Long-Term Actuarial Mathematics

Sample Written Answer Questions

October 13, 2018

There are 26 sample Written Answer questions in this Study Note for the Long-Term Actuarial Mathematics exam. Unlike the Multiple Choice questions, these have not been sorted by source material as most cover multiple chapters of the text or both the text and the study notes.

The first 20 questions are from the 2014 and 2015 MLC exams. Questions that have been modified have been modified to:

- Replace the Illustrative Life Table (ILT) which was used on the MLC exam with the Standard Ultimate Life Table (SULT) which will be used with the LTAM exam. All problems that previously used the ILT have been converted to the SULT.
- Remove portions of questions which covered material that is no longer covered by the Long-Term Actuarial Mathematics exam.

The solutions to these questions include comments from the graders concerning candidates' performance on the exam. Such comments are not included for the other questions.

There are five sample written questions which primarily cover the material that has been added to the Long-Term Actuarial Mathematics exam.

Different solutions show different levels of accuracy in intermediate results. These model solutions are not intended to imply that this is the best rounding for each question. Graders do not penalize rounding decisions, unless an answer is rounded to too few digits in the context of the problem and the given information. In particular, if a problem in one step asks you to calculate something to the nearest 1, and you calculate it as (for example) 823.18, you need not bother saying "that's 823 to the nearest 1", and you may use 823.18 or 823 in future steps.

Versions:

July 2, 2018	Original Set of Questions Published
July 11,2018	Correction to questions WA.21 and WA.22
August 10, 2018	Correction to question WA.25
October 13, 2018	Correction to question WA.6

WA.1. (*8 points*) Ben, age 40, purchases a special life insurance policy with the following features:

- Premiums are paid monthly for life.
- A death benefit of 20,000 is paid at the end of the year of death if death occurs after age 60.
- No benefit is paid if death occurs before age 60.

Premiums are calculated using the following assumptions:

- Mortality follows the Standard Ultimate Life Table.
- The Woolhouse formula with two terms is used to calculate annuity values.
- $i = 0.05$

(a) (*1 point*) Show that the expected present value of the death benefit is 2130 to the nearest 10. You should calculate the value to the nearest 1.

(b) (*2 points*) Calculate the annualized net premium for the policy.

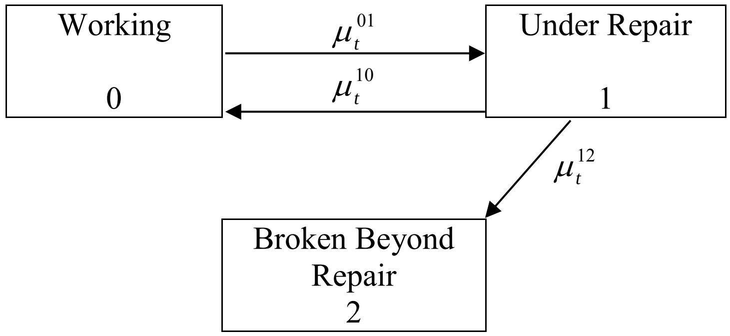
(c) (*2 points*) State two reasons why the annualized net premium rate will change if premiums are payable continuously, giving the direction of change for each reason.

The policy offers an optional feature under which the premium is waived if Ben's wife, Anne, dies. Anne is 50 years old at the issue date. Ben and Anne have independent future lifetimes.

(d) (*3 points*) Calculate the increase in the annualized net premium due to the premium waiver.

(This is a modified version of Question 1 from the Spring 2014 Written Answer exam.)

WA.2. (*7 points*) An actuarial student in your department is using the following 3-state Markov model to price a one-year consumer warranty for a television.



The product has the following features:

(i) The warranty is sold at the time that a consumer purchases the television and pays a replacement cost of 1000 at the end of the half-year during which the television becomes Broken Beyond Repair.

(ii) Premiums are payable at the beginning of each half-year, but are waived if the television is not Working at the time a premium is due.

(iii) All televisions are in the Working state at time of purchase.

(a) (*2 points*) Write down the Kolmogorov forward differential equations with associated boundary conditions (initial conditions) for ${}_tp_0^{00}$, ${}_tp_0^{01}$, and ${}_tp_0^{02}$ under this model.

The forces of transition in this model for a television purchased t years ago are:

$$\mu_t^{01} = 0.5 + 0.6t$$

$$\mu_t^{10} = 0.2$$

$$\mu_t^{12} = 2^t$$

The student has applied the Forward Euler approximation to the Kolmogorov forward differential equations with a step size of $h = 0.5$ to calculate probabilities for this model. You have verified that, using this approach, ${}_{0.5}p_0^{00} = 0.75$.

WA.2. **Continued**

(b) (*2 points*) Using the student's approach:

(i) Calculate ${}_{0.5}p_0^{01}$.

(ii) Show that the probability that a television will become Broken Beyond Repair within a year of purchase is 0.18 to the nearest 0.01. You should calculate the probability to the nearest 0.001.

(c) (*2 points*) Using the probabilities from the student's approach and $i^{(2)} = 8\%$:

(i) Calculate the actuarial present value at issue of the replacement cost payments for this policy.

(ii) Calculate the semi-annual net premium for this policy.

(d) (*1 point*) Suggest a change to the student's approach that would improve the accuracy of the probability calculations.

(This is Question 2 from the Spring 2014 Written Answer exam.)

WA.3. (5 *points*) PonceDeLeon Pharmaceuticals is conducting a one-year study of a "Fountain of Youth" drug. They have two versions under development.

(i) There is a control group of 1000 subjects with independent future lifetimes who will not receive either drug. For each of these subjects, $q = 0.20$.

(ii) Drug A will be given to 1000 subjects comprising Cohort A.

- With probability 80%, each subject in Cohort A will have $q = 0.20$.
- With probability 20%, each subject in Cohort A will have $q = 0.05$.
- Conditional on *q*, the lives have independent future lifetimes.

(iii) Drug B will be given to 1000 subjects with independent future lifetimes, comprising Cohort B. The drug will affect different subjects differently and independently.

- With probability 80%, any given subject in Cohort B will have $q = 0.20$.
- With probability 20%, any given subject in Cohort B will have $q = 0.05$.

(a) (*1 point*) Calculate the mean and variance of the number of deaths in the control group.

