} + 175,000\,{}_{30}E_{20}A_{50} - 250,000\,{}_{50}E_{20}A_{70}$$

$${}_{30}E_{20}\,{}_{10}E_{20} \cdot {}_{20}E_{30}$$

$${}_{50}E_{20} \Rightarrow (1.05)^{-50}\left(\frac{l_{70}}{l_{20}}\right)$$

$$125,000(0.04922) + 175,000(0.61224)(0.37254)(0.18931)$$
$$- 250,000(1.05)^{-50}\left(\frac{91,082.4}{100,000}\right)(0.42818)$$

$$= 6152.5 + 7556.2482 - 8502.2897 = 5206.46$$

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8. (6 points) You are given that ${}_tq_{90} = \dfrac{t^2}{100}$ for $0 \le t \le 10$.

   Calculate μ_{95} .

   Solution:

$$ {}_tp_{90} = 1 - \frac{t^2}{100} $$

$$ \mu_{90+t} = \frac{-\dfrac{d}{dt}\left({}_tp_{90}\right)}{{}_tp_{90}} = \frac{\dfrac{t}{50}}{1-\dfrac{t^2}{100}} $$

$$ \mu_{90+5} = \frac{\dfrac{5}{50}}{1-\dfrac{5^2}{100}} = \frac{2}{15} = 0.1333333 $$

9. (9 points) Under the Standard Ultimate Life Table:

   a. $e_{60:\overline{10}|} = 9.733$

   b. $e_{70:\overline{10}|} = 9.201$

   c. $e_{80} = 10.606$

   Determine e_{60}.

   Solution:

$$e_{60} = e_{60:\overline{10}|} + e_{70:\overline{10}|}\left({}_{10}p_{60}\right) + e_{80}\left({}_{20}p_{60}\right)$$

$$= 9.733 + 9.201\left(\frac{l_{70}}{l_{60}}\right) + 10.606\left(\frac{l_{80}}{l_{60}}\right)$$

$$= 9.733 + 9.201\left(\frac{91082.4}{96634.1}\right) + 10.606\left(\frac{75657.2}{96634.1}\right)$$

$$= 26.7091$$

10. (12 points) You are given the following mortality table:

x	q_x
90	0.2
91	0.4
92	0.6
93	0.8
94	1.0

Let Z be the present value random variable for a discrete three year term insurance policy issued to (90) with a death benefit of 5,000 paid at the end of the year of death.

You are also given that $d = 0.10$.

Calculate $Var[Z]$.

Solution:

$$v = 1 - 0.10 = 0.90$$

$$Var[Z] = {}^2A^1_{90:\overline{3}|} - (A^1_{90:\overline{3}|})^2$$

$$l_{90} = 1000; l_{91} = 1000(0.8) = 800; l_{92} = 900(0.6) = 480$$
$$l_{93} = 630(0.4) = 192; l_{94} = 315(0.2) = 38.4; l_{95} = 63(0) = 0$$

$$A^1_{90:\overline{3}|} = \frac{(1000-800)(.9) + (800-480)(.9)^2 + (480-192)(.9)^3}{1000} = 0.649152$$

$${}^2A^1_{90:\overline{3}|} = \frac{(1000-800)(.9)^2 + (800-480)(.9)^4 + (480-192)(.9)^6}{1000} = 0.525007$$

$$Var[Z] = 5000^2(0.525007 - 0.649152^2) = 2590217.22$$

11. (10 points) You are given the following select and ultimate mortality table:

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
100	0.05	0.15	0.025	102
101	0.10	0.30	0.50	103
102	0.20	0.40	0.80	104
103	0.25	0.50	1.00	105

During the first two years, deaths are assumed to be distributed uniformly between integral ages. After two years, it is assumed that we have a constant force of mortality between integral ages.

Calculate ${}_{0.7|0.9}q_{[101]+0.6}$.

Solution:

$$ {}_{0.7|0.9}q_{[101]+0.6} = \frac{l_{[101]+1.3} - l_{103.2}}{l_{[101]+0.6}} \Rightarrow \frac{0.7l_{[101]+1} + 0.3l_{[101]+2} - (l_{103})^{0.8}(l_{104})^{0.2}}{0.4l_{[101]} + 0.6l_{[101]+1}} $$

$$ l_{[101]} = 1000 $$
$$ l_{[101]+1} = 1000(1-0.1) = 900 $$
$$ l_{103} = 900(1-0.3) = 630 $$
$$ l_{104} = 630(1-0.5) = 315 $$

$$ {}_{0.7|0.9}q_{[101]+0.6} = \frac{0.7(900) + .3(630) - (630)^{0.8}(315)^{0.2}}{0.4(1000) + 0.6(900)} = 0.2878 $$

12. (10 points) There are currently 1000 independent lives all age 80 who own life insurance policies at Maxwell Life Insurance Company.

    You are given that mortality for these policies follows Gompertz Law with $B = 0.000005 \text{ and } c = 1.10$.

    Let L_{90} be the random variable representing the number of lives alive at the end of 10 years.

    Calculate the $Var(L_{90})$.

