Chapter 8

1. You are given a multiple decrement model with decrements of (1) death by natural causes and (2) death by accidental causes.

You are also given:

- \( \mu_x^{(1)} = 0.031 \)
- \( \mu_x^{(2)} = 0.015 \)
- \( \delta = 0.05 \)

a. Calculate the annual net benefit premium rate paid continuously for a whole life policy issued to \((x)\) that pays 100,000 at the moment of death when death is from an accident and pays 70,000 at the moment of death when death is from natural causes.

Solution:

\[
PVP = PVB \implies P\bar{a}_x^{00} = 100,000\bar{A}_x^{02} + 70,000\bar{A}_x^{01}
\]

\[
P\int_0^\infty v' \cdot p_x^{(z)} \cdot \mu_x^{(1)} \cdot dt + 70,000\int_0^\infty v' \cdot p_x^{(z)} \cdot \mu_x^{(2)} \cdot dt
\]

\[
P\int_0^\infty e^{-0.05t} \cdot e^{-(0.031+0.015)t} \cdot dt = 100,000\int_0^\infty e^{-0.05t} \cdot e^{-(0.031+0.015)t} \cdot 0.015 \cdot dt + 70,000\int_0^\infty e^{-0.05t} \cdot e^{-(0.031+0.015)t} \cdot 0.031 \cdot dt
\]

\[
P\int_0^\infty e^{-0.096t} \cdot dt = (100,000)(0.015)\int_0^\infty e^{-0.096t} \cdot dt + (70,000)(0.031)\int_0^\infty e^{-0.096t} \cdot dt \implies P = 3670
\]
b. A 10 year term insurance policy issued to \( x \) pays 50,000 at the moment of death for any death plus an additional 40,000 at the moment of death if the death is from accidental causes. The annual net benefit premium is paid continuously. Calculate the annual net benefit premium rate.

**Solution:**

\[
PVP = PVB \implies P\overline{a}^{10}_{x:10} = 90,000 \overline{A}^{02}_{x:30} + 50,000 \overline{A}^{01}_{x:30} \quad \text{where } \overline{A}^{0j}_{x:30} \text{ indicates a 10 year term insurance.}
\]

\[
P \int_0^{10} v^{'} \cdot p_x^{(r)} \, dt = 90,000 \int_0^{10} v^{'} \cdot p_x^{(r)} \cdot \mu_{x+t}^{(2)} \, dt + 50,000 \int_0^{10} v^{'} \cdot p_x^{(r)} \cdot \mu_{x+t}^{(1)} \, dt
\]

\[
P \int_0^{10} v^{'} \cdot p_x^{(r)} \, dt = 90,000 \int_0^{10} v^{'} \cdot p_x^{(r)} (0.015) \, dt + 50,000 \int_0^{10} v^{'} \cdot p_x^{(r)} (0.031) \, dt
\]

Note that \( \int_0^{10} v^{'} \cdot p_x^{(r)} \, dt \) can be cancelled out of each term.

\[
P = (90,000)(0.015) + (50,000)(0.031) \implies P = 2900
\]
c. A fully discrete whole policy issued to (x) pays 100,000 upon a death from natural causes. It also pays 300,000 upon death from accidental causes. The net benefit premium is paid annually for 20 years during the lifetime of the insured. Calculate the annual net benefit premium.

Solution:

\[ PVP = PVP \implies P^{[0]}_{a_{x:20}} = 300,000A^0_x + 100,000A^0_x \]

\[
P^{[0]}_{a_{x:20}} = 300,000 \sum_{i=0}^{19} v^i \cdot p^x_{x+i} \cdot q^x_{x+i} + 100,000 \sum_{i=0}^{19} v^i \cdot p^x_{x+i} \cdot q^x_{x+i} \\
\]

\[
v^i = e^{-0.05i} \cdot p^x_{x+i} \quad p^x_{x+i} = e^{-0.031+0.015i} = e^{-0.046i} \\
\]

\[
q_x^{[0]} = \int_0^1 \mu^x_{x+s} \cdot ds = \mu^x_{x+s} \int_0^1 e^{-0.031+0.015s} \cdot ds = \mu^x_{x+s} \left[ \frac{1-e^{-0.046}}{0.046} \right] \\
\]

\[
P^{[0]}_{a_{x:20}} = 300,000 \sum_{i=0}^{19} e^{-0.05i} \cdot e^{-0.096i} \cdot (0.015) \left[ \frac{1-e^{-0.046}}{0.046} \right] + 100,000 \sum_{i=0}^{19} e^{-0.05i} \cdot e^{-0.096i} \cdot (0.031) \left[ \frac{1-e^{-0.046}}{0.046} \right] \\
\]