(b) (*2 points*) Calculate the mean and variance of the number of deaths in Cohort A.

(c) (*2 points*) Calculate the mean and variance of the number of deaths in Cohort B.

(This is a modified version of Question 3 from the Spring 2014 Written Answer exam.)

WA.4. (*8 points*) An insurer issues fully discrete whole life insurance policies to 10,000 lives, each age 45, with independent future lifetimes.

The death benefit for each policy is 100,000. Gross premiums are determined using the equivalence principle. You are given the following information:

	Pricing and Reserve Assumptions	Policy Year 1 Actual Experience	Policy Year 2 Actual Experience
Interest	5%	7%	Same as pricing
Expense at Start of the Year	75% of premium + 100 per policy in the first year; 10% of premium + 20 per policy thereafter	75% of premium + 105 per policy	Same as pricing
Settlement Expense	200 per policy	Same as pricing	220 per policy
Mortality	Standard Ultimate Life Table	Same as pricing	10 deaths

(a) (*2 points*) Show that the gross premium for each policy is 1020 to the nearest 10. You should calculate the premium to the nearest 1.

(b) (*2 points*) Calculate the gross premium reserve for a policy inforce at the end of policy year 1.

(c) (*3 points*) For each of interest, expense and mortality, in that order, calculate the gain or loss by source in policy year 1 on this block of policies.

(d) (*1 point*) Explain the sources and direction of any gains or losses in policy year 2. Exact values are not necessary.

(This is a modified version of Question 4 from the Spring 2014 Written Answer exam.)

WA.5. (*9 points*) There are two whole life insurances of 100,000 on (35) with death benefits payable at the end of the year of death. Policy A has annual premiums and Policy Q has quarterly premiums.

You are given:

	Policy A	Policy Q
Premiums	587 per year	150 per quarter
Commissions on initial premium	100	100
Commissions on subsequent premiums in first 10 years	59 per year	16 per quarter
Commissions on subsequent premiums after 10 years	0 per year	0 per quarter
The symbol for gross premium reserve at time *t*	${}_tV^{\text{A}}$	${}_tV^{\text{Q}}$
The symbol for the loss random variable at time 0	${}_0L^{\text{A}}$	${}_0L^{\text{Q}}$

You are also given:

- Mortality follows the Standard Ultimate Life Table.
- $i = 0.05$
- ${}_4V^{\text{A}} = 984$
- ${}_4V^{\text{Q}} = 988$
- ${}_5V^{\text{A}} = 1539$
- $\ddot{a}^{(4)}_{40:\overline{5}|} = 4.4590$
- The Woolhouse formula with two terms is used to calculate annuity values.

(a) (*2 points*) Calculate ${}_0L^{\text{Q}}$ when $T_{35} = 0.30$.

(b) (*3 points*) Calculate $E\left[{}_0L^{\text{Q}}\right]$.

(c) (*2 points*) Show that ${}_5V^{\text{Q}}$ is 1540, to the nearest 10. You should calculate the value to the nearest 1.

(d) (*2 points*) Sketch the graph of ${}_tV^{\text{A}} - {}_tV^{\text{Q}}$ for $4 \le t \le 5$. An exact calculation of the difference at intermediate times is not expected. Label the scales on the *x* (time) and *y* (reserve difference) axes.

(This is a modified version of Question 5 from the Spring 2014 Written Answer exam.)

WA.6. (*7 points*)

(a) (*1 point*) List three reasons why employers sponsor pensions for their employees.

A benefit plan provides a retirement benefit if the employee lives to age 65, and a death benefit if the employee dies prior to age 65.

- The retirement benefit is an annual whole life annuity-due of 3% of the final 3-year average salary for each year of service.
- The death benefit is a lump sum payable at the end of the year of death equal to two times the employee's annual salary in the year of death.

You are given:

- Chris started her employment on January 2, 1990 at exact age 38 with a starting salary of 50,000.
- The company gives salary increases of 3% on January 1 each year.
- Employees can terminate employment only by retirement at 65 or death.

You are given the following:

- $q_{62+k} = 0.08 + 0.01k$ for $k = 0,1,2,3$
- The expected present value on January 1, 2017 of an annuity-due of 1 per year payable annually to Chris, if she survives, will be 4.7491.
- $i = 0.04$

(b) (*3 points*) Calculate the actuarial present value on January 1, 2014 of Chris' death benefit.

(c) (*3 points*) Calculate the actuarial present value on January 1, 2014 of Chris' retirement benefit.

(This is a modified version of Question 7 from the Spring 2014 Written Answer exam.)

WA.7. (*11 points*) For a fully discrete whole life insurance of 100,000 on (35), you are given:

(i)

Policy Year	Commission Rate	Per Policy Expenses
1	100%	100
2-10	10%	8
11 and later	0	8

(ii) Per policy expenses are incurred at the beginning of each policy year.

(iii) Mortality follows the Standard Ultimate Life Table.

(iv) $i = 0.05$

(v) G is the gross premium.

(vi) G is calculated such that the actuarial present value of gross premiums is 110% of the actuarial present value of benefits, commissions, and per policy expenses.

(vii) C is the random variable for the present value at issue of commissions.

(a) (*2 points*) $E[C]$ can be written in the form kG. Calculate k.

(b) (*1 point*) Calculate the expected present value at issue of per policy expenses.

(c) (*2 points*) Show that G is 640, rounded to the nearest 10. You should calculate G to the nearest 1.

(d) (*3 points*) Calculate $\text{Var}(C)$.

The company charges a premium of 57 per month for a policyholder who wishes to pay monthly. Commission rates during renewal policy years are as given above for annual premium policies.

C^* is the present value at issue of commissions for a policy paying premiums monthly.

WA.7. Continued

(e) (*3 points*) Using the 2-term Woolhouse formula, calculate the commission rate during the first policy year for a policy paying premiums monthly, so that $\mathrm{E}[C^*] = \mathrm{E}[C]$.

(This is a modified version of Question 1 from the Fall 2014 Written Answer exam.)

