STAT 472
Test 1
Fall, 2021

September 30, 2021

1. You are given the following select and ultimate mortality table:

Age $[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
88	0.05	0.10	0.20	90
89	0.07	0.15	0.35	91
90	0.12	0.25	0.50	92

You are also given that $i = 0.05$.

Let Z be the present value random variable for a four year term insurance with a death benefit of 100 paid at the end of the year of death issued to (88) who was just underwritten.

a. (6 points) Calculate the $E[Z]$.

Solution:

$$l_{[88]} = 1000; l_{[88]+1} = (1000)(1-0.05) = 950; l_{90} = (950)(1-0.1) = 855$$

$$l_{91} = (855)(1-0.20) = 684; l_{92} = (684)(1-0.35) = 444.6$$

$$1000A^{\ 1}_{[88]:\overline{4}|} = \frac{(1000-950)}{1.05} + \frac{(950-855)}{(1.05)^2} + \frac{(855-684)}{(1.05)^3} + \frac{(684-444.6)}{(1.05)^4} = 478.458$$

$$100A^{\ 1}_{[88]:\overline{4}|} = 47.8458$$

b. (6 points) Calculate the $Var[Z]$.

Solution:

$$1000 \cdot {}^2 A^{\ 1}_{[88]:\overline{4}|} = \frac{(1000-950)}{(1.05)^2} + \frac{(950-855)}{(1.05)^4} + \frac{(855-684)}{(1.05)^6} + \frac{(684-444.6)}{(1.05)^8} = 413.146$$

$$Var[Z] = (100)^2[{}^2 A^{\ 1}_{[88]:\overline{4}|} - (A^{\ 1}_{[88]:\overline{4}|})^2] = (100)^2[0.413146 - (0.478458)^2] = 1842.243$$

c. (5 points) Explain why select mortality rates revert to ultimate (or population mortality rates) after the select period.

Solution:

Over time, the effect of underwriting wears off. When the policy is underwritten, only the healthy lives can purchase insurance. However, at time passes those that were healthy when underwritten develop diseases and other health issues. By the end of the select period, they are no longer any healthier than the general population.

2. Let Z be the present value random variable for a whole life insurance with a death benefit of 1000 paid at the end of the year of death for a life age 100.

   You are given the following mortality table:

Age x	q_x
100	0.20
101	0.40
102	0.75
103	1.00

   You are also given that $d = 0.08$ and that the force of mortality is constant between integral ages.

   a. (2 points) Write an expression for Z.

      Solution:

      $$Z = 1000v^{K_{100}+1} = 1000(0.92)^{K_{100}+1}$$

   b. (6 points) Calculate the $Var[Z]$.

      Solution:

$$l_{100} = 1000; l_{101} = (1000)(1-0.2) = 800; l_{102} = (800)(1-0.4) = 480$$
$$l_{103} = (480)(1-0.75) = 120; l_{104} = (120)(1-1) = 0$$

$$A_{100} = \frac{200(0.92) + (320)(0.92)^2 + (360)(0.92)^3 + (120)(0.92)^4}{1000} = 0.8211428$$

$$^2A_{100} = \frac{200(0.92)^2 + (320)(0.92)^4 + (360)(0.92)^6 + (120)(0.92)^8}{1000} = 0.6783998$$

$$Var[Z] = (1000)^2\left[{}^2A_{100} - (A_{100})^2\right] = (1000)^2\left[0.6783998 - (0.8211428)^2\right] = 4124.26$$

c. (1 point) State ω for this table.

Solution:

$\omega = 104$ which is the first age where everyone is dead. See the prior part.

d. (6 points) Calculate ${}_{0.8|0.6}q_{100.8}$.

Solution:

$${}_{0.8|0.6}q_{100.8} = \frac{l_{101.6} - l_{102.2}}{l_{100.8}} = \frac{(800)^{1-0.6}(480)^{0.6} - (480)^{1-0.2}(120)^{0.2}}{(1000)^{1-0.8}(800)^{0.8}} = 0.269$$

e. (8 points) Calculate $Var[K_{100}]$.

