STAT 475
Test 2
Spring 2020
April 23, 2020

1. For the single factor longevity model expressed as $q(x,t) = q(x,0)(1-\varphi_x)^t$, you are given:

   i. Base mortality rates equal to $q_{80+t} = 0.05 + 0.01t$ for $t = 0,\ 1,\ 2,\ 3,\ \ldots$.

   ii. The following selected improvement factors:

x	80	81	82	$\geq$ 83
φ_x	0.04	0.03	0.02	0.01

   iii. $i = 0.05$

   iv. K_{80} denotes the curtate future lifetime of a life who is age 80 at time 0.

   a. Calculate $\Pr(K_{80} = k)$ for $k = 0,\ 1,\ \text{and}\ 2$ accurate to five decimal places.

   Solution:

$$q(80,0) = 0.05$$
$$q(81,0) = 0.06 ==> q(81,1) = (0.06)(1-0.03)^1 = 0.0582$$
$$q(82,0) = 0.07 ==> q(82,2) = (0.07)(1-0.02)^2 = 0.067228$$

$$\Pr(K_{80}) = 0 = q(80,0) = 0.05$$
$$\Pr(K_{80}) = 1 = p(80,0)q(81,1) = (1-0.05)(0.0582) = 0.05529$$
$$\Pr(K_{80}) = 2 = p(80,0)p(81,1)q(82,2) = (1-0.05)(1-0.0582)(0.067228) = 0.06015$$

b. Let Y denote the present value of a 3 year temporary life annuity due of 1 per year issued to a life who is age 80 at time 0.

i. Calculate the expected value of Y accurate to five decimal places.

Solution:

$$E[Y] = APV = 1 + vp(80,0) + v^2 p(80,0)\,p(81,1)$$

$$= 1 + (1.05)^{-1}(1-0.05) + (1.05)^{-2}(1-0.05)(1-0.0582) = 2.71629$$

ii. Calculate the variance of Y accurate to five decimal places.

Solution:

$$Var[Y] = \frac{{}^2A_{80:\overline{3}|} - \left(A_{80:\overline{3}|}\right)^2}{d^2} = \frac{0.758485071 - (0.870652845)^2}{\left(\frac{0.05}{1.05}\right)^2} = 0.19787$$

$$A_{80:\overline{3}|} = vq(80,0) + v^2 p(80,0)q(81,1) + v^3 p(80,0)\,p(81,1)$$

$$= (1.05)^{-1}(0.05) + (1.05)^{-2}(1-0.05)(0.0582) + (1.05)^{-3}(1-0.05)(1-0.0582) = 0.870652845$$

The last term is valid because if (80) lives to the end of the second year, we will pay at the end of the third year whether (80) lives (endowment) or dies (death benefit).

$${}^2A_{80:\overline{3}|} = v^2 q(80,0) + v^4 p(80,0)q(81,1) + v^6 p(80,0)\,p(81,1)$$

$$= (1.05)^{-2}(0.05) + (1.05)^{-4}(1-0.05)(0.0582) + (1.05)^{-6}(1-0.05)(1-0.0582) = 0.758485071$$

2. You are given the following three lives in a mortality study:

Life	Age at Entry	Age at Termination	Reason for Termination
1	70.5	80.0	End of Study
2	66.8	70.8	Death
3	69.0	72.0	Death

Let q_{70}^{e} be the estimate for q_{70} using the exact exposure method. Let q_{70}^{A} be the estimate for q_{70} using the actuarial exposure method.

Calculate the estimate for $(1000)(q_{70}^{A} - q_{70}^{e})$ obtained from these data.

Solution:

For the exact exposure, we calculate the exposure during age 70 prior to death or other termination. For the actuarial exposure method, we calculate the exposure during age 70 for other termination but for death, we count exposure until the end of the year.

Therefore, exposure is:

Life	Deaths	Exposure – Exact	Exposure – Actuarial
1	0	0.5	0.5
2	1	0.8	1.0
3	0	1.0	1.0

$$q_{70}^{e} = 1 - e^{-\frac{Deaths}{ExactExposure}} = 1 - e^{-\frac{1}{2.3}} = 0.3525946$$

$$q_{70}^{A} = \frac{Deaths}{ExactExposure} = \frac{1}{2.5} = 0.4$$

$$(1000)(q_{70}^{A} - q_{70}^{e}) = 1000(0.4 - 0.3525946) = 47.405392$$

3. The actuaries at Lee Life Insurance Company have set short term improvement factors for population mortality based on the experience in 2018 and 2019, and long term factors based on projected values in 2025 and 2026. Actuaries calculate the appropriate improvement factors for intermediate years using a cubic spline.

   You are given the following information:

   (i) There are no cohort effects.

