Wang Warranty Corporation is testing iPods. Wang starts with 100 iPods and tests them by dropping them on the ground. Wang records the number of drops before each iPod will no longer play. The following data is collected from this test:

Drops to Failure	Number	Drops to Failure	Number
1	6	9	7
2	2	10	6
4	3	11	7
6	4	12	7
7	4	13	48
8	5	22	1

Ledbetter Life Insurance Company is completing a mortality study on a 3 year term insurance policy. The following data is available:

Life	Date of Entry	Date of Exit	Reason for Exit
1	0	0.2	Lapse
2	0	0.3	Death
3	0	0.4	Lapse
4	0	0.5	Death
5	0	0.5	Death
6	0	0.5	Lapse
7	0	1.0	Death
8	0	3.0	Expiry of Policy
9	0	3.0	Expiry of Policy
10	0	3.0	Expiry of Policy
11	0	3.0	Expiry of Policy
12	0	3.0	Expiry of Policy
13	0	3.0	Expiry of Policy
14	0	3.0	Expiry of Policy
15	0	3.0	Expiry of Policy
16	0.5	2.0	Lapse
17	0.5	1.0	Death
18	1.0	3.0	Expiry of Policy
19	1.0	3.0	Expiry of Policy
20	2.0	2.5	Death

Schneider Trucking Company had the following losses during 2013:

Amount of Claim	Number of Payments	Total Amount of Losses
0 – 10	8	60
10-20	5	70
20-30	4	110
30 +	3	200
Total	20	440

Chapter 12

1. Using the data from Wang Warranty Corporation, calculate:

   a. $p_{100}(x)$

   Solution:

x	p100(x)	x	F100(x)
		x<1	0
1	0.06	1≤x<2	0.06
2	0.02	2≤x<4	0.08
4	0.03	4≤x<6	0.11
6	0.04	6≤x<7	0.15
7	0.04	7≤x<8	0.19
8	0.05	8≤x<9	0.24
9	0.07	9≤x<10	0.31
10	0.06	10≤x<11	0.37
11	0.07	11≤x<12	0.44
12	0.07	12≤x<13	0.51
13	0.48	13≤x<22	0.99
22	0.01	x≥22	1.00

   b. $F_{100}(x)$

   Solution:

   See Part a.

   c. The empirical mean

   Solution:

$$E[N]=\frac{\begin{array}{c}(1)(6)+(2)(2)+(4)(3)+(6)(4)+(7)(4)+(8)(5)\\ +(9)(7)+(10)(6)+(11)(7)+(12)(7)+(13)(48)+(22)(1)\end{array}}{100}=10.44$$

d. The empirical variance

Solution:

$$E[N^2] = \frac{\begin{array}{c}(1)^2(6)+(2)^2(2)+(4)^2(3)+(6)^2(4)+(7)^2(4)+(8)^2(5)\\ +(9)^2(7)+(10)^2(6)+(11)^2(7)+(12)^2(7)+(13)^2(48)+(22)^2(1)\end{array}}{100} = 123.4$$

$$Var[N] = 123.4 - (10.44)^2 = 14.4064$$

Note: $E[N] = 10.44$ comes from Part c.

2. Using the data from Schneider Trucking Company, calculate:

a. The ogive, $F_{20}(x)$

Solution:

$$F_{20}(0) = 0$$
$$F_{20}(10) = 8/20 = 0.4$$
$$F_{20}(20) = 13/20 = 0.65$$
$$F_{20}(30) = 17/20 = 0.85$$

We know these points as they are the break points in our grouped data. The ogive assumes that we have a straight line between these points. We use linear interpolation to find the points.

For the range from 0 to 10,

$$F_{20}(x) = \frac{10-x}{10-0}F_{20}(0) + \frac{x-0}{10-0}F_{20}(10) = \frac{10-x}{10}(0) + \frac{x}{10}(0.4) = 0.04x$$

