Long-Term Actuarial Mathematics

Solutions to Sample Written Answer Questions

February 11, 2020

There are 26 sample written answer questions in this Study Note for the Long-Term Actuarial Mathematics exam. Unlike the Multiple Choice questions, these have not been sorted by source material as most cover multiple chapters of the text or both the text and the study notes.

The first 20 questions are from the 2014 and 2015 MLC exams. Questions that have been modified have been modified to:

- Replace the Illustrative Life Table (ILT) which was used on the MLC exam with the Standard Ultimate Life Table (SULT) which will be used with the LTAM exam. All problems that previously used the ILT have been converted to the SULT.
- Remove portions of questions which covered material that is no longer covered by the Long-Term Actuarial Mathematics exam.

The solutions to these questions include comments from the graders concerning candidates' performance on the exam. Such comments are not included for the other questions.

There are five sample written questions which primarily cover the material that has been added to the Long-Term Actuarial Mathematics exam.

Different solutions show different levels of accuracy in intermediate results. These model solutions are not intended to imply that this is the best rounding for each question. Graders do not penalize rounding decisions, unless an answer is rounded to too few digits in the context of the problem and the given information. In particular, if a problem in one step asks you to calculate something to the nearest 1, and you calculate it as (for example) 823.18, you need not bother saying "that's 823 to the nearest 1", and you may use 823.18 or 823 in future steps.

Versions:

July 2, 2018	Original Set of Questions Published
July 11,2018	Correction to questions WA.21 and WA.22
August 10, 2018	Correction to question WA.20, WA.23, and WA.25
October 1, 2018	Correction to question WA.12
October 13, 2018	Correction to questions WA.6 and WA.13
July 31, 2019	Correct minor typos in solution to WA.2, WA.18 and WA.25
February 11, 2020	Correct minor typos in solution to WA.2

WA.1.

(a) EPV Death Benefit:

$$20,000\,{}_{20}E_{40}\,A_{60} = (0.36663)(0.29028) = 2128.5$$

Commentary:
Almost all candidates earned full marks for this part of the question.

(b) EPV Premium of P per year:

$$P\ddot{a}_{40}^{(12)} \approx P\left(\ddot{a}_{40} - \frac{11}{24}\right) = P\left(18.4578 - 0.45833\right) = 17.9995P$$

which gives the annual premium of $P = \dfrac{2128.5}{17.9995} = 118.25$

Commentary:
This part was also done well by most candidates. Some candidates used the 3-term Woolhouse formula rather than the 2-term. No points were deducted for doing this correctly, but this is a substantially more time-consuming calculation, and many candidates lost points through errors in the formula or calculation. Other candidates used the $\alpha(m)$ *and* $\beta(m)$ *formula based on an assumption of uniform distribution of deaths to calculate the annuity value. This resulted in a small deduction.*

WA.1. Continued

(c)

Reason 1: On average, premiums paid continuously are paid later than premiums paid monthly, leading to a loss of interest income on the premium payments.

This will lead to an increase in the annualized net premium.

Reason 2: In the year of death, on average the total premium received if premiums are continuous will be less than the total for monthly premiums.

This will lead to an increase in the annualized net premium.

Commentary:
Stronger candidates gave good answers to this part. Brief explanations were often better than longer ones. No credit was given for irrelevant comments.
A number of candidates stated that the annualized premium would increase because (i) $\bar{a}_x < \ddot{a}_x^{(12)}$ and (ii) the expected present value of benefits does not change. While this is true, it does not address the reasons for the change, which was what the question asked for. This answer received only partial credit.
Some candidates proposed that continuous premiums meant that the interest and/or mortality rates would change. This is incorrect and received no credit.
A few candidates wrote that continuous premium meant that the death benefits would now be paid at the moment of death. This is incorrect and received no credit.

(d) Let P^* denote the revised premium. Premiums are paid while both (40) and (50) survive. This can be valued as a joint life annuity, so the EPV of premiums is now

$$P^*\left(\ddot{a}_{40:50}^{(12)}\right) \approx P^*\left(\ddot{a}_{40:50} - \frac{11}{24}\right) = P^*\left(16.5558 - 0.45833\right) = 16.0975$$

The EPV of benefits is 2128.5 as above, so the revised premium is

$$P^* = \frac{2128.5}{16.0975} = 132.23$$

which gives an increase in the premium of $P^* - P = 132.23 - 118.25 = 13.98$

WA.1. Continued

Alternative solution:

$$P^*\ddot{a}_{40}^{(12)} = 2128.5 + P^*\ddot{a}_{50|40}^{(12)} = 2128.5 + P^*\left(\ddot{a}_{40}^{(12)} - \ddot{a}_{40:\overline{50|}}^{(12)}\right)$$

$$\Rightarrow P^* = \frac{2128.5}{\ddot{a}_{40:\overline{50|}}^{(12)}} \quad \text{as above}$$

Commentary:
A good proportion of candidates answered this part correctly, but most did not. Most candidates who used reversionary annuities did so incorrectly.
A candidate who made the same mistake twice did not have points deducted twice. For example, a candidate who used the UDD annuity formula in part (b) and in part (d) would have lost some credit in part (b), but would have received full credit in part (d) provided the rest of the calculation was correct.

WA.2.

(a)

$$\frac{d}{dt}\,{}_tp_0^{00} = {}_tp_0^{01}\mu_t^{10} - {}_tp_0^{00}\mu_t^{01}$$
$$\frac{d}{dt}\,{}_tp_0^{01} = {}_tp_0^{00}\mu_t^{01} - {}_tp_0^{01}\left(\mu_t^{10} + \mu_t^{12}\right)$$
$$\frac{d}{dt}\,{}_tp_0^{02} = {}_tp_0^{01}\mu_t^{12}$$

Boundary Conditions:

$${}_0p_0^{00} = 1 \quad {}_0p_0^{01} = 0 \quad {}_0p_0^{02} = 0$$

Commentary:
The question asks for the Kolmogorov forward differential equations as well as boundary conditions. Most candidates were able to give the differential equations, but quite a few did not provide boundary conditions.
For full credit, candidates were expected to use the specific model given in the question, which meant that subscripts and superscripts needed to correspond with the notation of the question for full credit.