   (ii) $\varphi(x,2018)=0.040 \quad \varphi(x,2019)=0.042$

   (iii) $\varphi(x,2025)=0.010 \quad \varphi(x,2026)=0.010$

   a. Assuming that the cubic spline takes the form of $C_a(x,t)=at^3+bt^2+ct+d$, determine the parameters accurate to five decimal places to be used to calculate the improvement factors.

   Solutions:

$$C_a(x,0)=a(0)^3+b(0)^2+c(0)+d=\varphi(x,2019) \text{ ==> } d=0.042$$

$$C_a'(x,t)=3at^2+2bt+c$$

$$C_a'(x,0)=3a(0)^2+2b(0)^1+c=\varphi(x,2019)-\varphi(x,2018) \text{ ==> } c=0.042-0.040=0.002$$

$$C_a(x,6)=a(6)^3+b(6)^2+c(6)+d=\varphi(x,2025)$$

$$\text{==> } 216a+36b+(0.002)(6)+0.042=0.010 \text{ ==> } 216a+36b=-0.044$$

$$C_a'(x,6)=3a(6)^2+2b(6)^1+c=\varphi(x,2026)-\varphi(x,2025)$$

$$108a+12b+0.002=0 \text{ ==> } 108a+12b=-0.002$$

Using algebra to solve we get

$$a=\frac{19}{54,000}=0.000351852 \quad \text{and} \quad b=-\frac{1}{300}=-0.00333333333$$

$$C_a(x,t)=\varphi(x,2019+t)=(0.000351852)(t)^3-(0.0033333333)(t)^2+(0.002)(t)+0.042$$

b. Calculate the improvement factor applying to a life age x in 2021 accurate to five decimal places.

Solution:

$$C_a(x,t) = \varphi(x, 2019+t) = (0.000351852)(t)^3 - (0.0033333333)(t)^2 + (0.002)(t) + 0.042$$

$$\varphi(x, 2021) = (0.000351852)(2)^3 - (0.0033333333)(2)^2 + (0.002)(2) + 0.042 = 0.03548$$

4. Emily is the Chief Actuary for Maxwell's Mouse Farm. Emily conducts a 3 year mortality study on the mice at her Farm. You are given the following information about 100 mice at the Farm:

   i. 60 mice were present at the start of the study
   ii. 20 mice were purchased at the end of 1 year
   iii. 15 mice were purchases at the end of 1.5 years
   iv. 5 mice were purchased at the end of 2 years
   v. 8 mice died at time 0.8
   vi. 6 mice died at time 1
   vii. 12 mice died at time 2
   viii. 3 mice died at time 2.5
   ix. 23 mice escaped from the Farm at time 0.6
   x. 13 mice escaped from the Farm at time 1
   xi. 10 mice escaped at time 2.4

The mice that escape the Farm are no longer observed.

Emily calculates $\hat{S}(2.4)$ using the Nelson Aalen estimator.

a. Determine the value of $\hat{S}(2.4)$ accurate to five decimal places.

Solution:

i	y_i	s_i	r_i
1	0.8	8	60 – 23 = 37
2	1.0	6	60 – 8 – 23 =29
3	2	12	60 + 20 + 15 – 8 – 6 – 23 – 13 = 45
4	2.5	3	60 + 20 + 15 + 5 – 8 – 6 – 12 – 23 – 13 - 10 = 28

$$\hat{H}(2.4) = \frac{8}{37} + \frac{6}{29} + \frac{12}{45} = 0.689779435$$

$$\hat{S}(2.4) = e^{-\hat{H}(2.4)} = e^{-0.689779435} = 0.50169$$

Emily wants to be sure that her answer is correct so she hires Michael of Bell Consultants to also calculate $S(2.4)$. Michael uses the Kaplan Meier Product Limit Estimator.

b. Calculate the value of $S(2.4)$ determined by Michael accurate to five decimal places.

Solution:

$$S(2.4) = \left(\frac{37-8}{37}\right)\left(\frac{29-6}{29}\right)\left(\frac{45-12}{45}\right) = 0.45586$$

In order to impress Emily and hopefully earn future consulting business, Michael also calculates the 80% linear confidence interval for $S(2.4)$.

c. Calculate the linear confidence interval determined by Michael.