For the range from 10 to 20,

$$F_{20}(x) = \frac{20-x}{20-10}F_{20}(10) + \frac{x-10}{20-10}F_{20}(20) = \frac{20-x}{10}(0.4) + \frac{x-10}{10}(0.65)$$

$$= 0.8 - 0.04x + 0.065x - 0.65 = 0.15 + 0.025x$$

For the range from 20 to 30,

$$F_{20}(x) = \frac{30-x}{30-20}F_{20}(20) + \frac{x-20}{30-20}F_{20}(30) = \frac{30-x}{10}(0.65) + \frac{x-20}{10}(0.85)$$

$$= 1.95 - 0.065x + 0.085x - 1.7 = 0.25 + 0.02x$$

Above 30, we cannot derive a function for the ogive as there is no upper limit on the amount of the claim.

b. The histogram, $f_{20}(x)$

Solution:

The histogram is the estimate of $f(x)$. We can get it two ways. We can take the dertivative of $F(x)$ from Part a.

$$f_{20}(x) = F'_{20}(x) = 0.04 \text{ for } 0 \le x \le 10$$
$$f_{20}(x) = F'_{20}(x) = 0.025 \text{ for } 10 \le x \le 20$$
$$f_{20}(x) = F'_{20}(x) = 0.02 \text{ for } 20 \le x \le 30$$
$$f_{20}(x) = F'_{20}(x) = \mathit{undefined} \text{ for } x > 30$$

Or we can use the formula that is

$$f_n(x) = \frac{n_j}{n(c_j - c_{j-1})}$$

$$f_{20}(x) = \frac{8}{(20)(10-0)} = 0.04 \text{ for } 0 \le x \le 10$$

$$f_{20}(x) = \frac{5}{(20)(20-10)} = 0.025 \text{ for } 10 \le x \le 20$$

$$f_{20}(x) = \frac{4}{(20)(30-20)} = 0.02 \text{ for } 20 \le x \le 30$$

3. Using the data from Wang Warranty Corporation, calculate:

a. Ĥ(x) where Ĥ(x) is the cumulative hazard function from the Nelson Åalen estimate

Solution:

i	y_i	s_i	r_i
1	1	6	100
2	2	2	94
3	4	3	92
4	6	4	89
5	7	4	85
6	8	5	81
7	9	7	76
8	10	6	59
9	11	7	53
10	12	7	56
11	13	48	49
12	22	1	1

x	**Ĥ(x)**	**Ŝ(x)**
x<1	0	$e^{-\hat{H}(x)} = e^{-0} = 1$
1<u><</u>x<2	6/100 = 0.060000	$e^{-\hat{H}(x)} = e^{-0.06} = 0.941765$
2<u><</u>x<4	6/100+2/94 = 0.081277	$e^{-\hat{H}(x)} = e^{-0.081277} = 0.921939$
4<u><</u>x<6	6/100+2/94+3/92 = 0.113885	$e^{-\hat{H}(x)} = e^{-0.113885} = 0.892360$
6<u><</u>x<7	0.158829	0.853142
7<u><</u>x<8	0.205888	0.813924
8<u><</u>x<9	0.267616	0.765201
9<u><</u>x<10	0.359722	0.697871
10<u><</u>x<11	0.446678	0.639750
11<u><</u>x<12	0.557789	0.572473
12<u><</u>x<13	0.682789	0.505206
13<u><</u>x<22	1.662381	0.189687
x<u>></u>22	2.662381	0.069782

b. Ŝ(x) where Ŝ(x) is the survival function from the Nelson Åalen estimate.

Solution:

See Part a above.

4. Using the data for Ledbetter Life Insurance Company, calculate the following where death is the decrement of interest:

a. $S_{20}(t)$ using the Kaplan Meier Product Limit Estimator

i	y_i	s_i	r_i
1	0.3	1	15-1=14
2	0.5	2	15-3=12
3	1.0	2	17-6=11
4	2.5	1	20-9=11

	$S_{20}(t)$
$0 \le t < 0.3$	1
$0.3 \le t < 0.5$	(1)(1-1/14) = 0.928571
$0.5 \le t < 1.0$	(0.928571)(1-2/12) = 0.773810
$1.0 \le t < 2.5$	(0.773810)(1-2/11) = 0.633117
$t \ge 2.5$	(0.633117)(1-1/11) = 0.575561

b. Ĥ(t) where Ĥ(t) is the cumulative hazard function from the Nelson Åalen estimate

Solution:

	$\hat{H}(t)$	$\hat{S}(t)$
$0 \le t < 0.3$	0	$e^{-\hat{H}(x)} = e^{-0} = 1$
$0.3 \le t < 0.5$	1/14 = 0.071429	$e^{-\hat{H}(x)} = e^{-0.071429} = 0.931063$
$0.5 \le t < 1.0$	0.071429+2/12 = 0.238095	0.788128
$1.0 \le t < 2.5$	0.238095+2/11 = 0.419913	0.657104
$t \ge 2.5$	0.419913+1/11 = 0.510823	0.600002

c. Ŝ(t) where Ŝ(t) is the survival function from the Nelson Åalen estimate.