(b) (i)

The Euler equation for ${}_{t+h}p_0^{01}$ is:

$${}_{t+h}p_0^{01} = {}_tp_0^{01} + h\left({}_tp_0^{00}\mu_t^{01} - {}_tp_0^{01}\left(\mu_t^{10} + \mu_t^{12}\right)\right)$$

so, using $t = 0, h = 0.5$ gives:

$$\Rightarrow {}_{0.5}p_0^{01} = {}_0p_0^{01} + h\left({}_0p_0^{00}\mu_0^{01} - {}_0p_0^{01}\left(\mu_0^{10} + \mu_0^{12}\right)\right)$$
$$= 0 + 0.5\left((1)(0.5) + (0)\left(0.2 + 2^0\right)\right) = 0.25$$

WA.2. Continued

(ii)

The Euler equation for ${}_{t+h}p_0^{02}$ is

$${}_{t+h}p_0^{02} = {}_{t}p_0^{02} + h\left({}_{t}p_0^{01}\mu_t^{12}\right)$$

so, using $t = 0.5$, $h = 0.5$ and then $t = 0.5, h = 0.5$ gives

$${}_{0.5}p_0^{02} = {}_{0}p_0^{02} + h\left({}_{0}p_0^{01}\mu_0^{12}\right) = 0 + 0.5\left((0)(2^0)\right) = 0$$

and ${}_{1}p_0^{02} = {}_{0.5}p_0^{02} + h\left({}_{0.5}p_0^{01}\mu_{0.5}^{12}\right) = 0 + 0.5\left((0.25)(2^{0.5})\right) = 0.177.$

Commentary:
Candidates were required to use Euler's method for full credit. Many candidates' solutions lacked clarity in this part, which made it difficult to award partial credit for incomplete answers.

(c)

(i) $APV = 1000(v \times {}_{0.5}p_0^{02} + v^2 \times {}_{1}p_0^{02})$ at 4%

$$= 1000\left[(1.04)^{-1}(0) + (1.04)^{-2}(0.177)\right] = 163.64$$

(ii) $P(1 + v \times {}_{0.5}p_0^{00}) = APV = 163.64$

$$\Rightarrow P = \frac{163.64}{1 + (1.04)^{-1}(0.75)} = 95.08$$

Commentary:
This part was done well by most candidates who attempted it. Note that all the probability values required were given in the question, so it was not necessary to answer (b) to answer (c) correctly. Full credit was given for answers using the rounded probability values given in the question.

(d) Use smaller *h for* greater accuracy.

Commentary:
Virtually all candidates who did not omit the question entirely answered this correctly.

WA.3.

(a) Let D^c denote deaths in the control group. Then:

$$D^c \sim Bin(1000, 0.2)$$
$$\Rightarrow E[D^c] = (1000)(0.2) = 200 \quad \text{and } V[D^c] = (1000)(0.20)(0.80) = 160$$

Commentary:
Performance on part (a) was very good, with most candidates receiving full credit.

(b) Let D^A denote deaths in cohort A. Then

$$D^A \mid q \sim Bin(1000, q)$$
$$\Rightarrow E[D^A \mid q] = 1000q \quad \text{and } V[D^A \mid q] = (1000)q(1-q)$$

$$where\ q = \begin{cases} 0.20 & \text{with probability of 0.8} \\ 0.05 & \text{with probability of 0.2} \end{cases}$$

So

$$E[D^A] = E[E[D^A \mid q]] = 0.8 \times 200 + 0.2 \times 50 = 170$$
$$V[D^A] = E[V[D^A \mid q]] + V[E[D^A \mid q]] = E[1000q(1-q)] + V[1000q]$$
$$= 1000(0.17 - 0.0325) + 1000^2(0.0036) = 3737.5$$

(c) Let D_i^B be a Bernoulli indicator function for the death of individual i and D^B denote the deaths in cohort B. Then $D_i^B \mid q$ are independent Bernoulli random variables with parameter q, and $D^B = \sum D_i^B$ so that

$$E[D_i^B] = E[E[D_i^B \mid q]] = E[q] = 0.17$$
$$\Rightarrow E[D^B] = E\left[\sum_{i=1}^{1000} D_i^B\right] = \sum_{i=1}^{1000} E[D_i^B] = (1000)(0.17) = 170$$

$$V[D_i^B] = E[V[D_i^B \mid q]] + V[E[D_i^B \mid q]] = E[q(1-q)] + V[q]$$
$$= E[q] - E[q^2] + V[q] = 0.17 - 0.0325 + 0.0036 = 0.1411$$

$$V[D^B] = V\left[\sum_{i=1}^{1000} D_i^B\right] = \sum_{i=1}^{1000} V[D_i^B] = (1000)(0.1411) = 141.1$$

WA.3. Continued

Commentary on (b) and (c):
Parts (b) and (c) tested candidates' understanding of the difference between an uncertain mortality rate applying to all subjects, and uncertainty in the mortality rate applying to each individual subject. In Cohort A, if you know the mortality of one life, then you know the mortality of all lives in the cohort. In Cohort B, knowing the mortality rate for one life gives no information about the other lives.

The main point was to understand how these cases are different, and to be able to calculate the mean and variance under each form of uncertainty. While the better candidates did very well on these parts, many candidates failed to distinguish the cases correctly. Some candidates calculated the two parts identically, while some switched the calculations for these two parts.

WA.4.

Commentary on Question:
All parts of this question were answered well by most candidates.

(a)

$$\text{EPV Benefits} = (100,000)A_{45} = (100,000)(0.15161) = 15,161$$

$$\text{EPV Premiums} = P\ddot{a}_{45} = 17.8162P$$

$$\text{EPV Expenses} = 20\ddot{a}_{45} + 80 + 200A_{45} + 0.1P\ddot{a}_{45} + 0.65P$$

$$= (20)(17.8162) + 80 + (200)(0.15161) + [(0.1)(17.8162) + 0.65]P = 466.646 + 2.43162P$$

$$P = \frac{15,161 + 466.646}{17.8162 - 2.43162} = 1015.80$$

Commentary:
It is easiest to deal with recurring expenses starting in year 1 (for all years) and add the extra first year expense. Candidates who tried to incorporate expenses year by year were more likely to make calculation errors. A common error was to omit the settlement expenses (200), or fail to discount them. The answer was given to nearest 10. Candidates who found a different answer to (a) were expected to answer the remaining parts with the given answer.