WA.8. (*9 points*) XYZ Company sponsors a defined benefit pension plan. Key attributes of the pension plan are:

- Benefits are payable as a single life annuity, paid at the beginning of each month.
- The annual benefit payable at age 65 is calculated as 2% of the final 3-year average salary up to 100,000 multiplied by years of service, plus 3% of the final 3-year average salary over 100,000 multiplied by years of service. The monthly benefit is the annual benefit divided by 12.
- You are given:

  (i) $i = 5\%$

  (ii) $A_{65}^{(12)} = 0.470$

Tom joined XYZ Company on January 1, 2009 at exact age 40. He is planning to retire on January 1, 2034 at age 65. Tom's salary for 2014 is 80,000 and he receives annual raises of 4% each January 1.

(a) (*2 points*) Assuming Tom works until age 65, show that the monthly accrued benefit at age 65 is 8050, to the nearest 10. You should calculate the benefit to the nearest 1.

(b) (*1 point*) Calculate the replacement ratio at age 65.

(c) (*2 points*) Calculate the expected present value of the benefit at age 65.

In addition to accruing a benefit in the defined benefit plan, Tom will contribute a constant percentage of his annual salary to a defined contribution plan.

You are also given:

(iii) Contributions are made at the start of each year, beginning on January 1, 2014.

(iv) Contributions earn an investment return of 7% per year.

(v) The contribution balance is converted to a monthly life annuity using assumptions (i) and (ii) above.

(d) (*4 points*) Calculate the percentage of each year's salary that Tom needs to contribute so that his total replacement ratio is 80% at age 65.

(This is Question 2 from the Fall 2014 Written Answer exam.)

WA.9. (*6 points*) For (x) and (y) with independent future lifetimes, you are given that $q_x = 0.2$ and $q_y = 0.1$.

(a) (*1 point*) Explain in words the meaning of the probability described by the symbol q_{xy}.

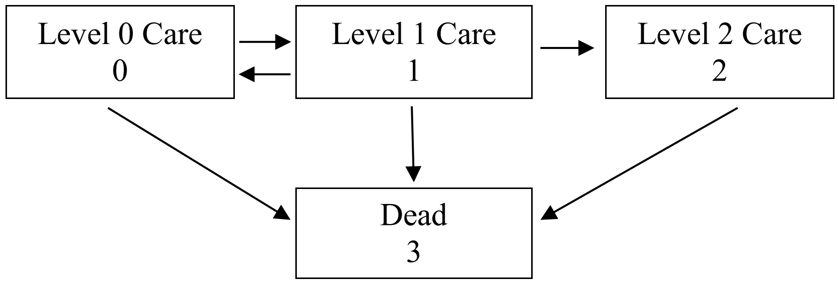
You are also given that mortality within integral ages follows a uniform distribution of deaths assumption for each of (x) and (y) individually.

(b) (*2 points*) Sketch the graph of ${}_s p_x$ as a function of s for $0 \le s \le 1$. You should mark numerical values on each axis.

(c) (*3 points*) Show that ${}_s q_{xy} = s q_{xy} + g(s) q_{\overline{xy}}$ for $0 \le s \le 1$, where $g(s)$ is a function of s that you should specify.

(This is Question 3 from the Fall 2014 Written Answer exam.)

WA.10. (*8 points*) A long-term care provider offers three care levels. Transitions are modeled as a Markov multiple-state model. Transitions and states are shown in the following diagram:



(a) (*1 point*) Write down Kolmogorov's forward differential equations with the associated boundary conditions for this model for:

(i) ${}_t p_x^{10}$

(ii) ${}_t p_x^{11}$

(b) (*3 points*) Estimate ${}_1 p_{80}^{10}$ using Euler's forward method, a step size of $h = \frac{1}{3}$, and the transition intensities and probabilities in the table below:

t	${}_t p_{80}^{11}$	μ_{80+t}^{01}	μ_{80+t}^{03}	μ_{80+t}^{10}	μ_{80+t}^{12}	μ_{80+t}^{13}	μ_{80+t}^{23}
0	1.00000	0.10000	0.02981	0.08000	0.15000	0.05962	0.11924
⅓	0.90346	0.10000	0.03082	0.08000	0.15000	0.06164	0.12328
⅔	0.81652	0.10000	0.03186	0.08000	0.15000	0.06373	0.12746
1	--	0.10000	0.03294	0.08000	0.15000	0.06589	0.13178

(c) (*4 points*) You are given the following annuity values at 5%:

x	$\bar{a}_x^{00}$	$\bar{a}_x^{01}$	$\bar{a}_x^{02}$	$\bar{a}_x^{11}$	$\bar{a}_x^{10}$	$\bar{a}_x^{12}$	$\bar{a}_x^{22}$
80	5.5793	1.3813	0.6109	3.0936	1.6719	1.7206	4.4712
85	4.8066	1.0396	0.3403	2.6723	0.8834	1.0883	3.2367

WA.10. Continued

You are also given:

- ${}_5p_{80}^{00} = 0.53880$ ${}_5p_{80}^{01} = 0.17327$ ${}_5p_{80}^{02} = 0.06956$

- $i = 0.05$

- Residents pay a service fee of 8000 per year continuously while in Level 0 or Level 1 Care.

- Level 2 Care costs are paid at a continuous rate of 30,000 per year for lives age 80 to 85, and 40,000 per year for lives older than 85.

Ada, who is age 80, is currently in Level 0 Care.

(i) Calculate the expected present value of Ada's future Level 0 and Level 1 service fees.

(ii) Calculate the expected present value of Ada's future Level 2 Care costs.

(This is Question 4 from the Fall 2014 Written Answer exam.)

WA.11. (*11 points*) For a special 3-year term life insurance issued to (50) with a premium refund feature, you are given:

(i) The death benefit is 100,000.

(ii) The premium refund feature refunds the last premium payment, without interest, at the end of the 3-year term if the insured is still alive.

(iii) The mortality rates are:

x	q_x
50	0.00592
51	0.00642
52	0.00697

(iv) Pre-contract expenses are 155.

(v) Commissions are 5% of each premium.

(vi) The hurdle rate is 14%.

(vii) The reserves of this policy have been set to:

t	${}_tV$
0	0
1	400
2	800

(viii) The annual premium for this policy is 1100.

(ix) The earned interest rates are:

Year 1	Year 2	Year 3
0.01	0.02	0.03

(a) (*1 point*) Show that the expected profit in policy year 2 for a policy in force at the start of year 2 is 37 to the nearest 1. You should calculate your answer to the nearest 0.01.

(b) (*4 points*) Calculate the profit vector of this policy.