We note that each of the sums is a geometric sum. The sum of a geometric sequence is

First Term – Next Term after last
\[
1 - \text{ratio} \quad \text{They will not cancel as the upper limit is not the same.} \\
\]

\[
P \left[ \frac{1-e^{-0.096(20)}}{1-e^{-0.096}} \right] = [(300,000)(0.015) + (100,000)(0.031)] e^{-0.05} \left[ \frac{1-e^{-0.046}}{0.046} \right] \left[ \frac{1-0}{1-e^{-0.096}} \right] \\
\]

\[ P = 8279.41 \]
2. Mayfawny purchases a whole life insurance policy.

There are three ways that Mayfawny’s policy can terminate:
   a. Death (1)
   b. Diagnosis of a critical illness (2); and
   c. Lapse (3).

The policy pays a death benefit of 10,000 at the moment of death. The policy will also pay a critical illness benefit of 20,000 if Mayfawny is diagnosed with a critical illness. Only one benefit will be paid.

There is no benefit paid upon lapse.

You are also given:
   i. $\mu_s^{(1)} = 0.01$
   ii. $\mu_s^{(2)} = 0.015$
   iii. $\mu_s^{(3)} = 0.06$
   iv. $\delta = 0.035$

Mayfawny pays a net premium continuously for her lifetime as long as the policy is in force. The net premium is determined using the equivalence principle.

Calculate the net premium paid by Mayfawny.
Solution:

\[ PVP = PVB \implies P\bar{a}_x^{00} = 10,000\bar{A}_x^{01} + 20,000\bar{A}_x^{02} + 0\bar{A}_x^{03} \]

\[
\bar{a}_x^{00} = \int_0^\infty v' \cdot p_x^{(r)} \cdot dt = \int_0^\infty e^{-0.01t} \cdot e^{-0.015t} \cdot dt = \int_0^\infty e^{-0.035t} \cdot e^{-0.015t} \cdot dt = 
\]

\[
\int_0^\infty e^{-0.035t} \cdot e^{-0.085t} \cdot dt = \int_0^\infty e^{-0.12t} \cdot dt = \left[ \frac{e^{-0.12t}}{-0.12} \right]_0^\infty = \frac{1}{0.12} 
\]

\[
\bar{A}_x^{01} = \int_0^\infty v' \cdot p_x^{(r)} \cdot \mu_x^{(1)} \cdot dt = \int_0^\infty e^{-0.035t} \cdot e^{-0.015t} \cdot 0.01 \cdot dt = \frac{0.01}{0.12} 
\]

\[
\bar{A}_x^{02} = \int_0^\infty v' \cdot p_x^{(r)} \cdot \mu_x^{(2)} \cdot dt = \int_0^\infty e^{-0.035t} \cdot e^{-0.085t} \cdot 0.015 \cdot dt = \frac{0.015}{0.12} 
\]

\[ P\bar{a}_x^{00} = 10,000\bar{A}_x^{01} + 20,000\bar{A}_x^{02} + 0\bar{A}_x^{03} \implies P = \frac{10,000\bar{A}_x^{01} + 20,000\bar{A}_x^{02} + 0\bar{A}_x^{03}}{\bar{a}_x^{00}} = 
\]

\[
\frac{10,000\left( \frac{0.01}{0.12} \right) + 20,000\left( \frac{0.015}{0.12} \right) + 0}{\left( \frac{1}{0.12} \right)} = 400 
\]
3. Jeff is receiving a salary paid continuously for as long as he is in employed at Purdue. Jeff can leave employment through death (1), retirement (2), or disability (3). Once Jeff leaves employment, the salary stops.