(b)

$$_1V = (100,200)A_{46} - 0.9P\ddot{a}_{46} + 20\ddot{a}_{46}$$

$$= (100,200)(0.15854) - [(0.9)(1015.80) - 20](17.6706) = 84.30$$

Alternative Solution:

$$_1V = \left[(P - E)(1.05) - q_{45}(100,200)\right] / p_{45}$$

$$= \frac{\left[(1015.80(1 - 0.75) - 100\right](1.05) - (0.000771)(100,200)}{1 - 0.000771} = 84.46$$

Commentary:
Candidates could answer using the prospective approach or the recursion/retrospective approach, which gives a slightly different answer due to rounding. A few candidates forgot expenses completely, others subtracted expenses when they should have added them (prospective) or vice versa (recursion). Another common mistake was to use the wrong expenses, for example, use the renewal expenses instead of first year in the recursive formula, or to omit settlement expenses. Some candidates used the actual interest rate (1.07) instead of the pricing rate (1.05), which led to partial credit if the rest of the solution was correct.

WA.4. Continued

(c)

Interest Gain:

$(10,000)(0.07-0.05)[P-(0.75P+100)]=30,790$

Expense Gain:

$(10,000)[(0.75P+100)-(0.75P+105)](1.07)=-53,500$

Mortality Gain: 0

Commentary:
Candidates were expected to indicate clearly whether the amount calculated is a gain or a loss. A few candidates calculated only the total gain/loss; this received small partial credit. One relatively common mistake was to use a factor of .07 instead of 1.07 when calculating the expense gain and loss.

(d) Interest: No gain or loss, as experience matches the assumption.
Expenses: Loss, as settlement expenses exceed the assumption.
Mortality: Expected deaths = $10000\, p_{45}\, q_{46} = 8.4$. There will be a loss, as actual deaths exceed expected deaths.

Commentary:
A few candidates compared the experience of year 2 with year 1. For example, "interest in year 2 was lower than in year 1 so there was a loss due to interest". This did not directly answer the question and did not receive full credit. A few candidates proposed that there would be a gain due to mortality because there were more deaths than expected.

WA.5.

(a)

Solution:

Given $T_{35} = 0.3$ then the benefit is paid at $t = 1$ and two premiums are paid, at $t = 0$ and $t = 0.25$. Allowing, also, for the expenses at $t = 0$ and $t = 0.25$ we have:

$$ {}_0L^Q \mid (T_{33} = 0.3) = 100{,}000v + 100 + 16v^{0.25} - 150(1 + v^{0.25}) = 95{,}056 $$

Commentary

Many candidates answered this correctly, but most did not, with common errors including (i) assuming the benefit is paid at the moment of death; (ii) omitting the initial expenses (iii) omitting the renewal expenses and (iv) only allowing for a single premium payment.

(b)

$E[{}_0L^Q] =$ EPV of Death Benefit + EPV of Expenses $-$ EPV of Premiums at issue:

EPV of Death Benefits = $(100{,}000)A_{35} = 9653$

EPV of Expenses = $84 + (4)(16)\ddot{a}^{(4)}_{35:\overline{10}|}$

$$ \text{where } \ddot{a}^{(4)}_{35:\overline{10}|} = \ddot{a}_{35:\overline{10}|} - \frac{3}{8}(1 - {}_{10}E_{35}) = 8.0926 - 0.375(1 - 0.61069) = 7.9466 $$

$\Rightarrow$ EPV of Expenses = 592.58

EPV of Premiums = $(4)(150)\ddot{a}^{(4)}_{35} = 600(\ddot{a}_{35} - 3/8) = 600(18.9728 - 0.375) = 11{,}158.68$

$\Rightarrow E[{}_0L^Q] = 9653 + 592.58 - 11{,}158.68 = -913.10$

Commentary:

As for part (a), there were relatively few fully correct answers to this question. Common errors included omitting the factor of 4 for the quarterly premiums and expenses, and omitting the 10-year term on the commissions. Candidates who made errors on intermediate calculations, such as the annual term annuity-due, could receive partial credit for that part if they showed their working.

WA.5. Continued

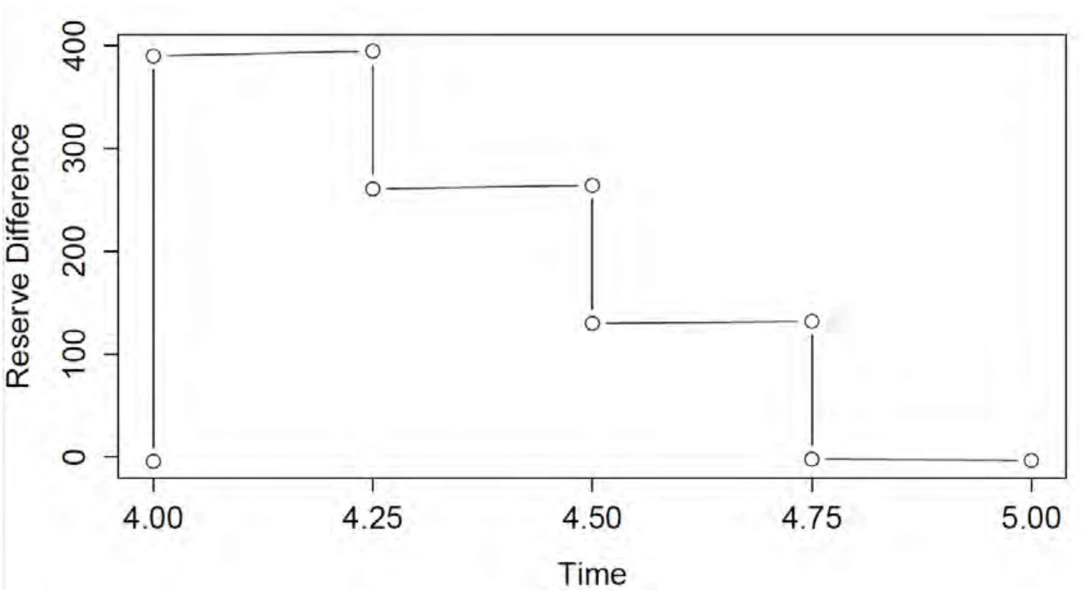
(c) **Solution:**

$$
\begin{aligned}
{}_5V^Q &= (100{,}000)A_{40} + (4)(16)\ddot{a}^{(4)}_{40:\overline{5|}} - (4)(150)\ddot{a}^{(4)}_{40} \\
&= (100{,}000)(0.12106) + 64(4.4590) - 600(18.4578 - 3/8) \\
&= 1542
\end{aligned}
$$

Commentary:
This part was done better than (b), though many of the comments for (b) apply also to (c). Errors that were penalized in (b) were not penalized again in (c).