(c) (*3 points*) Calculate the profit signature and Net Present Value (NPV) of this policy.

WA.11. Continued

(d) (*3 points*) Rank from low to high the Internal Rate of Return (IRR) of the following products, explaining your order.

Product A: The special 3-year term life insurance described above.

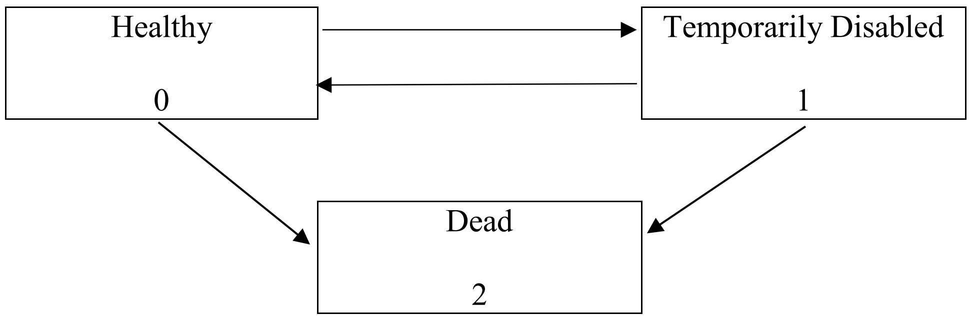
Product B: A 3-year term life insurance policy with the following profit signature: $[-155, 0, 0, 210]$

Product C: The same special 3-year term life insurance as Product A, except that the reserves of the product have been set to:

t	${}_tV$
0	0
1	300
2	800

(This is a modified version of Question 5 from the Fall 2014 Written Answer exam.)

WA.12. (*9 points*) You are using the following 3-state Markov model to price a 10-year disability insurance product.



(a) (*2 points*) Show that for this model, $\sum_{j=0}^{2} \bar{a}^{0j}_{x:\overline{10}|} = \bar{a}_{\overline{10}|}$.

The product has the following features:

- The product is issued to individuals age x who are in the Healthy state.
- The product pays a continuous disability benefit at a rate of 1000 per year while the insured is in the Temporarily Disabled state.
- The product pays a death benefit of 10,000 at the moment of death.
- Net premiums are payable continuously while the insured is in the Healthy state.

You are also given the following information:

$$\delta = 0.1 \qquad \bar{a}^{00}_{x:\overline{10}|} = 4.49 \qquad \bar{a}^{02}_{x:\overline{10}|} = 1.36 \qquad \bar{A}^{02}_{x:\overline{10}|} = 0.3871$$

$$\mu^{01}_{x+t} = 0.04 \qquad \mu^{02}_{x+t} = 0.02t \qquad \mu^{10}_{x+t} = 0.05 \qquad \mu^{12}_{x+t} = 0.04t$$

(b) (*2 points*) Show that the net premium rate for this policy is 970 per year to the nearest 10. You should calculate the rate to the nearest 1.

Let ${}_tV^{(i)}$ denote the net premium reserve for a policy in state i at time t. You are also given:

$${}_3V^{(0)} = 1304.54 \qquad {}_3V^{(1)} = 7530.09$$

(c) (*2 points*) Calculate $\frac{d}{dt}\,{}_tV^{(0)}$ at $t = 3$.

WA.12. **Continued**

(d) (*3 points*) Your company is considering adding an additional feature to this product. Under this additional feature, the insurer would return the sum of the premiums paid at the end of 10 years without interest if no benefits were paid during the life of the policy.

Calculate the increase in the net premium rate payable continuously for the product as a result of including this feature.

(This is Question 1 from the Spring 2015 Written Answer exam.)

WA.13. (*6 points*) You are given the following excerpt from a triple decrement table:

x	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
60	1000	$d_{60}^{(1)}$	60	45
⋮	⋮	⋮	⋮	⋮

You are also given the following information about the decrements:

- $\mu_{60+t}^{(1)} = 1.2t$ for $0 \le t \le 1$
- Decrement 2 happens exactly halfway through the year.
- Decrement 3 happens at the end of the year.

(a) (*2 points*) Calculate $q_{60}'^{(2)}$.

(b) (*2 points*) Calculate $d_{60}^{(1)}$.

Now suppose instead that Decrement 2 occurs at the start of the year, and that each $q_{60}'^{(i)}$ remains unchanged.

(c) (*2 points*) State with reasons the effect (increase, decrease, no change, cannot be determined) that this change would have on the following probabilities:

(i) $q_{60}^{(1)}$

(ii) $q_{60}^{(2)}$

(iii) $q_{60}^{(3)}$

(This is Question 2 from the Spring 2015 Written Answer exam.)

WA.14. (*8 points*) A life insurance company issues a whole life annuity immediate with annual payments. The annuity is issued to (65). The annuity pays 50,000 at the end of each year for the life of the annuitant with the first payment being made at age 66.

A single premium is paid to purchase this annuity.

You are given:

(i) Mortality follows the Standard Ultimate Life Table.

(ii) $i = 0.05$

(iii) An expense of 100 is incurred to make each annuity payment.

(iv) Issue expense incurred at issue is 3000 per policy.

(v) Commissions are 10% of the single premium.

(a) (*2 points*) Assume that the single gross premium, *G*, is 110% of the expected present value at issue of benefits and expenses. Show that *G* is 780,000 to the nearest 5,000. You should calculate *G* to the nearest 100.

(b) (*2 points*) Using *G* from Part (a), calculate the probability that this policy will generate a profit.

Another life insurance company issued 8000 independent annuities identical to that described above. This company determines the single gross premium per policy, G^P, using the portfolio percentile premium principle such that the probability that the present value of the loss at issue on the portfolio is negative is 90%.

(c) (*3 points*) Show that G^P is 700,000 to the nearest 10,000. You should calculate G^P to the nearest 100.

(d) (*1 point*) G^P is calculated so that the portfolio of annuities has a 90% chance of generating a profit. *G* is such that the probability of an individual policy having a profit is less than 90%.

Explain why G^P is less than *G*.

(This is a modified version of Question 3 from the Spring 2015 Written Answer exam.)

WA.15. (*9 points*) For a three-year special decreasing term insurance policy on (x), the death benefit is paid at the end of the year of death. The level annual net premium is P.