Solution:

$$E[K_{100}] = e_{100} = \sum_{1}^{4} {}_k p_{100} = {}_1 p_{100} + {}_2 p_{100} + {}_3 p_{100} + {}_4 p_{100} = \frac{l_{101} + l_{102} + l_{103} + l_{104}}{l_{100}}$$

$$= \frac{800 + 480 + 120 + 0}{1000} = 1.4$$

$$E[(K_{100})^2] = 2\sum_{1}^{4} k \cdot {}_k p_{100} - e_{100} = 2\left[(1)_1 p_{100} + (2)_2 p_{100} + (3)_3 p_{100} + (4)_4 p_{100}\right] - 1.4$$

$$= 2\left[\frac{(1)l_{101} + (2)l_{102} + (3)l_{103} + (4)l_{104}}{l_{100}}\right] - 1.4$$

$$= 2\left[\frac{(1)(800) + (2)(480) + (3)(120) + (4)(0)}{1000}\right] - 1.4 = 2.84$$

$$Var[K_{100}] = 2.84 - (1.4)^2 = 0.88$$

3. You are given that $F_0(x) = \dfrac{x^2}{100}$ for $0 \le x \le 10$.

a. (4 points) Calculate $\overset{\circ}{e}_0$.

Solution:

$$\overset{\circ}{e}_0 = \int_0^{10} {}_x p_0 dt = \int_0^{10} \left(1 - \frac{x^2}{100}\right) dt = \left[x - \frac{x^3}{300}\right]_0^{10} = 10 - \frac{1000}{300} = 6.66667$$

b. (6 points) Calculate $Var[T_0]$.

Solution:

$$E[T_0{}^2] = 2\int_0^{10} x \cdot {}_x p_0 \cdot dt = 2\int_0^{10} x\left(1 - \frac{x^2}{100}\right) dt = 2\left[\frac{x^2}{2} - \frac{x^4}{400}\right]_0^{10} = 2[50 - 25] = 50$$

$$Var[T_0] = 50 - (6.66667)^2 = 5.55556$$

c. (4 points) Calculate ${}_3p_5$.

Solution:

$$ {}_3p_5 = \frac{S_0(8)}{S_0(5)} = \frac{1-\frac{8^2}{100}}{1-\frac{5^2}{100}} = 0.48 $$

Or do it with l's.

$$ {}_3p_5 = \frac{l_8}{l_5} = \frac{36}{75} = 0.48 $$

$$ l_0 = 100; l_5 = (100)\left(1-\frac{5^2}{100}\right) = 75; l_8 = (100)\left(1-\frac{8^2}{100}\right) = 36 $$

d. (4 points) Calculate μ_5 .

Solution:

$$ \mu_x = \frac{-\frac{d}{dx}S_0(x)}{S_0(x)} = \frac{-\frac{d}{dx}\left(1-\frac{x^2}{100}\right)}{1-\frac{x^2}{100}} = \frac{\frac{2x}{100}}{1-\frac{x^2}{100}} $$

$$ \mu_5 = \frac{\frac{2(5)}{100}}{1-\frac{5^2}{100}} = \frac{0.1}{0.75} = 0.13333 $$

4. (12 points) You are given that mortality follows Makeham Law with $A = 0.01,\ B = 0.000003, \text{ and } C = 1.10$.

You are further given that:

i. $i = 0.04$

ii. $A_{79} = 0.460$

iii. ${}^2A_{81} = 0.273$

Let Z be the present value random variable for a whole life policy for (80). The death benefit is 1 payable at the end of the year of death.

Calculate the $Var[Z]$ to five decimal places.

Solution:

$$Var[Z] = {}^2A_{80} - (A_{80})^2$$

$$A_{79} = vq_{79} + vp_{79}A_{80}$$

$$p_{79} = \exp\left[-0.01 - \frac{0.000003}{\ln(1.10)}(1.10)^{79}(1.1-1)\right] = 0.9842637$$

$$0.46 = (1.04)^{-1}(1-0.9842637) + (1.04)^{-1}(0.9842637)A_{80}$$

$$A_{80} = \frac{0.46 - (1.04)^{-1}(1-0.9842637)}{(1.04)^{-1}(0.9842637)} = 0.4700607$$