(d)

Solution:



Commentary:
Key points for credit were:

- *The differences are -4 at time 4 and -3 at time 5.*
- *There is a jump up immediately after time 4 as the annual premium (net of expenses) is much greater than the quarterly premium (net of expenses).*
- *The curve steps down each 1/4 year; the steps are approximately equal to 134, which is the quarterly premium, net of expenses.*

This part was answered correctly by a relatively small set of the very best candidates. Many candidates sketched a smooth curve, and many others omitted this part.

WA.6.

Commentary on Question:
This was a relatively straightforward question that many students omitted. Those who attempted this question generally did well.

(a) Reasons include:
- To compete for new employees
- To retain employees in productive years
- To facilitate turnover of employees at older ages
- To offer tax efficient remuneration
- As a tool in negotiations with unions (or other employee collective bargaining units)
- To fulfill responsibility to provide for long-serving employees.
- To improve morale of employees

Commentary on Question:
Most candidates offered at least one valid reason, and many offered the three requested.

(b) Let S_x denote the salary earned in year of age x to $x+1$. We have
$S_x = (50,000)(1.03)^{x-38}$.

The EPV of the death benefit is:

$$(2S_{62})q_{62}(1.04)^{-1} + (2S_{63})_{1|}q_{62}(1.04)^{-2} + (2S_{64})_{2|}q_{62}(1.04)^{-3}$$

Where $q_{62} = 0.08$; ${}_{1|}q_{62} = (0.92)(0.09) = 0.0828$; ${}_{2|}q_{62} = (0.92)(0.91)(0.10) = 0.08372$

So the EPV of the death benefit is

$$(2)(50,000)(1.03)^{24}[0.08(1.04)^{-1} + (1.03)(0.0828)(1.04)^{-2} + (1.03)^2(0.08372)(1.04)^{-3}]$$
$$= 47,716$$

WA.6. **Continued**

(c) The EPV of the retirement benefit is

$$(0.03)(27)(FAS)({}_3p_{62})(1.04)^{-3}\ddot{a}_{65}$$

Where FAS is the final average salary =

$$FAS = 50,000\left(\frac{(1.03)^{24}+(1.03)^{25}+(1.03)^{26}}{3}\right)=104,719$$

So EPV is

$$(0.03)(27)(104,719)(0.75348)(0.88900)(4.7491) = 269,833$$

Commentary on Question:

Parts (b) and (c) were answered well. The most common minor error was counting years of service incorrectly, which resulted in a small penalty for part (b), and none for (c) if the answer was consistent with (b). A few candidates justified their use of 26 years of service by proposing that Chris was 1 day short of 27 years. Although this interpretation is incorrect, candidates were not penalized if they gave this justification.

WA.7.

(a) The EPV of the commission payments is

$$E[C] = 0.1G\ddot{a}_{35:\overline{10|}} + 0.9G = [(0.1)(8.0926) + 0.9]G = 1.70926G$$

(b) The EPV is

$$8\ddot{a}_{35} + 92 = (8)(18.9728) + 92 = 243.78$$

(c) The Equation of Value is

$$G\ddot{a}_{35} = 1.1\left[(100,000)A_{35} + 1.70926G + 243.78\right]$$

$$\Rightarrow G = \frac{(1.1)[(100,000)(0.09653) + 243.78]}{18.9728 - 1.1(1.70926)} = 636.91$$

Comments:
Parts (a), (b) and (c) were all done well, with a large majority of candidates scoring full marks. A few candidates did not allow for the 10-year term on commissions.

(d) Consider the random variable C:

$$C = 0.1G\ddot{a}_{\overline{\min(K_{35}+1,10)|}} + 0.9G = 0.1G\frac{1 - v^{\min(K_{35}+1,10)}}{d} + 0.9G$$

$$\Rightarrow Var[C] = \left(\frac{0.1G}{d}\right)^2 \left({}^2A_{35:\overline{10|}} - (A_{35:\overline{10|}})^2\right)$$

where ${}^2A_{35:\overline{10|}} = {}^2A_{35} - v^{10}{}_{10}E_{35} \cdot {}^2A_{45} + v^{10}{}_{10}E_{35}$

$$= 0.01601 - (0.61391)(0.61069)(0.03463) + (0.61391)(0.61069) = 0.37794$$

$$\Rightarrow Var[C] = \left(\frac{(0.1)(636.91)}{0.047619}\right)^2 (0.37794 - (0.61464)^2) = 282.06$$

Depending on the decimals carried, you may get a slightly different answer.

WA.7. Continued

Comments:

This part proved more challenging to candidates, with relatively few gaining full credit. Most candidates earned partial credit, although a good number omitted this part entirely.

It was not necessary to write down the random variable C for full credit, but candidates who included this step were more likely to get to the correct answer than those who tried to write the variance down directly.

A number of candidates tried to calculate the annuity variance directly, and found that there was no feasible way to get the correct answer. It is worth remembering that it is always easier to calculate the variance of a death (or endowment) benefit than to calculate, directly, the variance of a life annuity.

Some candidates calculated a negative variance. Candidates who noted that this answer is impossible could receive partial credit for their working.