You are given:

(i) $d = 0.10$

(ii) $q_{x+k} = 0.1\left(2^k\right), k = 0,1,2$

(iii) ${}_1V^n = 8147.08$ and ${}_2V^n = 12,480.86$, are the net premium reserves at the end of year 1 and year 2, respectively.

(iv) The death benefit during the first year is 180,000.

(a) (*2 points*) Show that P is 23,000 to the nearest 500. You should calculate P to the nearest 1.

(b) (*2 points*) Show that the death benefits, to the nearest 5000, during the second and third year, are 120,000 and 100,000, respectively. You should calculate the death benefits to the nearest 1000.

You calculate reserves for this policy using the Full Preliminary Term (FPT) reserve method.

(c) (*3 points*)

(i) Calculate the net premium for the first year using the FPT reserve method.

(ii) Calculate the net premium for years 2 and 3 using the FPT reserve method.

(d) (*1 point*) Calculate the reserve at the end of the second year using the FPT reserve method.

(e) (*1 point*) FPT is a modified net premium reserve method. Explain the purpose of modified net premium reserve methods.

(This is Question 4 from the Spring 2015 Written Answer exam.)

WA.16. (*6 points*) Susie begins work at age 40 at ABC Life on January 1, 2014, with a starting salary of 30,000. She will switch jobs to XYZ Re at some time before age 55 at her then-current salary and will remain at XYZ Re until retirement.

- Both companies offer 2% annual salary raises on January 1 of each year.
- The annual retirement benefit at ABC Life is 900 per year of service.
- The annual retirement benefit at XYZ Re is 3% of the final 3-year average salary for each year of service.

Susie will retire on her 65th birthday. She will receive retirement benefits from both companies.

(a) (*2 points*) Assume Susie stays at ABC Life for 9.5 years and then switches to XYZ Re. Show that the replacement ratio would be 63% to the nearest 1%. You should calculate the replacement ratio to the nearest 0.1%.

(b) (*1 point*) Calculate the maximum length of time that Susie can remain at ABC Life and still attain a replacement ratio of at least 65%.

Susie switches to XYZ Re after seven years on January 1, 2021 (and gets the annual raise). Later, on January 1, 2029, XYZ Re decides to stop all benefit accruals. No further benefits accrue after this date.

On January 1, 2029, Susie purchases a 10-year deferred whole life annuity due, with premiums payable annually during the deferred period. The annuity payments, combined with her accrued retirement benefits, will give her a replacement ratio of 65%.

You are given:

- Susie's mortality follows the Standard Ultimate Life Table
- $i = 5\%$

(c) (*3 points*) Calculate the annual net premium for the annuity.

(This is a modified version of Question 7 from the Spring 2015 Written Answer exam.)

WA.17. (*7 points*) You are given the following survival function for (40):

$$S_{40}(t) = \begin{cases} 1-(0.02t)^2, & 0 \le t < 25 \\ 0.75e^{b(t-25)}, & t \ge 25 \end{cases}$$

(a) (*3 points*)

(i) List three necessary and sufficient conditions that a survival function for a lifetime distribution must satisfy in order to be valid.

(ii) State with reasons whether each of the following values for b will result in $S_{40}(t)$ being valid.

- $b = -0.2$
- $b = \ \ 0.0$
- $b = \ \ 0.2$

(b) (*4 points*) You are given that $b = -0.1$.

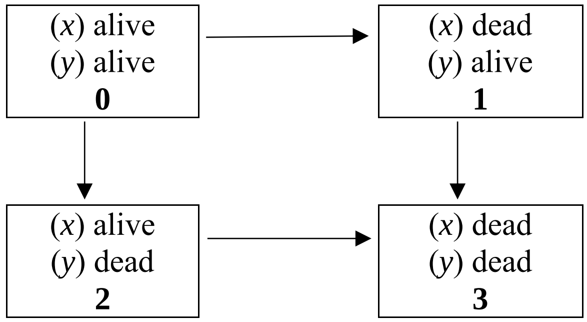
(i) Calculate μ_{60}.

(ii) Calculate μ_{70}.

(iii) Calculate $\mathring{e}_{40:\overline{35}|}$.

(This is Question 1 from the Fall 2015 Written Answer exam.)

WA.18. (*11 points*) The mortality of a couple, (x) and (y), is modeled using the Markov multiple-state model described in the following diagram:



You are given:

- $\mu^{01}_{x+t:y+t} = A + Bc^{x+t}$ $\mu^{02}_{x+t:y+t} = A + Bc^{y+t}$ $\mu^{13}_{y+t} = D + Bc^{y+t}$ $\mu^{23}_{x+t} = E + Bc^{x+t}$
- $A = 0.0001$ $B = 10^{-5}$ $c = 1.12$ $D = 0.00015$ $E = 0.0002$

(a) (*1 point*) State with reasons whether (x) and (y) have independent future lifetimes under this model.

(b) (*3 points*)

(i) Write down the Kolmogorov forward differential equation for ${}_tp^{00}_{xy}$, and give the associated boundary condition.

(ii) Starting from the equation in (i), prove that

$${}_tp^{00}_{xy} = \exp\left\{-\int_0^t \left(\mu^{01}_{x+r:y+r} + \mu^{02}_{x+r:y+r}\right) dr\right\}.$$

A couple, who are ages $x = 50$ and $y = 55$, purchases a special single premium, deferred joint and last survivor annuity. The annuity will pay 50,000 per year while both are alive, and will pay 30,000 per year while only one is alive. Payments are continuous, and begin 10 years after the annuity purchase.

If neither life survives the deferred period, then a sum of 3 times the single premium is paid immediately on the second death.

There are no other benefits.

WA.18. **Continued**

You are given:

(i) ${}_{10}p^{00}_{50:55} = 0.86041 \quad {}_{10}p^{01}_{50:55} = 0.04835 \quad {}_{10}p^{02}_{50:55} = 0.08628$

(ii) $\bar{A}^{03}_{50:55:\overline{10}|} = 0.003421$

(iii) $\bar{a}^{00}_{60:65} = 8.8219 \quad \bar{a}^{01}_{60:65} = 1.3768 \quad \bar{a}^{02}_{60:65} = 3.0175$

(iv) $\bar{a}^{11}_{65} = 10.1948 \quad \bar{a}^{22}_{60} = 11.8302$

(v) $i = 0.05$

(c) (*3 points*)

(i) Show that the expected present value of the future benefits at time 10, if both lives survive to time 10, is 573,000 to the nearest 1000. You should calculate the value to the nearest 100.