    Solutions:

$$L_{90} \sim Bin(1000, {}_{10}p_{80})$$

$${}_{10}p_{80} = e^{-\frac{-0.000005}{\ln(1.1)}(1.1)^{80}(1.1^{10}-1)} = 0.8426$$

$$Var[L_{90}] = npq = 1000(0.8426)(1-0.8426) = 132.6253$$

13. Jeff is (70). He wants to buy a life insurance policy from Lai Life Insurance Company. However, he is not sure which policy to buy. All calculations assume:

    i. Mortality follows the Standard Ultimate Life Table

    ii. $i = 0.05$

    iii. Deaths are uniformly distributed between integral ages.

    First, he decides to consider a whole life insurance policy that pays a death benefit of 100,000 at the moment of death. Jeff asks Jake who is the chief actuary at Lai Life to do several things for him.

    h. (4 points) Write the present value random variable Z for this policy.

    $$Z = 100{,}000 v^{T_{70}} = 100{,}000(1.05)^{-T_{70}}$$

    i. (5 points) Jeff estimates that the expected present value of this whole life policy is 44,000 to the nearest 1000. Calculate it to the nearest 1.

    $$EPV = 100{,}000\bar{A}_{70} = 100{,}000\left(\frac{i}{\delta}\right)A_{70}$$
    $$= 100{,}000(1.02480)(0.42818) = 43{,}880$$

    j. (10 points) Calculate the probability that Z is less than 51,000. The probability needs to be accurate to 3 decimal places.

    Solution:

    $$\Pr(Z < 51{,}000) = \Pr(100{,}000 v^{T_{70}} < 51{,}000)$$

    $$v^{T_{70}} < 0.51 \Rightarrow (1.05)^{-T_{70}} < 0.51 ==> T_{70} > \frac{-\ln(0.51)}{\ln(1.05)} \Rightarrow T_{70} > 13.8008$$

    $$P(T_{70} > 13.8008) = {}_{13.8008}p_{70}$$

    $$\frac{l_{83.8008}}{l_{70}} = \frac{0.1992 l_{83} + 0.8008 l_{84}}{l_{70}}$$

    $$= \frac{0.1992(67{,}614.6) + 0.8008(64{,}506.5)}{91{,}082.40} = 0.71502$$

k. (10 points) Jeff decides to purchase a 13 year endowment insurance policy with a death benefit of 100,000 payable at the end of the year of death.

Determine the expected present value of this endowment insurance.

Solution:

$$EPV = 100,000 A_{70:\overline{13}|}$$

$$= 100,000\left(A^{1}_{70:\overline{13}|} + {}_{13}E_{70}\right) = 100,000[{}_{70} - A_{83}({}_{13}E_{70}) + {}_{13}E_{70}]$$

$$= 100,000(0.42818 - 0.64336({}_{13}p_{70})v^{13} + ({}_{13}p_{70})v^{13}$$

$$= 100,000\left[0.42818 - 0.64336\left(\frac{67614.6}{91082.4}\right)(1.05)^{-13} + \left(\frac{67614.6}{91082.4}\right)(1.05)^{-13}\right]$$

$$= 56,858.26$$

l. (5 Points) Explain why the expected present value of the endowment insurance is greater than the expected present value of the whole life insurance.

Solution:

The expected present value of endowment insurance is greater because the longest you will have to wait to get paid is the 13 year term which is given; if you die before then the death benefit will be paid when you die but if you die after 13 years you still receive it as a pure endowment at the end of the set term (in this case 13 years). For the whole life, if you survive 13 years, the death benefit will not be paid until you die. Since this death benefit would be paid after the time of the pure endowment, it has a smaller present value.

14. (10 points) Jimmy (30) buys a special 45 year term insurance policy with a non-level death benefit. The death benefits are paid at the end of the year of death and are listed in the following table:

Years	Death Benefit
1-20	100,000
21-35	50,000
36-45	25,000

Using the Standard Ultimate Life Table with $i = 5\%$, calculate the expected present value of this insurance.

Solution:

$$EPV = 100,000A_{30} - 50,000A_{50}({}_{20}E_{30}) - 25,000A_{65}({}_{35}E_{30}) - 25,000A_{75}({}_{45}E_{30})$$

$$= 100,000(0.07698) - 50,000(0.18931)(0.37254)$$
$$- 25,000(0.35477)(1.05)^{-35}\left(\frac{94,579.7}{99,727.3}\right) - 25,000(0.50868)(1.05)^{-45}\left(\frac{85,203.5}{99,727.3}\right)$$

$$= 1,437.577$$

15. (9 points) You are given:

    i. Z is the present value for a whole life policy sold to (x) with a death benefit of 1 payable at the end of the year of death.

    ii. ${}^2A_x = 0.41$

    iii. $Var(Z) = 0.05$

    iv. $q_x = 0.035$

    v. $q_{x+1} = 0.037$

    vi. $i = 0.06$

Calculate A_{x+2} accurate to 4 decimal places.

Solution:

$$A_{x+1} = vq_{x+1} + vp_{x+1}A_{x+2}$$

$$A_x = vq_x + vp_xA_{x+1}$$

$$Var[Z] = {}^2A_x - (A_x)^2$$

$$0.05 = 0.41 - \left(A_x\right)^2 \Rightarrow A_x = 0.6$$

$$0.6 = (1.06)^{-1}(0.035) + (1.06)^{-1}(1 - 0.035)A_{x+1} \Rightarrow A_{x+1} = 0.62280$$

$$0.62280 = (1.06)^{-1}(0.037) + (1.06)^{-1}(1 - 0.037)A_{x+2} \Rightarrow A_{x+2} = 0.64711$$