(e)

$$E[C] = 1.70926G = (1.70926)(636.91) = 1088.64$$

Let c denote the first year commission rate. Then

$$E[C^*] = (0.1)(12)(57)(\ddot{a}^{(12)}_{35:\overline{10}|}) + (12)(57)(c-0.1)(\ddot{a}^{(12)}_{35:\overline{1}|})$$

$$\ddot{a}^{(12)}_{35:\overline{10}|} = \ddot{a}_{35:\overline{10}|} - \frac{11}{24}(1 - {}_{10}E_{35}) = 7.91417$$

$$\ddot{a}^{(12)}_{35:\overline{1}|} = 1 - \frac{11}{24}(1 - {}_{1}E_{35}) = 0.97800$$

$$\Rightarrow E[C^*] = 541.33 + 668.95(c - 0.1)$$

$$c = \frac{1088.64 - 541.33}{668.95} + 0.1 = 0.9182$$

Comments:

This part also proved quite challenging. A relatively small number of candidates earned full credit. Another group earned nearly full credit, but did not make the adjustment for the 10% renewal commission. Many candidates omitted this part.

WA.8.

(a) Let S_x denote the salary in year of age x to $x+1$. The projected salaries in the final three years are:

$$S_{62} = (80,000)(1.04)^{17} = 155,832$$
$$S_{63} = (80,000)(1.04)^{18} = 162,065$$
$$S_{64} = (80,000)(1.04)^{19} = 168,548$$
$$FAS = \frac{155,832 + 162,065 + 168,548}{3} = 162,148$$

So the projected benefit per month is
25[(0.02)(100,000)+(0.03)(62,148)]/12=8050.92

Comment:
This part was done well, with most candidates earning full credit. The most common error was miscounting years of service.

(b) The Replacement Ratio (RR) is

$$\frac{(12)(8051)}{168,548} = 57.3\%$$

Comment:
Again, this part was done well. The most common error was using the final average salary instead of the final year's salary in the denominator.

(c) The EPV is

$$EPV = (8051)(12)\ddot{a}_{65}^{(12)}$$

$$\ddot{a}_{65}^{(12)} = \frac{1 - A_{65}^{(12)}}{d^{(12)}} = \frac{1 - 0.470}{0.04869} = 10.885$$

$$\Rightarrow EPV = 1,051,620$$

Comment:
Many candidates gained full credit on this part. The most common error was forgetting to multiply the monthly benefit by 12.

WA.8. Continued

(d) We require (0.8-0.573) = 0.227 replacement ratio for the DC plan.
The cost of adding 22.7% RR is

$$(0.227)(168,548)\ddot{a}_{65}^{(12)} = 416,464$$

The accumulated contributions to age 65, where c is the contribution rate, are

$$cS_{45}\left((1.07)^{20} + (1.04)(1.07)^{19} + \ldots + (1.04)^{19}(1.07)\right)$$
$$= cS_{45}\left(\frac{(1.07)^{20} - (1.04)^{20}}{1-(1.04)/(1.07)}\right) = (4,789,500)c$$

Equating the contributions and the benefit value gives

$$c = 8.70\%$$

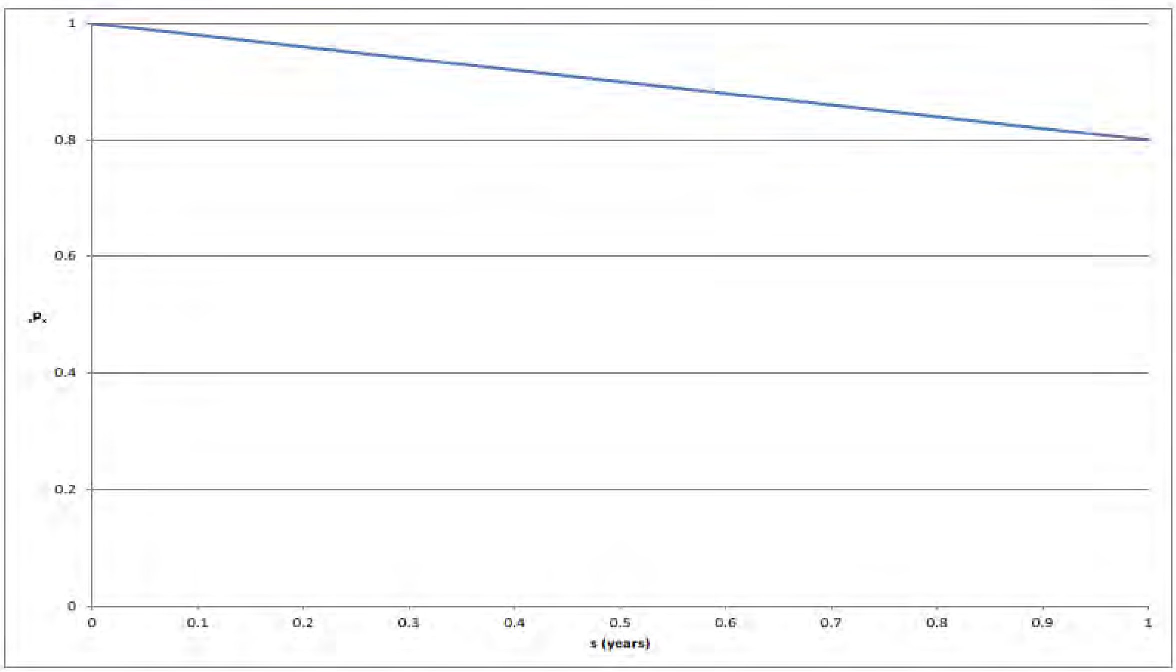
Comment:
This part was less well done, although a number of strong candidates scored full marks. Quite a few candidates omitted this part entirely. The most common errors, which earned partial credit, were miscalculating the geometric series, or omitting entirely either the 4% or 7% terms in the series.

WA.9.

(a) The symbol q_{xy} is the probability that at least one of two lives, currently age x and y, dies within one year.

Comments:
Part (a) was well done by almost all candidates. Some candidates mistakenly described the last survivor status. For full credit, candidates were required to specify the 1-year time period for the mortality probability.