Solution:

See Part b above.

5. Using the data for Ledbetter Life Insurance Company, and treating all expiries as lapses, calculate the following where lapse is the decrement of interest:

a. $S_{20}(t)$ using the Kaplan Meier Product Limit Estimator

Solution:

Since we are studying lapses (and expiries), those will become out s_i.

i	y_i	s_i	r_i
1	0.2	1	15-0 = 15
2	0.4	1	15-2 = 13
3	0.5	1	15-3 = 12
4	2.0	1	19-8 = 11
5	3.0	10	20-10 = 10

	$S_{20}(t)$
$0 \le t < 0.2$	1
$0.2 \le t < 0.4$	(1)(1-1/15) = 0.933333
$0.4 \le t < 0.5$	0.933333)(1-1/13) = 0.861538
$0.5 \le t < 2.0$	(0.861538)(1-1/12) = 0.789744
$2.0 \le t < 3.0$	(0.789744)(1-1/11) = 0.717949
$t \ge 3.0$	(0.717949)(1-10/10) = 0

b. Ĥ(t) where Ĥ(t) is the cumulative hazard function from the Nelson Åalen estimator

	$\hat{H}(t)$	$\hat{S}(t)$
$0 \le t < 0.2$	0	$e^{-\hat{H}(x)} = e^{-0} = 1$
$0.2 \le t < 0.4$	1/15 = 0.066667	$e^{-\hat{H}(x)} = e^{-0.066667} = 0.935507$
$0.4 \le t < 0.5$	0.066667+1/13 = 0.143590	0.866243
$0.5 \le t < 2.0$	0.143590+1/12 = 0.226923	0.796982
$2.0 \le t < 3.0$	0.226923+1/11 = 0.317832	0.727725
$t \ge 3.0$	0.317832+10/10 = 1.317832	0.267715

c. Ŝ(t) where Ŝ(t) is the survival function from the Nelson Åalen estimator

Solution:

See Part b above.

6. * Three hundred mice were observed at birth. An additional 20 mice were first observed at age 2 (days) and 30 more were first observed at age 4.

There were 6 deaths at age 1, 10 at age 3, 10 at age 4, a at age 5, b at age 9, and 6 at age 12.

In addition, 45 mice escaped and were lost to observation at age 7, 35 at age 10, and 15 at age 13.

The following product-limit estimates were obtained: $S_{350}(7)=0.892 \text{ and } S_{350}(13)=0.856$.

Determine a and b.

Solution:

i	y_i	s_i	r_i
1	1	6	300
2	3	10	300+20 - 6 = 314
3	4	10	300+20-6-10 = 304
4	5	a	300+20+30-6-10-10 = 324
5	9	b	300+20+30-6-10-10-a-45 = 279 - a
6	12	6	300+20+30-6+10-10-a-b-45-35 = 244 – a – b

$$S_{350}(7)=(1)(1-6/300)(1-10/314)(1-10/304)(1-a/324)=0.892$$

$$1-a/324=\frac{0.892}{(1-6/300)(1-10/314)(1-10/304)}=0.972122727$$

$$a=9$$

$$S_{350}(13)=(1)(1-6/300)(1-10/314)(1-10/304)(1-9/324)(1-b/(279-a))(1-6/(244-a-b))=0.856$$

$$(1)(1-6/300)(1-10/314)(1-10/304)(1-9/324)(1-b/270)(1-6/(235-b))=0.856$$

$$(1-b/270)(1-6/(235-b))=\frac{0.856}{(1-6/300)(1-10/314)(1-10/304)(1-9/324)}=0.959543$$

$$b=4$$

7. * There are n lives observed from birth. None are censored and no two lives die at the same age. At the time of the ninth death, the Nelson Åalen estimate of the cumulative hazard rate is 0.511 and at the time of the tenth death it is 0.588. Estimate the value of the survival function at the time of the third death.