You are given:

i. The salary pays at an annual rate of 70,000 per year.
ii. \( \delta = 0.05 \)
iii. Jeff is currently age 59.
iv. Jeff will retire at age 65 if he is still teaching. He will not retire prior to age 65.
v. \( \mu_{59+t}^{(1)} = 0.01 + 0.001t \)
vi. \( \mu_{59+t}^{(3)} = 0.03 - 0.001t \)

Calculate the present value of Jeff’s future earnings while employed at Purdue.

Solution:

\[
PV = 70,000 \int_0^6 v' \cdot p_{59}^{(7)} \cdot dt \quad \text{The limits on the integral are determined by Jeff’s retirement date.}
\]

\[
v' = e^{-\delta t} = e^{-0.05t} ; \quad p_{59}^{(7)} = e^{\int_0^t (\mu_{59+s}^{(1)} + \mu_{59+s}^{(2)}) \cdot ds} - \int_0^t (0.01 + 0.001r - 0.003 - 0.001t) \cdot ds - \int_0^t 0.04 \cdot ds = e^{0.04t} - e^{-0.04t}
\]

\[
\therefore PV = 70,000 \int_0^6 e^{-0.05t} \cdot e^{-0.04t} \cdot dt = 70,000 \int_0^6 e^{-0.09t} \cdot dt =
\]

\[
70,000 \left[ \frac{e^{-0.09t}}{-0.09} \right]_0^6 = 70,000 \left[ \frac{e^{-0.09(6)} - 1}{-0.09} \right] = 324,529.14
\]
4. You are given the following table where decrement (1) is death, decrement (2) is lapse, and decrement (3) is diagnosis of critical illness:

<table>
<thead>
<tr>
<th></th>
<th>( q^{(1)}_x )</th>
<th>( q^{(2)}_x )</th>
<th>( q^{(3)}_x )</th>
<th>( p^{(x)}_x )</th>
<th>( l^{(x)}_x )</th>
<th>( d^{(1)}_x )</th>
<th>( d^{(2)}_x )</th>
<th>( d^{(3)}_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>0.02</td>
<td>0.15</td>
<td>0.010</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>56</td>
<td>0.03</td>
<td>0.06</td>
<td>0.015</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>57</td>
<td>0.04</td>
<td>0.04</td>
<td>0.020</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>58</td>
<td>0.05</td>
<td>0.03</td>
<td>0.025</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>59</td>
<td>0.06</td>
<td>0.02</td>
<td>0.030</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Complete the table using a radix of 10,000.

See answers at end.

b. Calculate:
   i. \( 3p^{(x)}_{55} \)

**Solution:**
\[
3p^{(x)}_{55} = \frac{l^{(x)}_{58}}{l^{(x)}_{55}} = \frac{6605.1}{10,000} = 0.66051
\]

ii. \( 2q^{(2)}_{56} \)

**Solution:**
\[
2q^{(2)}_{56} = \frac{d^{(2)}_{56} + d^{(2)}_{57}}{l^{(x)}_{56}} = \frac{492 + 293.56}{8200} = 0.0958
\]

iii. \( \eta q^{(3)}_{55} \)

**Solution:**
\[
\eta q^{(3)}_{55} = \frac{d^{(3)}_{56} + d^{(3)}_{57}}{l^{(x)}_{55}} = \frac{123 + 146.78}{10,000} = 0.026978
\]
iv. The probability that a person age 55 will decrement from death or critical illness before age 60.