(b) The function is



Comments:
Most candidates did well on this part. Some candidates lost marks for failing to mark key numerical values on the axes. Candidates who sketched a non-linear function for ${}_tp_x$*, or who sketched a different probability, or who sketched a line from 1 to 0, received no credit for this part.*

WA.9. **Continued**

(c)

$${}_s q_{xy} = 1 - {}_s p_x \cdot {}_s p_y \quad \text{(independence)}$$
$$= 1 - (1 - {}_s q_x)(1 - {}_s q_y) = 1 - (1 - s \cdot q_x)(1 - s \cdot q_y) \quad \text{(UDD)}$$
$$= s(q_x + q_y) - s^2 q_x q_y$$

Also $q_x + q_y = q_{xy} + q_{\overline{xy}}$ and $q_{\overline{xy}} = q_x q_y$

$$\Rightarrow {}_s q_{xy} = s(q_{xy} + q_{\overline{xy}}) - s^2 q_{\overline{xy}}$$
$${}_s q_{xy} = s(q_{xy}) + (s - s^2) q_{\overline{xy}}$$

$$\Rightarrow g(s) = s - s^2$$

Comments:
Performance on this part was mixed. Many candidates received full credit. A number of candidates made a reasonable start but could not derive the final result. Partial credit was awarded in these cases. The most common error was assuming that UDD applied to the joint life status.

WA.10.

(a)

$$\frac{d}{dt}\,{}_tp_x^{10} = {}_tp_x^{11}\mu_{x+t}^{10} - {}_tp_x^{10}\left(\mu_{x+t}^{01} + \mu_{x+t}^{03}\right)$$

$$\frac{d}{dt}\,{}_tp_x^{11} = {}_tp_x^{10}\mu_{x+t}^{01} - {}_tp_x^{11}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13}\right)$$

Boundary Conditions: ${}_0p_x^{10} = 0 \qquad {}_0p_x^{11} = 1$

Comments:

Most candidates achieved full credit for this part. Common errors included omitting the boundary conditions, or omitting terms on the right hand side. A few candidates forgot the $\frac{d}{dt}$ *on the left hand side.*

WA.10. **Continued**

(b) From the Kolmogorov equation for $\frac{d}{dt}\,{}_tp_{x80}^{10}$, we have

$$\lim_{h\to 0}\frac{{}_{t+h}p_{80}^{10}-{}_tp_{80}^{10}}{h}={}_tp_{80}^{11}\mu_{80+t}^{10}-{}_tp_{80}^{10}\left(\mu_{80+t}^{01}+\mu_{80+t}^{03}\right)$$

So, for small h, $\frac{{}_{t+h}p_{80}^{10}-{}_tp_{80}^{10}}{h}\approx{}_tp_{80}^{11}\mu_{80+t}^{10}-{}_tp_{80}^{10}\left(\mu_{80+t}^{01}+\mu_{80+t}^{03}\right)$

$$\Rightarrow{}_{t+h}p_{80}^{10}\approx{}_tp_{80}^{10}+h\left[{}_tp_{80}^{11}\mu_{80+t}^{10}-{}_tp_{80}^{10}\left(\mu_{80+t}^{01}+\mu_{80+t}^{03}\right)\right]$$

$$\Rightarrow{}_{1/3}p_{80}^{10}\approx 0+\frac{1}{3}(0.08)=0.02667$$

$${}_{2/3}p_{80}^{10}\approx{}_{1/3}p_{80}^{10}+\frac{1}{3}\left[(0.90346)(0.08)-(0.02667)(0.13082)\right]=0.04960$$

$${}_{1}p_{80}^{10}\approx{}_{2/3}p_{80}^{10}+\frac{1}{3}\left[(0.81652)(0.08)-(0.04960)(0.13186)\right]=0.06919$$

ALTERNATIVE SOLUTION

$${}_1p_{80}^{10}=\left({}_{1/3}p_{80}^{11}\cdot{}_{1/3}p_{80\ 1/3}^{10}\cdot{}_{1/3}p_{80\ 1/3}^{00}\right)+\left({}_{2/3}p_{80}^{11}\cdot{}_{1/3}p_{80\ 2/3}^{10}\right)+\left({}_{1/3}p_{80}^{10}\cdot{}_{1/3}p_{80\ 1/3}^{00}\cdot{}_{1/3}p_{80\ 1/3}^{00}\right)$$

We approximate the probabilities as follows

$${}_{1/3}p_{80+t}^{10}\approx\frac{1}{3}\mu_{80+t}^{10}=0.02667\quad\text{and}\quad{}_{1/3}p_{80+t}^{01}\approx\frac{1}{3}\mu_{80+t}^{01}=0.03333$$

$${}_{1/3}p_{80\ 1/3}^{00}\approx 1-\frac{1}{3}\left(\mu_{80\ 1/3}^{01}+\mu_{80\ 1/3}^{03}\right)=0.95606$$

$${}_{1/3}p_{80\ 2/3}^{00}\approx 1-\frac{1}{3}\left(\mu_{80\ 2/3}^{01}+\mu_{80\ 2/3}^{03}\right)=0.95605$$

$$\Rightarrow{}_1p_{80}^{10}=(0.90346)(0.02667)(0.95605)+(0.81652)(0.02667)$$
$$+(0.02667)(0.95606)(0.95605)=0.06920$$

Comments:

- *This part was quite well done, with many candidates receiving full credit.*
- *Those who followed the alternative method were more prone to numerical errors.*
- *Some candidates lost marks by using the wrong values from the tables. Deductions for this were fairly small if the rest of the solution was correct.*
- *Many candidates calculated values for* ${}_hp_{80}^{11}$ *, not realizing these had been given to them in the question.*

WA.10. Continued

(c) (i) The expected present value of the service fees is

$$EPV = 8000\left(\bar{a}_{80}^{00} + \bar{a}_{80}^{01}\right) = (8000)(5.5793 + 1.3813) = 55,685$$

(c) (ii) The expected present values of the level 2 care costs is

$$EPV = (30,000)\bar{a}_{80}^{02} + (10,000)_{5|}\bar{a}_{80}^{02}$$

$$= (30,000)\bar{a}_{80}^{02} + (10,000)(v^5 \cdot {}_5p_{80}^{00} \cdot \bar{a}_{85}^{02} + v^5 \cdot {}_5p_{80}^{01} \cdot \bar{a}_{85}^{12} + v^5 \cdot {}_5p_{80}^{02} \cdot \bar{a}_{85}^{22}) = 23,005$$