Solution:

$$\hat{H}(death\ 1) = 1/n$$

$$\hat{H}(death\ 2) = \hat{H}(death\ 1) + [1/(n-1)]$$

……

$$\hat{H}(death\ 9) = \hat{H}(death\ 8) + [1/(n-8)]$$

$$\hat{H}(death\ 10) = \hat{H}(death\ 9) + [1/(n-9)]$$
$$0.588 = 0.511 + [1/(n-9)] ==> (0.588-0.511)(n-9) = 1 ==> n-9 = 13 ==> n = 22$$

$$\hat{H}(death\ 3) = (1/22) + (1/21) + (1/20) = 0.14307$$

$$\hat{S}(death\ 3) = e^{-\hat{H}(death\ 3)} = e^{-0.14307} = 0.8667$$

8. Astleford Ant Farm is studying the life expectancy of ants. The farm is owned by two brothers who are both actuaries. They isolate 100 ants and record the following data:

Number of Days till Death	**Number of Ants Dying**
1	4
2	8
3	12
4	20
5	40
6	8
7	4
8	2
9	1
10	1

a. One of the brothers, Robert, uses the Nelson-Åalen estimator to determine $\hat{H}(5)$. Determine the 90% linear confidence interval for $\hat{H}(5)$.

Solution:

$$\hat{H}(5) = \frac{4}{100} + \frac{8}{96} + \frac{12}{88} + \frac{20}{76} + \frac{40}{56} = 1.2371406$$

$$Var = \frac{4(100-4)}{100^3} + \frac{8(96-8)}{96^3} + \frac{12(88-12)}{88^3} + \frac{20(76-20)}{76^3} + \frac{40(56-40)}{56^3} = 0.0087137$$

$$\textit{90\% Confidence Interval} = 1.2371406 \pm (1.645)(\sqrt{0.0087137}) =$$

$$(1.0836\ ,\ 1.3907)$$

b. The other brother, Daniel, decides that since he has complete data for these 100 ants, he will just use the unbiased estimator of $\hat{S}(5)$. Using this approach, determine the 90% confidence interval for $\hat{S}(5)$.

Solution:

$$\hat{S}(5) = S_{100}(5) = \frac{\textit{Number who survive past time 5}}{n} = \frac{8+4+2+1+1}{100} = 0.16$$

$$Var(S(5)) = \frac{\hat{S}(5)[1-\hat{S}(5)]}{n} = \frac{(0.16)(0.84)}{100} = 0.001344$$

$$\textit{90\% Confidence Interval} = 0.16 \pm 1.645\left(\sqrt{0.001344}\right) = (0.09969\ ,\ 0.22031)$$

9. The following information on students in the actuarial program at Purdue is used to complete an analysis of students leaving the program because they are switching majors.

Student	**Time of Entry**	**Time of Exit**	**Reason for Exit**
1	0	.5	Switching Major
2-5	0	1	Switching Major
6	0	2	Switching Major
7	0	3	Graduation
8	0	3	Switching Major
9-12	0	3.5	Graduation
13-23	0	4	Graduation
24	0.5	2	Switching Major
25	0.5	3	Switching Major
26	1	3.5	Graduation
27	1	4	Switching Major
28	1.5	4	Graduation
29	2	5	Graduation
30	3	5	Graduation

$\hat{S}(x)$ is estimated using the product limit estimator.

Estimate $Var[S_{30}(2)]$ using the Greenwood approximation.

Solution:

$$Var[S_{30}(2)] = \left[\hat{S}(2)\right]^2 \left[\sum \frac{s_i}{r_i(r_i - s_i)}\right]$$

$$\hat{S}(2) = \left(\frac{22}{23}\right)\left(\frac{20}{24}\right)\left(\frac{21}{23}\right) = 0.72779$$

$$Var[S_{30}(2)] = (0.72779)^2 \left[\frac{1}{(23)(22)} + \frac{4}{(24)(20)} + \frac{2}{(23)(21)}\right] = 0.007654$$

10. A mortality study is conducted on 50 lives, all from age 0. At age 15, there were two deaths; at age 17, there were three censored observations; at age 25 there were four deaths; at age 30, there were c censored observations; at age 32 there were eight deaths; and at age 40 there were two deaths.