Solution:
\[
\frac{d_{55}^{(1)} + d_{56}^{(1)} + \ldots + d_{59}^{(1)} + d_{55}^{(3)} + d_{56}^{(3)} + \ldots + d_{59}^{(3)}}{l_{55}^{(r)}} = \frac{2136.763305}{10,000} = 0.21368
\]

c. Assuming uniform distribution of each decrement between integer ages, calculate:

i. \( 0.25 q_{55}^{(2)} \)

Solution:
\[
0.25 q_{55}^{(2)} = (0.25)q_{55}^{(2)} = (0.25)\left(\frac{1500}{10,000}\right) = 0.0375
\]

ii. \( 0.5 p_{56}^{(r)} \)

Solution:
\[
0.5 p_{56}^{(r)} = 1 - 0.5 q_{56}^{(r)} = 1 - (0.5)(q_{56}^{(r)}) = 1 - (0.5)(0.105) = 0.9475
\]

iii. \( 0.5 p_{56.8}^{(r)} \)

Solution:
\[
0.5 p_{56.8}^{(r)} = \frac{l_{57}^{(r)}}{l_{56.8}^{(r)}} = \frac{(0.7)l_{57}^{(r)} + (0.3)l_{58}^{(r)}}{(0.2)l_{56}^{(r)} + (0.8)l_{57}^{(r)}} = \frac{(0.7)(7339) + (0.3)(6605.1)}{(0.2)(8200) + (0.8)(7339)} = 0.94776
\]

iv. \( 0.5 q_{55.6}^{(1)} \)

Solution:
\[
0.5 q_{55.6}^{(1)} = \frac{0.4d_{55.6}^{(1)} + 0.1d_{56}^{(1)}}{l_{55.6}^{(r)}} = \frac{(0.4)(200) + (0.1)(246)}{(0.4)(10,000) + (0.6)(8200)} = 0.01173
\]
d. Assuming a constant force of decrement for each decrement between integer ages, calculate:

i. \( q_{55}^{(2)} \)

Solution:

\[
0.25 q_{55}^{(2)} = \frac{q_{55}^{(2)}}{q_{55}^{(1)}} \left( 1 - \left( p_{55}^{(2)} \right)^{0.25} \right) = \frac{0.15}{0.18} \left( 1 - (0.82)^{0.25} \right) = 0.04034
\]

ii. \( p_{56}^{(2)} \)

Solution:

\[
0.5 p_{56}^{(2)} = \left( p_{56}^{(2)} \right)^{0.5} = (0.895)^{0.5} = 0.94604
\]

iii. \( p_{56.8}^{(2)} \)

Solution:

\[
0.5 p_{56.8}^{(2)} = \frac{p_{57.3}^{(2)}}{p_{56.8}^{(2)}} = \frac{p_{57}^{(2)} \left( 0.3 p_{57}^{(2)} \right)}{p_{56}^{(2)} \left( 0.8 p_{56}^{(2)} \right)} = \frac{(7339)(0.9)^{0.3}}{(8200)(0.895)^{0.8}} = 0.94763
\]

iv. \( q_{55.6}^{(1)} \)

Solution:

\[
0.5 q_{55.6}^{(1)} = \frac{q_{56.8}^{(1)}}{q_{55.6}^{(1)}} = \frac{q_{56}^{(1)} + 0.1 q_{56}^{(1)}}{q_{55.6}^{(1)}} = \frac{q_{55}^{(1)} - 0.6 q_{55}^{(1)} + 0.1 q_{55}^{(1)}}{(1 - (0.82)^{0.6})} = \\
\frac{200 - (10,000) \left( \frac{0.02}{0.18} \right) (1 - (0.82)^{0.6}) + (8200) \left( \frac{0.03}{0.105} \right) (1 - (0.895)^{0.1})}{(10,000)(0.82)^{0.6}} = \\
\frac{101.1184537}{8877.45156} = 0.011390
\]
5. A fully discrete 3 year term pays a benefit of 1000 upon any death. It pays an additional 1000 (for a total of 2000) upon death from accident. You are given:

<table>
<thead>
<tr>
<th>x</th>
<th>( q_x^{(1)} )</th>
<th>( q_x^{(2)} )</th>
<th>( p_x^{(1)} )</th>
<th>( d_x^{(1)} )</th>
<th>( d_x^{(2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.030</td>
<td>0.010</td>
<td>1000</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>21</td>
<td>0.025</td>
<td>0.020</td>
<td>960</td>
<td>24</td>
<td>19.2</td>
</tr>
<tr>
<td>22</td>
<td>0.020</td>
<td>0.030</td>
<td>916.8</td>
<td>18.336</td>
<td>27.504</td>
</tr>
</tbody>
</table>

Decrement (1) is death from accidental causes while decrement (2) is death from non-accidental causes.