Solution:

$$Var[S(2.4)] = \left[S(2.4)\right]^2 \sum_{i=1}^{3} \frac{s_i}{r_i(s_i - r_i)}$$

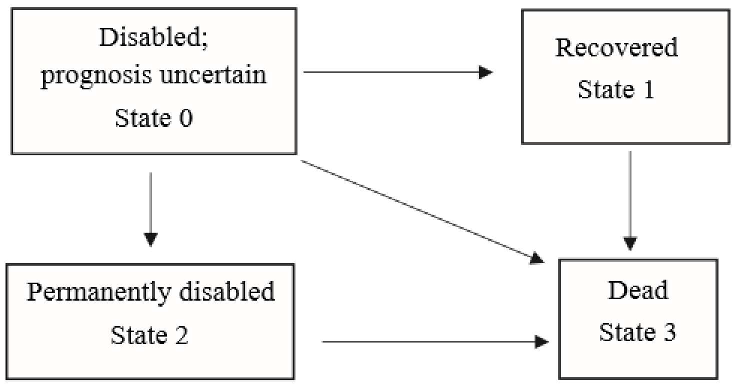
$$= (0.45586)^2 \left(\frac{8}{37(37-8)} + \frac{6}{29(29-6)} + \frac{12}{45(45-12)}\right) = 0.00509787$$

$$80\% CI = 0.45586 \pm 1.282\sqrt{0.00509787} = (0.36432; 0.54739)$$

5. David, who is age 45, has recently suffered a disabling injury on the job. His prognosis is uncertain.

David receives a structured settlement from Rahn & Uppal Insurance. The settlement is a life annuity, starting immediately, and payable continuously at a rate of 90,000 per year while David is disabled. An additional annuity of 20,000 per year is payable continuously while David's prognosis is uncertain for up to two years, to offset medical expenses.

Rahn & Uppal uses the following multiple state model:



You are given the following:

i. Rahn & Uppal hold reserves equal to the expected present value of future benefits.

ii. $i = 0.04$

You are also given the following table of annuity values and transition probabilities:

x	$\bar{a}_x^{00}$	$\bar{a}_x^{01}$	$\bar{a}_x^{02}$	$\bar{a}_x^{11}$	$\bar{a}_x^{22}$	${}_2p_x^{00}$	${}_2p_x^{01}$	${}_2p_x^{02}$
45	0.559	5.540	7.161	19.948	10.703	0.0301	0.2766	0.6164
47	0.559	5.420	7.104	19.528	10.623	0.0301	0.2766	0.6162

a. Show that the expected present value of future benefits at $t = 0$, the start date for the annuity payments, is 706,000 to the nearest 1000. You should calculate the value to the nearest 1.

Solution:

$$APV = 90,000(\ddot{a}_{45}^{00} + \ddot{a}_{45}^{02}) + 20,000(\ddot{a}_{45:\overline{2|}}^{00})$$

$$\ddot{a}_{45:\overline{2|}}^{00} = \ddot{a}_{45}^{00} - v^2 \bullet {}_2p_{45}^{00} \bullet \ddot{a}_{47}^{00}$$

$$= 90,000(0.559 + 7.161) + 20,000[0.559 - (1.04)^{-2}(0.0301)(0.559)] = 705,668.87$$

b. Show that ${}_2V^{(0)}$, the reserve at $t=2$ if David is in State 0 at that time, is 690,000 to the nearest 1000. You should calculate the reserve to the nearest 1.

Solution:

$${}_2V^{(0)} = 90,000(\ddot{a}_{47}^{00} + \ddot{a}_{47}^{02}) = 90,000(0.559 + 7.104) = 689,670$$

c. Calculate ${}_2V^{(2)}$, the reserve at $t=2$ if David is in State 2 at that time.

Solution:

$${}_2V^{(2)} = 90,000(\ddot{a}_{47}^{22}) = 90,000(10.623) = 956,070$$

6. Ethan (45) was injured in a work injury. He is permanently disabled. His employer's workers compensation plan has agree to pay him the following benefits during his remaining lifetime:

    i. Benefit A - A life annuity due with annual payments of 200,000 for 20 years and then 100,000 per year thereafter;

    ii. Benefit B - A lump sum payment of 1,000,000 today and another 1,000,000 at age 55 if he is alive at age 55; and

    iii. Benefit C - A death benefit of 1,000,000 paid at the end of the year of death if he dies before age 65.

   You are given that $\mu'_{45+t} = \mu^{SULT}_{45+t} + 0.005 \text{ for } t \geq 0$ where μ^{SULT}_{45+t} is the force of mortality in the Standard Ultimate Life Table. Ethan's force of mortality is assumed to be μ'_{45+t} between age 45 and 65. After age 65, Ethan's force of mortality is assumed to be the force of mortality in the Standard Ultimate Life Table.

   You are also given that $i = 5\%$.

   Finally, you are given that $\ddot{a}^{\mu'}_{45} = 16.6586 \text{ and } \ddot{a}^{\mu'}_{65} = 12.9334$ where these values are calculated using μ'_{45+t} for all ages and $i = 5\%$.

   a. Calculate the actuarial present value of Benefit A.