Let S be the product limit estimate of $S(35)$ and let V be the Greenwood estimate of this estimator's variance. You are given $V/S^2 = 0.011467$.

Determine c.

Solution:

$$V = S^2\left[\frac{2}{(50)(48)} + \frac{4}{(45)(41)} + \frac{8}{(41-c)(33-c)}\right]$$

$$0.011467 = \frac{V}{S^2} = 0.003001 + \frac{8}{(41-c)(33-c)}$$

Using the quadratic, c = 6

11. * Fifteen cancer patients were observed from the time of diagnosis until the earlier of death of 36 months from diagnosis. Deaths occurred as follows: at 15 months, there were two deaths; at 20 months there were three deaths; at 24 months there were 2 deaths; at 30 months there were d deaths; at 34 months there were two deaths; and at 36 months there were one death.

The Nelson Åalen estimate of $H(35)$ is 1.5641.

Determine the variance of this estimator.

Solution:

$$\hat{H}(35) = 1.5641 = \frac{2}{15} + \frac{3}{13} + \frac{2}{10} + \frac{d}{8} + \frac{2}{8-d}$$

$$(1.5641 - 0.5641)(8)(8-d) = d(8-d) + 16$$

$$0 = d^2 - 16d + 48 ==> d = 4$$

$$Var[\hat{H}(35)] = \frac{2(15-2)}{15^3} + \frac{3(13-3)}{13^3} + \frac{2(10-2)}{10^3} + \frac{(4)(8-4)}{8^3} + \frac{2(4-2)}{4^3} = 0.1311$$

1.

x	$p_{100}(x)$	x	$F_{100}(x)$	$\hat{H}(x)$	$\hat{S}(x)$
		x<1	0	0	1
1	0.06	$1 \le x < 2$	0.06	0.060000	0.941765
2	0.02	$2 \le x < 4$	0.08	0.081277	0.921939
4	0.03	$4 \le x < 6$	0.11	0.113885	0.892360
6	0.04	$6 \le x < 7$	0.15	0.158829	0.853142
7	0.04	$7 \le x < 8$	0.19	0.205888	0.813924
8	0.05	$8 \le x < 9$	0.24	0.267616	0.765201
9	0.07	$9 \le x < 10$	0.31	0.359722	0.697871
10	0.06	$10 \le x < 11$	0.37	0.446678	0.639750
11	0.07	$11 \le x < 12$	0.44	0.557789	0.572473
12	0.07	$12 \le x < 13$	0.51	0.682789	0.505206
13	0.48	$13 \le x < 22$	0.99	1.662381	0.189687
22	0.01	$x \ge 22$	1.00	2.662381	0.069782

Empirical Mean = 10.44 and Empirical Variance = 14.4064

2.

a. 0.04x for $0 \le x \le 10$

0.15 + 0.025x for $10 \le x \le 20$

0.25 + 0.02x for $20 \le x \le 30$

Undefined for x > 30

b. 0.04 for $0 \le x \le 10$

0.025 for $10 \le x \le 20$

0.02 for $20 \le x \le 30$

Undefined for x > 30

4.

	$S_{20}(t)$	$\hat{H}(t)$	$\hat{S}(t)$
$0 \le t < 0.3$	1	0	1
$0.3 \le t < 0.5$	0.928571	0.071429	0.931063
$0.5 \le t < 1.0$	0.773810	0.238095	0.788128
$1.0 \le t < 2.5$	0.633117	0.419913	0.657104
$t \ge 2.5$	0.575561	0.510823	0.600002

5.

	$S_{20}(t)$	$\hat{H}(t)$	$\hat{S}(t)$
$0 \le t < 0.2$	1	0	1
$0.2 \le t < 0.4$	0.933333	0.066667	0.935507
$0.4 \le t < 0.5$	0.861538	0.143590	0.866243
$0.5 \le t < 2.0$	0.789744	0.226923	0.796982
$2.0 \le t < 3.0$	0.717949	0.317832	0.727725
$t \ge 3.0$	0	1.317832	0.267715

6. $a = 9$ and $b = 4$
7. 0.8667
8.
    a. $(1.0836\ ,\ 1.3907)$
    b. (0.09969, 0.22031)
9. 0.007654
10. 6
11. 0.1311