Alternative solutions for (c)(ii)

$$EPV = (30,000)\bar{a}_{80:\overline{5}|}^{02} + (40,000)_{5|}\bar{a}_{80}^{02}$$

$${}_{5|}\bar{a}_{80}^{02} = v^5 \cdot {}_5p_{80}^{00} \cdot \bar{a}_{85}^{02} + v^5 \cdot {}_5p_{80}^{01} \cdot \bar{a}_{85}^{12} + v^5 \cdot {}_5p_{80}^{02} \cdot \bar{a}_{85}^{22} = 0.4678$$

$$\bar{a}_{80:\overline{5}|}^{02} = \bar{a}_{80}^{02} - {}_{5|}\bar{a}_{80}^{02} = 0.14308$$

$$\Rightarrow EPV = 23,005$$

$$EPV = (40,000)\bar{a}_{80}^{02} - (10,000)_{5|}\bar{a}_{80}^{02}$$

$$\bar{a}_{80:\overline{5}|}^{02} = \bar{a}_{80}^{02} - (v^5 \cdot {}_5p_{80}^{00} \cdot \bar{a}_{85}^{02} + v^5 \cdot {}_5p_{80}^{01} \cdot \bar{a}_{85}^{12} + v^5 \cdot {}_5p_{80}^{02} \cdot \bar{a}_{85}^{22}) = 0.14308$$

$$\Rightarrow EPV = 23,005$$

Comments:

- *This part is designed to test understanding of the multiple state model annuity factor notation, and ability to manipulate the probabilities and annuities to create term annuity factors. It was one of the more challenging parts of the exam overall.*
- *Many candidates achieved full credit for (c)(i). Those who did not tended to com- bine, for example, $\bar{a}_{80}^{00}$ and $\bar{a}_{80}^{11}$, which would require Ada to be in state 0 and state 1 simultaneously at age 80.*
- *The best candidates correctly noted for (c)(ii) that there are three cases to allow for in creating the five-year deferred annuity, corresponding to Ada being in state 0, state 1 or state 2 in five years. However, most candidates did not take all three cases into consideration.*
- *Some answers allowed appropriately for one or two cases, and these received partial credit.*
- *Some candidates combined, for example, ${}_5p_{80}^{02}$ with $\bar{a}_{80}^{02}$, which is a more serious error, as it indicates a lack of understanding of the functions involved.*

WA.11.

(a)

$$\Pr_2 = ({}_1V + P - E)(1+i_2) - q_{51}S - p_{51}({}_2V)$$
$$= (400+1100-55)(1.02) - (0.00642)(100,000) - (0.99358)(800) = 37.04$$

Comments:
Most candidates earned full credit for this part. Those who did not tended to struggle with the rest of the question.

(b)

t	${}_{t-1}V$	P	E_t	I_t	EDB_t	EMB	E_tV	$\Pr_t$
0			155					-155.00
1	0	1100	55	10.45	592		397.63	65.82
2	400	1100	55	28.90	642		794.86	37.04
3	800	1100	55	55.35	697	1092.33	0.00	111.02

The profit vector is the final column.

P is the premium
E_t denotes expenses
I_t denotes interest on funds in year t
$EDB_t = 100{,}000\ q_{50+t-1}$
EMB in year 3 is 1100 p_{52}
$E_tV = p_{50+t-1}\ {}_tV$.

Comments:
Many candidates received full credit for this part, and a larger number received partial credit. The most common errors included (i) ignoring the maturity benefit and (ii) incorrect use of probabilities in E_tV. Some candidates used a profit table, others calculated each term in the profit vector individually. Either approach was acceptable. It is not necessary for candidates to define all terms, but it can be helpful when graders are considering partial credit.

WA.11. **Continued**

(c) The profit signature at t is Π_t where:

$\Pi_0 = \Pr_0$ and $\Pi_t = {}_{t-1}p_{50} \cdot \Pr_t$

$$\Rightarrow (\Pi_0, \Pi_1, \Pi_2, \Pi_3) = (-155.00,\ 65.82,\ 36.82,\ 109.65)$$

$$\Rightarrow NPV = \sum_{k=0}^{3} \Pi_k \cdot v_{14\%}^k = 5.08$$

Comments:

Most candidates did this part well. Some candidates used incorrect probabilities ($_tp_{50}$ instead of $_{t-1}p_{50}$ in the profit signature), and others used incorrect rates for discounting the profits for the NPV.

(d) The IRR of B is j where $155 = 210v_j^3 \Rightarrow j = 10.65\%$.

and the IRR of A is greater than 14%, because the NPV at 14% is positive.
Hence IRR of B is less than IRR of A.

Product C has lower reserves in year 1,which allow an earlier release of surplus compared to Product A, which give a higher NPV than A at an 14% hurdle rate, but does not necessarily mean that C has a higher IRR.

The lower reserve in year 1 results in the following profit signature for C:
(-155.00, 165.23, -64.58, 109.65)

We note that the NPV of A is a little larger than 14%, because the NPV at 14% is close to zero. Calculating the NPV for A and C at 16% gives -0.65 for A and 9.7 for C. Hence, the IRR for C is greater than 16% , and for A is less than 16%.

That is $\Rightarrow IRR(B) < IRR(A) < IRR(C)$

Comments:

Many candidates evaluated the IRR for all three cases, presumably using the financial functions on the BA2 calculator. This was awarded full credit if correct. However, it was not necessary to determine the IRR to answer the question.

Candidates who demonstrated understanding of the relationship between the release of surplus and the return to the insurer gained partial credit, even if the justification was incomplete.

No credit was awarded for the IRR ordering if there was no accompanying explanation or justification.

WA.12.