(ii) Calculate the single net premium for the policy.

(d) (*4 points*)

(i) Determine ${}_{10}V^{(0)}$, ${}_{10}V^{(1)}$ and ${}_{10}V^{(2)}$ for the policy.

(ii) Write down Thiele's differential equation for the reserve at t, $t \geq 10$, assuming both lives survive to t.

(iii) Using Euler's forward method, with a step size of $h = 0.5$, calculate the reserve required at 10.5 assuming both lives are alive at that time.

(This is Question 2 from the Fall 2015 Written Answer exam.)

WA.19. (*10 points*) For a special fully discrete two-year term insurance on Elizabeth, age 60, you are given:

(i) The death benefit is 1000 plus the return of gross premiums paid with interest at 6%.

(ii) The following double decrement table, where decrement (d) is death and decrement (w) is withdrawal:

x	$q_x^{(d)}$	$q_x^{(w)}$
60	0.06	0.04
61	0.12	0.00

(iii) There are no withdrawal benefits.

(iv) $i = 0.06$

(v) G denotes the annual gross premium.

(vi) L_0 denotes the insurer's loss at issue random variable for Elizabeth's policy.

(a) (*4 points*) Calculate the values in the following table. Express L_0 in terms of G where appropriate:

Event	Value of L_0, Given that the Event Occurred	Probability of Event
Death in year 1		
Withdrawal in year 1		
Death in year 2		
Neither death or withdrawal		

WA.19. Continued

(b) (*4 points*)

(i) Show that $E\left[L_0\right] = a - b\,G$, where a is 150 to the nearest 10 and b is 1.58 to the nearest 0.01. You should calculate a to the nearest 1 and b to the nearest 0.001.

(ii) Show that $Var\left(L_0\right) = c\,G^2 + d\,G + e$, where c is 0.5 to the nearest 0.1, d is 480 to the nearest 10, and e is 116,000 to the nearest 1000. You should calculate c to the nearest 0.01, d to the nearest 1, and e to the nearest 100.

(c) (*2 points*) The insurer expects to issue 200 such policies to insureds with independent future lifetimes. The premium for each policy is $G = 130$.

Let L_{agg} denote the insurer's aggregate future loss random variable at issue for these 200 policies.

Calculate $\Pr\left(L_{agg} > 0\right)$ using the normal approximation without continuity correction.

(This is Question 4 from the Fall 2015 Written Answer exam.)

WA.20. (*6 points*) ABC Life Insurance Company sells 5-year pure endowment policies of 1 to 100 independent lives age 85. You are given:

(i) Mortality follows the Standard Ultimate Life Table.

(ii) There is a 60% chance that the force of interest over the next five years will be $\delta_t = 0.03\sqrt{t}$, for $0 \le t \le 5$, and there is a 40% chance that the force of interest over the next five years will be $\delta_t = 0.02$, for $0 \le t \le 5$.

(iii) The present value of benefits random variable for the portfolio of policies is denoted by Y.

(a) (*3 points*) Show that the mean of Y is 60 to the nearest 10. You should calculate the value to the nearest 1.

(b) (*3 points*) Calculate the probability that Y is less than 50 using the normal approximation without continuity correction.

(This is a modified version of Question 6 from the Fall 2015 Written Answer exam.)

WA.21. *(8 points)* On December 31, 2015, Nancy, who is age 55, has 25 years of service. Her salary in 2015 was 50,000.

The annual accrued benefit as of any date is 1.6% of the three-year final average salary as of that date, multiplied by years of service as of that date. The pension is payable only to retired participants as a monthly single life annuity, with the first payment due at retirement.

The valuation assumptions are as follows:

- There are no benefits paid upon death.
- There are no benefits paid upon disability prior to age 60.
- Exits from employment follow the Standard Service Table, except that all lives surviving in employment to age 61 retire at that time.
- There are retirements at age 60 that occur at exact age 60 while other decrements at age 60 occur throughout the year after the retirement at exact age 60 and prior to exact age 61. Those that retire or become disabled between exact age 60 and exact age 61 are assumed to retire at exact age 61.
- After retirement, mortality follows the Standard Ultimate Life Table.
- Annuities are valued using the 2-term Woolhouse formula.
- Nancy has received a 3% salary increase on January 1 of each year for the last five years.
- Future salaries are expected to increase by 3% each year on January 1.
- All contributions are paid on January 1 each year.
- $i = 0.05$

(a) *(1 point)* A pension plan may be classified as a Defined Contribution plan or a Defined Benefit plan. State which type of plan Nancy has and briefly describe the other type of plan.

(b) *(3 points)*

(i) Show that the actuarial accrued liability for Nancy at December 31, 2015 using the Projected Unit Credit (PUC) funding method is 220,000 to the nearest 1000. Calculate the actuarial accrued liability to the nearest 1.

(ii) Calculate the normal cost for 2016 using PUC.

(c) *(3 points)*

(i) Show that the actuarial accrued liability for Nancy at December 31, 2015 using the Traditional Unit Credit (TUC) funding method is 186,000 to the nearest 1000. Calculate the actuarial accrued liability to the nearest 1.

(ii) Calculate the normal cost for 2016 using TUC.

WA.21. **Continued**

(d) *(1 point)* For Nancy, the normal cost during 2016 using PUC is less than the normal cost during 2016 using TUC. Explain why this will be true for all employees near retirement.

WA.22. *(7 points)* On 1/1/2009, Sheila purchased a 10-year hybrid long term care and life insurance product. The product has the following features:

(i) The product uses a reimbursement approach where the caregiver is reimbursed up to a maximum rate of 2000 per month when the insured requires assistance to perform 2 or more activities of daily living (ADLs), of the 6 commonly used ADLs.

(ii) The product has a 3 month waiting period and a 6 month off period.

(iii) The sum insured for the life insurance portion of the product is 100,000.

(iv) Premiums are payable at a continuous rate of 150 per month while the insured is active (able to perform at least 5 of 6 ADLs).