$${}^2A_{80} = v^2q_{80} + v^2p_{80}\,{}^2A_{81}$$

$$p_{80} = \exp\left[-0.01 - \frac{0.000003}{\ln(1.10)}(1.10)^{80}(1.1-1)\right] = 0.9836869$$

$${}^2A_{80} = (1.04)^{-2}(1-0.9836869) + (1.04)^{-2}(0.9836869)^2(0.273) = 0.26337$$

$$Var[Z] = {}^2A_{80} - (A_{80})^2 = 0.26337 - (0.4700607)^2 = 0.04247$$

5. (10 points) Avena Assurance Company provides warranty protection on Christmas tree lights. Under the warranty coverage, if a strand of lights burns out during the year, Avena will replace the strand of lights at end of the year. The coverage provided has a three year term and coverage continues on the replaced lights. In other words, if the strands that replace the burned out strands also stop working, Avena will replace them also.

   The city of West Lafayette purchases coverage and has 50,000 strands of Christmas lights which are all new.

   Christmas light strands burn out at the following rates:

   i. The probability that a new strand of lights will fail in the first year of service is 20%.
   ii. The probability that a one year old strand of lights will fail during the next year is 45%.
   iii. The probability that a two year old strand of lights will fail during the next year is 60%.

   A cost to replace a strand of lights is 2.

   Using an interest rate of 8%, calculate the Actuarial Present Value of the coverage offered by Avena to West Lafayette.

   Solution:

   50,000 new bulbs start year 1. Bulbs to be replaced $= (50,000)(0.2) = 10,000$

   50,000 bulbs start year 2. 40,000 are one year old and 10,000 are new.
   Bulbs to be replaced $= (10,000)(0.2) + (40,000)(0.45) = 2000 + 18,000 = 20,000$

   50,000 bulbs start year 3. 22000 are two years old, 8,000 are one year old and 20,000 are new.
   Bulbs to be replaced $= (20,000)(0.2) + (8,000)(0.45) + (22,000)(0.6) = 20,800$

   $$EPV = (10,000)(2)(1.08)^{-1} + (20,000)(2)(1.08)^{-2} + (20,800)(2)(1.08)^{-3} = 85,835.49$$

6. (10 points) You are given that $\mu_{90+t} = 0.1t^2$.

You are also given that N_{93} is the random variable representing the number of people who will be alive at age 93 if there are 5,000 people alive at age 90.

Calculate $Var[N_{93}]$.

Solution:

$$Var[N_{93}] = npq = (5000)({}_3p_{90})(1 - {}_3p_{90})$$

$${}_3p_{90} = e^{-\int_0^3 \mu_{90+t} \cdot dt} = e^{-\int_0^3 0.1t^2 \cdot dt} = e^{-\left.\frac{t^3}{30}\right]_0^3} = e^{-0.9}$$

$$Var[N_{93}] = (5000)(e^{-0.9})(1 - e^{-0.9}) = 1206.35$$

7. (10 points) Improving maintenance protocols will extend the lifetime of an industrial robot. The robot's mortality rates and improvement factors are given below:

x	$q(x,0)$	$\varphi(x,1)$	$\varphi(x,2)$	$\varphi(x,3)$	$\varphi(x,4)$
0	0.1	0.25	0.21	0.19	0.17
1	0.2	0.20	0.17	0.14	0.13
2	0.3	0.15	0.10	0.07	0.05
3	0.4	0.10	0.08	0.06	0.02
4	0.5	0.05	0.03	0.01	0.00

Calculate the probability that a robot placed into service today is still functioning at the end of four years.

Solution:

$${}_4q_0 = [1-q(0,0)][1-q(1,1)][1-q(2,2)][(1-q(3,3)]$$

$$= [1-q(0,0)][1-q(1,0)\{1-\varphi(1,1)\}][1-q(2,0)\{1-\varphi(2,1)\}\{1-\varphi(2,2)\}]\times$$
$$[(1-q(3,0)\{1-\varphi(3,1)\}\{1-\varphi(3,2)\}\{1-\varphi(3,3)\}]$$

$$= [1-0.1][1-0.2\{1-0.2\}][1-0.3\{1-0.15\}\{1-0.10\}][(1-0.4\{1-0.1\}\{1-0.08\}\{1-0.06\}]$$

$$= 0.40115$$