The annual effective interest rate is 10%.

a. Calculate the level annual net premium for this insurance.

**Solution:**
Columns above in yellow were added to facilitate the calculation.

\[
PVP = PVB
\]

\[
P = \frac{P(1000 + 960v + 916.8v^2) + 2000(30v + 24v^2 + 18.336v^3) + 1000(10v + 19.2v^2 + 27.504v^3)}{(1000 + 960v + 916.8v^2)} = 63.64
\]
b. Calculate the net premium reserve at the end of year 0, 1, 2, and 3.

Solution:
By Definition \( V = 0 \) and \( V = 0 \)

We will use the recursive formula to get the other two reserves.

\[
\begin{align*}
\nu V &= \left( V + P \right) \left( 1 + i \right) - b_1^{(1)} \cdot q_x^{(1)} - b_1^{(2)} \cdot q_x^{(2)} \\
&= \left( 0 + 63.64 \right) (1.1) - (2000)(0.03) - (1000)(0.01) \\
&= 0.00
\end{align*}
\]

\[
\begin{align*}
\nu V &= \left( V + P \right) \left( 1 + i \right) - b_2^{(1)} \cdot q_x^{(1)} - b_1^{(2)} \cdot q_x^{(2)} \\
&= \left( V + P \right) \left( 1 + i \right) - b_2^{(1)} \cdot q_x^{(1)} - b_1^{(2)} \cdot q_x^{(2)} \\
&= \left( 0 + 63.64 \right) (1.1) - (2000)(0.025) - (1000)(0.02) \\
&= 0.00
\end{align*}
\]

Do not erroneously draw the conclusion that all reserves are zero.
It just so happens here. A poorly developed question. ;)

6. You are given:
   a. \( q_x^{(1)} = 0.200 \)
   b. \( q_x^{(2)} = 0.080 \)
   c. \( q_x^{(3)} = 0.125 \)

Assuming that each decrement is uniformly distributed over each year of age in the associated single decrement table, calculate \( q_x^{(1)} \).

Solution:

\[
q_x^{(1)} = q_x^{(1)} \left\{ 1 - 1/2 \left[ q_x^{(2)} + q_x^{(3)} \right] + 1/3 \left[ q_x^{(2)} \cdot q_x^{(3)} \right] \right\} =
\]

\[
(0.2) \left\{ 1 - 1/2 \left[ 0.08 + 0.125 \right] + 1/3 \left[ 0.08 \cdot 0.125 \right] \right\} = 0.180167
\]
7. You are given:
   a. \( q_x^{(1)} = 0.200 \)
   b. \( q_x^{(2)} = 0.080 \)
   c. \( q_x^{(3)} = 0.125 \)

Assuming that each decrement in the multiple decrement table is uniformly distributed over each year of age, calculate \( q_x^{(1)} \).

**Solution:**
\[
p_x^{(r)} = p_x^{(r)} \cdot p_x^{(2)} \cdot p_x^{(3)} = (1 - q_x^{(1)})(1 - q_x^{(2)})(1 - q_x^{(3)}) = (0.8)(0.92)(0.875) = 0.644
\]
\[
p_x^{(1)} = \left( p_x^{(r)} \right)^{\frac{q_x^{(1)}}{q_x^{(r)}}} \implies 0.8 = (0.644)^{\frac{q_x^{(1)}}{q_x^{(r)}}} \implies q_x^{(1)} = \frac{\ln(0.8)}{\ln(0.644)} (1 - 0.644) = 0.180520
\]

8. You are given the following for a double decrement table:
   a. \( q_x^{(1)} = 0.200 \)
   b. \( q_x^{(2)} = 0.080 \)

Assuming that each decrement in the multiple decrement table is uniformly distributed over each year of age, calculate \( 0.4q_x^{(1)} \).