Sheila's experience during the term of this policy was as follows:

- From 1/1/2012 to 6/30/2012, Sheila was only able to perform 4 ADLs, and required outpatient care costing 1000 per month.
- From 3/31/2013 to 11/30/2013, Sheila was only able to perform 3 ADLs, and required inpatient care costing 2500 per month.
- For the remainder of the time from 1/1/2009 through 6/30/2018, Sheila was able to perform at least 5 ADLs.
- Sheila died on 7/1/2018.

(a) *(1 point)* List the six ADLs in common use for long term care products.

(b) *(2 points)* Draw a sketch of a Markov model that could be used to model this hybrid insurance product.

(c) *(2 points)* Calculate the amount of the death benefit payable upon Sheila's death, assuming:

  (i) The long term care and life insurance benefits are combined using the "return of premium" approach.

  (ii) The long term care and life insurance benefits are combined using the "accelerated benefit" approach.

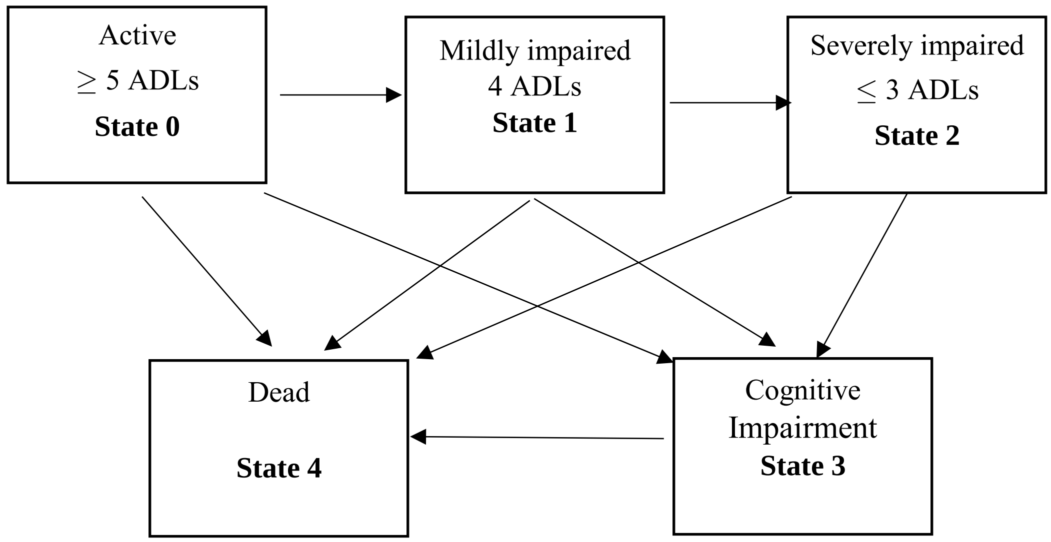
Now suppose that instead of using a 6 month off period, the insurer uses a 12 month off period.

(d) *(2 points)* Calculate the change in total benefits paid as a result of this change, assuming:

  (i) The long term care and life insurance benefits are combined using the "return of premium" approach.

  (ii) The long term care and life insurance benefits are combined using the "accelerated benefit" approach.

WA.23. *(10 points)* You are using the following multiple state Markov model to analyze a long term care product which provides coverage until the death of the insured. Assume all relevant transition intensities exist.



You are given the following information:

(i) The insurance pays a benefit of 3000 per year, payable continuously, while the policyholder is in State 2.

(ii) Policyholders may opt to have benefits start immediately on transition into State 2 or, for a lower premium, may select a 6-month waiting period before benefits begin.

(iii) Benefits are valued at a force of interest of $\delta = 0.05$

(iv) Transition intensities for all ages $x \geq 90$ are

$$\mu_x^{12} = 0.10 \quad \mu_x^{13} = 0.04 \quad \mu_x^{14} = 0.10$$
$$\mu_x^{23} = 0.04 \quad \mu_x^{24} = 0.30 \quad \mu_x^{34} = 0.20$$

(a) *(2 points)* Derive the Kolmogorov forward differential equation for ${}_tp_x^{12}$.

(b) *(1 point)* Show that ${}_tp_{90}^{12} = e^{-0.24t} - e^{-0.34t}$

(c) *(2 points)* Calculate the expected present value of the LTC benefit in State 2 for a life currently age 90 and in State 1, assuming benefits begin immediately on transition.

(d) *(1 point)* $\bar{a}_{90:\overline{0.5|}}^{\overline{22}} = 0.45$ to the nearest 0.01. Calculate $\bar{a}_{90:\overline{0.5|}}^{\overline{22}}$ to the nearest 0.001.

WA.23. **Continued**

(e) *(2 points)* Calculate the expected present value of the LTC benefit in State 2 for a life currently age 90 and in State 1, assuming a waiting period of 6 months between transition to State 2 and the start of the benefit payments.

(f) *(2 points)* Suggest two reasons why an insurer uses waiting periods in long term health products.

WA.24. *(9 points)* As part of a mortality study, ten newborns were observed until each either died or withdrew from the study. The table below shows the age at death or withdrawal of each individual.

Individual	Age at Death	Age at Withdrawal
1	12	--
2	35	--
3	--	55
4	59	--
5	--	61
6	--	66
7	73	--
8	73	--
9	80	--
10	--	90

(a) *(3 points)* Show that the Kaplan-Meier estimate of $S(80)$ is 0.2 to the nearest 0.1. You should calculate the value to the nearest 0.001.

(b) *(4 points)*

(i) Construct an approximate 95% linear confidence interval for the Kaplan-Meier estimate of $S(80)$, using a normal approximation and Greenwood's approximation to the variance of this estimator.

(ii) Construct an approximate 95% log-transformed confidence interval for the Kaplan-Meier estimate of $S(80)$.

(iii) Explain briefly why the log-transformed interval is a better estimate in this case.

(c) *(2 points)* Let $\hat{S}(y)$ denote the Kaplan Meier estimate of $S(y)$ for $y \geq 80$, using Brown, Hollander and Korwar's tail correction, and let $\hat{S}^*(y)$ denote the Kaplan-Meier estimate of $S(y)$ for $y \geq 80$ using Efron's tail correction.

Sketch a graph of $\hat{S}(y) - \hat{S}^*(y)$ for $y \geq 80$. Clearly label the axes and show all key values.