   Solution:

$$APV = 200,000(\ddot{a}^{\mu'}_{45} - v^{20}\,{}_{20}p^{\mu'}_{45}\ddot{a}^{\mu'}_{65}) + 100,000(v^{20}\,{}_{20}p^{\mu'}_{45})(\ddot{a}^{SULT}_{65})$$

$$= 200,000\left[16.6586 - (1.05)^{-20}\left(\frac{94,579.7}{99,033.9}\right)e^{-0.005(20)}(12.9334)\right] + 100,000\left[(1.05)^{-20}\left(\frac{94,579.7}{99,033.9}\right)e^{-0.005(20)}\right](13.5498)$$

$$= 2,930,575$$

b. Calculate the actuarial present value of Benefit B.

Solution:

$$APV = 1,000,000 + 1,000,000 v^{10} {}_{10}p_{45}^{\mu'}$$

$$= 1,000,000\left[1 + (1.05)^{-10}\left(\frac{97,846.2}{99,033.9}\right)e^{-0.005(10)}\right] = 1,576,969$$

c. Calculate the actuarial present value of Benefit C.

Solution:

$$1,000,000\left(A_{45}^{u'} - (1.05)^{-20}\left(\frac{94,579.7}{99,033.9}\right)e^{-0.005(20)} A_{65}^{u'}\right)$$

$$= 1,000,000\left(0.20673333 - (1.05)^{-20}\left(\frac{94,579.7}{99,033.9}\right)e^{-0.005(20)}(0.38412381)\right)$$

$$= 81,630$$

$$A_{45}^{u'} = 1 - d \cdot \ddot{a}_{45}^{u'} = 1 - \left(\frac{0.05}{1.05}\right)16.6586 = 0.20673333$$

$$A_{65}^{u'} = 1 - d \cdot \ddot{a}_{65}^{u'} = 1 - \left(\frac{0.05}{1.05}\right)12.9334 = 0.38412381$$

7. The Cooper Life Insurance Company sells three year term insurance policies to (90). The death benefit for the policy is 50,000 paid at the end of the year. The annual premium for this policy is 4550.

Cooper holds full preliminary term reserves. The reserve basis for the reserves is mortality equal to the Standard Ultimate Life Table with interest at 5%. Further, you are given the reserve at the end of the second year is 336.36.

The pricing basis is:

i. Mortality is 75% of the Standard Ultimate Life Table;

ii. The earned interest rate is $i = 7\%$;

iii. Issue expenses are 250 per policy;

iv. Commissions are 20% in the first year and 3% thereafter; and

v. Maintenance expenses are 25 per policy for every year including the first.

You are also given:

i. Lapse rates at the end of year 1 are 8%, at the end of year 2 are 5% and there are no lapses at the end year 3.

ii. There are no cash values.

iii. The discount rate used to calculate profit measures is 10%.

a. $\Pr_0$ is equal to -1020 to the nearest 10. Calculate $\Pr_0$ to the nearest 0.01.

Solution:

$$\Pr_0 = 250 + (0.20 - 0.03)(4550) = 1023.50$$

b. $\Pr_1$ is equal to 910 to the nearest 10. Calculate $\Pr_1$ to the nearest 0.01.

Solution:

$$\Pr_1 = (0 + 4550(1 - 0.03) - 25)(1.07) - 50,000(0.75)(0.100917))$$
$$- 0(1 - 0.75(0.100917)(1 - 0.08) = 911.31$$

Note that since it is FPT, ${}_0V = {}_1V = 0$

c. Calculate Pr_2.

Solution:

$$\mathrm{Pr}_2 = (0 + 4550(1-0.03) - 25)(1.07) - 50,000(0.75)(0.112675))$$
$$-336.36(1-0.75(0.112675)(1-0.05) = 177.84$$

Note that since it is FPT, ${}_0V = {}_1V = 0$

You are given that $\mathrm{Pr}_3 = 341.55$

d. Calculate the Net Present Value.

$$\pi_0 = \mathrm{Pr}_0 = -1023.50$$
$$\pi_1 = \mathrm{Pr}_1 = 911.31$$
$$\pi_2 = \mathrm{Pr}_2(p_{90}^{(\tau)}) = 177.84(1-(0.75)(0.100917))(1-0.08) = 151.23$$
$$\pi_3 = \mathrm{Pr}_3({}_2p_{90}^{(\tau)}) = 341.55(1-(0.75)(0.100917))(1-0.08)(1-(0.75)(0.112675)(1-0.05) = 252.60$$

$$NPV = -1023.50 + 911.31(1.1)^{-1} + 151.23(1.1)^{-2} + 252.60(1.1)^{-3} = 119.73$$

e. Calculate the Profit Margin.

Solution:

$$PM = \frac{NPV}{PVP} = \frac{119.73}{4550(1+(1.1)^{-1}(1-(0.75)(0.100917))(1-0.05) + (1.1)^{-2}(1-(0.75)(0.100917))(1-0.8)((1-(0.75)(0.112675))(1-0.05)}$$

$$= \frac{119.73}{10,848.50} = 0.01104$$