**Solution:**
\[
p_x^{(r)} = p_x^{(r)} \cdot p_x^{(2)} = (1 - q_x^{(1)})(1 - q_x^{(2)}) = (0.8)(0.92) = 0.736
\]
\[
p_x^{(1)} = \left( p_x^{(r)} \right)^{\frac{q_x^{(1)}}{q_x^{(r)}}} \implies 0.8 = (0.736)^{\frac{q_x^{(1)}}{0.736}} \implies q_x^{(1)} = \frac{\ln(0.8)}{\ln(0.736)} (1 - 0.736) = 0.192186173
\]
\[
0.4q_x^{(1)} = \frac{0.4d_x^{(1)}}{1 - 0.4(1 - 0.736)} = 0.085951 \quad \text{This assumes } l_x^{(r)} = 1
\]
9. You are given:
   a. \( q_x^{(1)} = 0.200 \)
   b. \( q_x^{(2)} = 0.080 \)

Decrement (1) is uniformly distributed over the year. Decrement (2) occurs at time 0.6.

Calculate \( q_x^{(2)} \).

**Solution:**

\[
q_x^{(2)} = \frac{d_x^{(2)}}{l_x^{(r)}}
\]

Let \( l_x^{(r)} = 1000 \). Then, if there was only decrement (1), we would have \( l_x^{(r)} \cdot q_x^{(1)} = (1000)(0.2) = 200 \) which would be distributed uniformly over the year. Since decrement (2) all occurs at time 0.6, for the first 0.6 of the year, there is only decrement (1). During this time, 0.6 of the 200 decrements occur since they occur uniformly. This means that 120 decrement (1)s occur during the first 0.6 of the year. This leaves us with 880.

\[
d_x^{(2)} = 880 \cdot q_x^{(2)} = (880)(0.08) = 70.4
\]

\[
q_x^{(2)} = \frac{d_x^{(2)}}{l_x^{(r)}} = \frac{70.4}{1000} = 0.0704
\]
10. For a double decrement table with $l_{40}^T = 2000$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q_x^{(1)}$</th>
<th>$q_x^{(2)}$</th>
<th>$q_x^{(1)}$</th>
<th>$q_x^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.24</td>
<td>0.10</td>
<td>0.25</td>
<td>$y$</td>
</tr>
<tr>
<td>41</td>
<td>--</td>
<td>--</td>
<td>0.20</td>
<td>$2y$</td>
</tr>
</tbody>
</table>

Calculate $l_{42}^T$.

**Solution:**

\[ p_{40}^{(r)} = 1 - q_{40}^{(1)} - q_{40}^{(2)} = p_{40}^{(1)} \cdot p_{40}^{(2)} \implies 1 - 0.24 - 0.10 = 0.66 = (1 - 0.25)(1 - y) \]

\[ y = 1 - \frac{0.66}{0.75} = 0.12 \implies 2y = 0.24 \]

\[ l_{42}^{(r)} = l_{40}^{(r)} \cdot p_{40}^{(r)} \cdot p_{41}^{(r)} = l_{40}^{(r)} \cdot p_{40}^{(1)} \cdot p_{40}^{(2)} \cdot p_{41}^{(1)} \cdot p_{41}^{(2)} = \]

\[ 2000(1 - 0.25)(1 - 0.12)(1 - 0.2)(1 - 0.24) = 802.56 \]
11. You are given the following excerpt from a double decrement table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( l_x^{(r)} )</th>
<th>( q_x^{(1)} )</th>
<th>( q_x^{(2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>53</td>
<td>---</td>
<td>0.025</td>
<td>0.030</td>
</tr>
<tr>
<td>54</td>
<td>5000</td>
<td>---</td>
<td>0.040</td>
</tr>
<tr>
<td>55</td>
<td>4625</td>
<td>0.055</td>
<td>0.050</td>
</tr>
</tbody>
</table>

Calculate \( \ddot{q}_{53}^{(1)} \).