WA.25. *(10 points)* You are modelling the mortality improvement in a group of annuity holders with an average age of 75. You are using the Cairns-Blake-Dowd (CBD) model specified below to describe the logit of the mortality rate:

$$lq(x,t) = K_t^{(1)} + K_t^{(2)}(x - \bar{x})$$
$$K_t^{(1)} = K_{t-1}^{(1)} + c^{(1)} + \sigma_{k_1} Z_t^{(1)}$$
$$K_t^{(2)} = K_{t-1}^{(2)} + c^{(2)} + \sigma_{k_2} Z_t^{(2)}$$

Where $Z_t^{(1)}$ and $Z_t^{(2)}$ are standard normal random variables with correlation $\rho = 0.3$ within a given year, but are independent from year to year. The fitted parameters from the study are:

$$K_{2018}^{(1)} = -1.9 \quad K_{2018}^{(2)} = 0.3 \quad c^{(1)} = -0.03 \quad c^{(2)} = 0.001 \quad \sigma_{k_1} = \sigma_{k_2} = 0.02$$

(a) *(1 point)* Describe one advantage of using a stochastic mortality improvement model over using a deterministic model when modelling a block of policies such as annuities.

(b) *(4 points)*

(i) Show that the mean of $lq(72, 2019)$ is -2.8 to the nearest 0.1. You should calculate the value to the nearest 0.001.

(ii) Show that the standard deviation of $lq(72, 2109)$ is 0.06 to the nearest 0.01. You should calculate the value to the nearest 0.001.

(c) *(2 point)* Calculate $\mathrm{E}\left[\dfrac{q(72,2019)}{1-q(72,2019)}\right]$.

(d) *(2 points)* Calculate $\Pr\left[p(72,2019) \geq 0.94\right]$.

(e) *(1 point)* Your colleague suggests that you consider using the CBD M7 model rather than the original CBD model to describe the mortality improvement of this block of policyholders. Describe one advantage of using the CBD M7 model rather than the original CBD model.

WA.26. *(12 points)* Diana, who is now 65 years old, is entering a Continuing Care Retirement Community (CCRC) under a Full Lifecare (Type A) contract. She will move into an Independent Living Unit (ILU). Diana pays a one-time fee of F immediately on entry and a level monthly fee of M at the start of each month that she is in the CCRC, including the first.

You are given:

(i) The entry fee, F, is equal to 25% of the expected present value of all future costs.

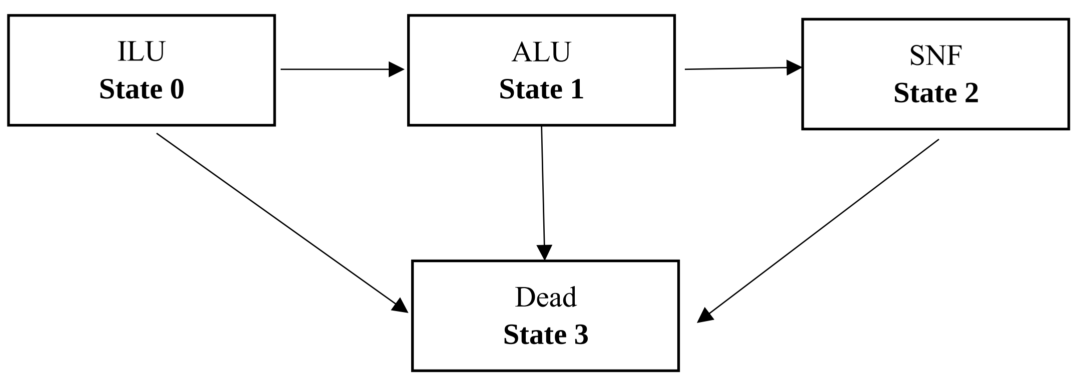
(ii) The CCRC operates three types of accommodation; they are listed here, with the monthly costs incurred at the beginning of the month by the CCRC for each resident in each category:

Independent Living Unit (ILU):	3,000
Assisted Living Unit (ALU):	7,500
Specialized Nursing Facility (SNF):	15,000

(iii) $i = 0.05$

(iv) The monthly fee M is determined so that the expected present value of the monthly costs is equal to the expected present value of M plus the expected present value of F.

(v) The CCRC uses the following multiple state model to determine the fee structure.



(vi) The following actuarial functions have been evaluated for the model at $i = 0.05$

x	$\ddot{a}_x^{(12)00}$	$\ddot{a}_x^{(12)01}$	$\ddot{a}_x^{(12)02}$	$\ddot{a}_x^{(12)11}$	$\ddot{a}_x^{(12)12}$	$\ddot{a}_x^{(12)22}$	$A_x^{(12)03}$
65	11.4106	1.3570	0.3745	11.8352	0.7979	10.6905	0.3601
70	9.5210	1.7037	0.4942	10.1754	0.9960	9.1961	0.4294

WA.26. **Continued**

(a) (3 points)

(i) You are given that *F* is 150,000 to the nearest 1000. Calculate *F* to the nearest 10.

(ii) Calculate *M*.

(b) (3 points)

(i) Calculate ${}_5V^{(0)}$, the reserve five years after entry, assuming Diana is in state 0.

(ii) Calculate ${}_5V^{(1)}$, the reserve five years after entry, assuming Diana is in state 1.

(c) *(2 points)* You are given that

${}_{\frac{1}{12}}p^{00}_{69\frac{11}{12}} = 0.94937$ ${}_{\frac{1}{12}}p^{01}_{69\frac{11}{12}} = 0.00906$ ${}_{\frac{1}{12}}p^{02}_{69\frac{11}{12}} = 0.00003$

Calculate ${}_{4\frac{11}{12}}V^{(0)}$.

The CCRC introduces an option under which 50% of the initial fee is refunded at the end of the month of death. The revised entry fee, *F*, will be equal to 25% of the expected present value of all future costs plus the refund. The revised monthly fee M is determined so that the expected present value of the monthly costs plus the expected present value of the refund benefit are equal to the expected present value of M plus the expected present value of F.

(d) *(4 points)*

(i) Calculate the revised entry fee assuming Diana selects the refund option.

(ii) Calculate the revised monthly fee assuming Diana selects this option.

(iii) State with reasons whether ${}_5V^{(0)}$ will increase, decrease or stay the same under the 50% refund contract.