Solution:

\[
\ddot{q}_{53}^{(1)} = \frac{d_{53}^{(1)} + d_{54}^{(1)}}{l_{53}^{(r)}}
\]

\[
l_{54}^{(r)} = l_{53}^{(r)} (1 - q_{53}^{(1)} - q_{53}^{(2)}) \iff l_{53}^{(r)} = \frac{l_{54}^{(r)}}{(1 - q_{53}^{(1)} - q_{53}^{(2)})} = \frac{5000}{1 - 0.025 - 0.030} = 5291.05291
\]

\[
l_{55}^{(r)} = l_{54}^{(r)} - d_{54}^{(1)} - d_{54}^{(2)} \iff d_{54}^{(1)} = l_{54}^{(r)} - d_{54}^{(2)} - l_{55}^{(r)} = 5000 - (5000)(0.04) - 4625 = 175
\]

\[
\ddot{q}_{53}^{(1)} = \frac{5291.05291(0.025) + 175}{5291.05291} = 0.058075
\]

Or

\[
\ddot{q}_{53}^{(1)} = q_{53}^{(1)} + p_{53}^{(r)} \cdot q_{54}^{(1)} = 0.025 + (1 - 0.025 - 0.030) \left( \frac{175}{5000} \right) = 0.058075
\]
12. For iPhones, the phone may cease service for mechanical failure or for other reasons (lost, stolen, dropped in a pitcher of beer, etc). You are given the following double decrement table:

<table>
<thead>
<tr>
<th>Year of Service</th>
<th>For an iPhone at the beginning of the year of service, probability of</th>
<th>Failure for Other Reasons</th>
<th>Survival through the year of service</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mechanical Failure</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.30</td>
<td>--</td>
</tr>
<tr>
<td>2</td>
<td>--</td>
<td>0.40</td>
<td>--</td>
</tr>
<tr>
<td>3</td>
<td>--</td>
<td>--</td>
<td>0.20</td>
</tr>
</tbody>
</table>

You are also given:

a. The number of iPhones that terminate for other reasons in year 3 is 40% of the number of iPhones that survive to the end of year 2.

b. The number of iPhones that terminate for other reasons in year 2 is 80% of the number of iPhones that survive to the end of year 2.

Calculate the probability that an iPhone will cease to function due to mechanical failure during the three year period following its entry into service.
Solution:
Let (m) be mechanical failure and (o) be failure for other reasons.

Now we will build a table assuming that we start with 1000 phones.

<table>
<thead>
<tr>
<th>Year</th>
<th>( d_1^{(m)} )</th>
<th>( d_1^{(o)} )</th>
<th>Surviving Phones</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200(^a)</td>
<td>300(^b)</td>
<td>500(^c)</td>
</tr>
<tr>
<td>2</td>
<td>50(^f)</td>
<td>200(^d)</td>
<td>250(^e)</td>
</tr>
<tr>
<td>3</td>
<td>100(^g)</td>
<td>100(^g)</td>
<td>50(^h)</td>
</tr>
</tbody>
</table>

\[ a \Rightarrow 1000(q_1^{(m)}) = (1000)(0.2) = 200 \]
\[ b \Rightarrow 1000(q_1^{(o)}) = (1000)(0.3) = 300 \]
\[ c \Rightarrow 1000 - 200 - 300 = 500 \]
\[ d \Rightarrow 500(q_1^{(o)}) = (500)(0.4) = 200 \]
\[ e \Rightarrow \text{From Given (b)} \Rightarrow 200 = (0.8)(e) \Rightarrow 200/0.8 = 250 \]
\[ f \Rightarrow 500 - 200 - 250 = 50 \]
\[ g \Rightarrow \text{From Given (a)} \Rightarrow g = (0.4)(250) = 100 \]
\[ h \Rightarrow 250(0.2) = 50 \]
\[ i = 250 - 100 - 50 = 100 \]

\[ q_x = \frac{d_1^{(m)} + d_2^{(m)} + d_3^{(m)}}{l_0^{(x)}} = \frac{200 + 50 + 100}{1000} = 0.35 \]
13. *Your actuarial student has constructed a multiple decrement table using independent mortality and lapse tables. The multiple decrement table values, where decrement \(d\) is death and decrement \(w\) is lapse, are as follows:

<table>
<thead>
<tr>
<th></th>
<th>(l_{60}^{(z)})</th>
<th>(d_{60}^{(d)})</th>
<th>(d_{60}^{(w)})</th>
<th>(l_{61}^{(z)})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>950,000</td>
<td>2,580</td>
<td>94,742</td>
<td>852,678</td>
</tr>
</tbody>
</table>

You discover that an incorrect value of \(q_{60}^{(w)}\) was taken from the independent lapse table. The correct value is 0.05.

Decrements are uniformly distributed over each year of age in the multiple decrement table.

You correct the multiple decrement table, keeping \(l_{60}^{(z)} = 950,000\).

Calculate the correct values of \(d_{60}^{(w)}\).
Solution:

\[ p_{60}^{(d)} = \left( p_{60}^{(r)} \right) \frac{q_{60}^{(d)}}{q_{60}^{(d)}} \]

\[ q_{60}^{(r)} = q_{60}^{(d)} + q_{60}^{(w)} = 1 - \frac{l_{61}^{(r)}}{l_{60}^{(r)}} \]

\[ p_{60}^{(r)} = \frac{l_{61}^{(r)}}{l_{60}^{(r)}} \]

\[ p_{60}^{(d)} = \left( p_{60}^{(r)} \right) \frac{q_{60}^{(d)}}{q_{60}^{(d)}} = \left( \frac{852.678}{950.000} \right) = 0.997138907 \]

Note that we can use the incorrect values to derive \( p_{60}^{(d)} \) since this value was correct in the calculations.

\[ p_{60}^{(w)} = \left( p_{60}^{(r)} \right) \frac{q_{60}^{(w)}}{q_{60}^{(w)}} = \left( \frac{852.678}{950.000} \right) = 0.997138907 \]

\[ q_{60}^{(w)} = \frac{\ln(0.95)}{\ln(0.997138907)(1-0.05)} \frac{q_{60}^{(w)}}{q_{60}^{(w)}} = 0.049929 \]

\[ \Rightarrow d_{60}^{(w)} = (950,000)(0.049929) = 47,433 \]
14. A person age 60 is subject to three decrements. You are given:

i. The following excerpt from a triple decrement table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q_x^{(1)}$</th>
<th>$q_x^{(2)}$</th>
<th>$q_x^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0.05</td>
<td>0.10</td>
<td>0.08</td>
</tr>
<tr>
<td>61</td>
<td>0.00</td>
<td>0.14</td>
<td>0.12</td>
</tr>
</tbody>
</table>

ii. Decrement 1 occurs exactly one quarter of the way through the year.

iii. $q_x^{(2)} = t q_x^{(2)}$ for integer $x$ and $0 \leq x \leq 1$.

iv. $q_x^{(3)} = \begin{cases} 0, & \text{for integer } x \text{ and } 0 \leq t \leq 0.5 \\ 2(t - 0.5) q_x^{(3)}, & \text{for integer } x \text{ and } 0.5 \leq t \leq 1 \end{cases}$

a. Calculate $p_x^{(r)}$.

**Solution:**

$$p_x^{(r)} = p_x^{(r)} \cdot p_x^{(r)} = (1 - .05 - .10 - .08)(1 - .00 - .14 - .12) = 0.5698$$
a. Calculate $0.8 \, p_{60}^{(r)}$.

Solution:

Let $l_{60}^{(r)} = 1000$

\[d_{60}^{(1)} = l_{60}^{(r)} \cdot q_{60}^{(1)} = (1000)(0.05) = 50.\] These all occur at time 0.25.

\[d_{60}^{(2)} = l_{60}^{(r)} \cdot q_{60}^{(2)} = (1000)(0.10) = 100.\] These decrements are uniformly distributed throughout the year.

\[d_{60}^{(3)} = l_{60}^{(r)} \cdot q_{60}^{(3)} = (1000)(0.08) = 80.\] These decrements begin at time 0.5 and are uniformly distributed throughout the second half of the year.

\[0.8 \, p_{60}^{(r)} = \frac{l_{60}^{(r)} - d_{60}^{(1)} - d_{60}^{(2)} - d_{60}^{(3)}}{l_{60}^{(r)}}\]

\[= \frac{1000 - 50 - (100)(0.8) - (80)(0.6)}{1000}\]

\[= 0.822\]