(a)

$$\bar{a}^{0j}_{x:\overline{10|}} = \int_0^{10} {}_tp_x^{0j} \cdot e^{-\delta t} \cdot dt$$

$$\Rightarrow \bar{a}^{00}_{x:\overline{10|}} + \bar{a}^{01}_{x:\overline{10|}} + \bar{a}^{02}_{x:\overline{10|}} = \int_0^{10} ({}_tp_x^{00} + {}_tp_x^{01} + {}_tp_x^{02}) \cdot e^{-\delta t} \cdot dt$$

but ${}_tp_x^{00} + {}_tp_x^{01} + {}_tp_x^{02} = 1$

$$\Rightarrow \bar{a}^{00}_{x:\overline{10|}} + \bar{a}^{01}_{x:\overline{10|}} + \bar{a}^{02}_{x:\overline{10|}} = \int_0^{10} e^{-\delta t} \cdot dt = \bar{a}_{\overline{10|}}$$

Comments:
Many candidates did well on this part. Some candidates gave a verbal explanation, rather than mathematical proof. Generally, this earned partial credit, but full credit was available if the verbal proof was sufficient detailed.

In the mathematical proof, the candidate was required to indicate clearly that ${}_tp_x^{00} + {}_tp_x^{01} + {}_tp_x^{02} = 1$ *for full credit.*

In the verbal explanations, candidates were required to state that $\bar{a}_x^{0j}$ *is the actuarial present value of a benefit of 1 per year paid 'while' the person is in state j, not 'when' or 'if' the person transitions to state j.*

Some candidates wrote out the sum and the result that had to be proved, but gave no details on the intermediate steps. These answers received no credit.

(b)

EPV Premiums: $P\bar{a}^{00}_{x:\overline{10|}} = 4.49P$

EPV Disability Annuity: $1000\bar{a}^{01}_{x:\overline{10|}} = 1000\left(\bar{a}_{\overline{10|}} - (\bar{a}^{00}_{x:\overline{10|}} + \bar{a}^{02}_{x:\overline{10|}})\right)$ where $\bar{a}_{\overline{10|}} = \dfrac{1-v^{10}}{\delta} = 6.32121$

$\Rightarrow EPV$ Disability Annuity $= 471.21$

EPV Death Benefit: $1000\bar{A}^{02}_{x:\overline{10|}} = 3871.0$

So the Premium is $P = \dfrac{3871 + 471.21}{4.49} = 967.10$

WA.12. Continued

Comments:
This part was done well by many candidates. Some candidates did not realize that the result in (a) could be used to find the missing annuity value in (b). Other candidates did notice this, but could not calculate the required annuity-certain, $\bar{a}_{\overline{10}|}$.

(c) The Thiele equation at time t is

$$\frac{d}{dt}\,{}_tV^{(0)} = \delta \cdot {}_tV^{(0)} + P - \mu_{x+t}^{01}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu_{x+t}^{02}\left(10{,}000 - {}_tV^{(0)}\right)$$

so at $t = 3$ we have

$$\frac{d}{dt}\,{}_tV^{(0)} = (0.1)(1304.54) + 967.1 - 0.04\left(7530.09 - 1304.54\right) - 0.06\left(10{,}000 - 1304.54\right) = 326.80$$

Comments:
This part proved more challenging to many candidates. The responses indicated that many candidates are memorizing Thiele's formula rather than understanding the intuition behind it. Common errors included the following:

> *Putting the State 1 annuity rate (1000) in the term which values the instantaneous cost of transition to state 1 (the one with* μ_{x+t}^{01} *). Thiele's equation allows for instantaneous payments after transfer, such as a death benefit; The annuity cost is captured in* ${}_tV^{(1)}$.
>
> *Using wrong signs for the release of* ${}_tV^{(0)}$ *terms, or for the* ${}_tV^{(1)}$ *term.*
>
> *Subtracting the premium rather than adding it.*

Candidates making one or two of these common errors would receive partial credit.

A few candidates wrote down a generic formula for Thiele's equation, but did not adapt it to this problem, nor indicate any appropriate numerical values. These answers received no credit.

WA.12. Continued

(d) Let P^* denote the new premium. The EPV of return of premium is

$$10P^* e^{-10\delta} {}_{10}p_x^{\overline{00}}$$

where ${}_{10}p_x^{\overline{00}} = e^{-\int_0^{10}(\mu_{x+t}^{01}+\mu_{x+t}^{02})dt} = e^{-0.4-1} = 0.24660$

So, the EPV is $(10)(P^*)(e^{-1})(0.24660) = 0.90718P^*$

Hence $P^* = \dfrac{3871+471.21}{4.49-0.90718} = 1211.95$

$\Rightarrow P^* - P = 244.87$

Comments:

This part was done reasonably well by many candidates who attempted it. Most candidates received partial credit, but only a few candidates received full credit.

A common error was to omit the discount factor, $e^{-10\delta}$. *The reason may have been that the question said the premiums were returned without interest. That wording means that the benefit is* $10P$, *i.e. a simple sum, not* $P\bar{s}_{\overline{10}|j}$ *which would be the payment if premiums were returned with interest credited at an interest rate of j per year.*

The wording does not mean that the insurer does not earn any interest between inception and time 10, which is what the omission of $e^{-10\delta}$ *implies.*

Another, related error was to replace $e^{-10\delta}$ *in the return of premium benefit with* $\bar{a}_{\overline{10}|}$. *This would be correct if premiums are returned at time 10, with interest at the same rate as the valuation (i.e.* $j = i = e^{\delta} - 1$*). It is incorrect in this case as the returned premiums were not credited with interest.*

Candidates who carried forward an error, for example in the annuity values, were not penalized again here.

WA.13.

(a) Let $l_{x+t^-}^{(\tau)}$ denote the value immediately before exits at exact age $x+t$, and $l_{x+t^+}^{(\tau)}$ denote the value immediately after. At $t = 0.5,$ just before decrement 2 exits, we have

$$q_{60}'^{(2)} = \frac{d^{(2)}}{l_{60.5^-}^{(\tau)}}$$

$$l_{60.5^-}^{(\tau)} = 1000\exp\left(-\int_0^{0.5} 1.2t\ dt\right) = (1000)(0.86071) = 860.71$$

$$\Rightarrow q_{60}'^{(2)} = \frac{60}{860.71} = 0.0697$$

Comment:
Many candidates gave $60/1000$ *as the answer, i.e. did not adjust the exposure to allow for decrement (1) departures before decrement (2) applied.*

(b)

$$l_{60.5+}^{(\tau)} = 800.71 \Rightarrow l_{61^-}^{(\tau)} = 800.71e^{-\int_{0.5}^{1} 1.2t\ dt} = 800.71(0.63763) = 510.6$$

$$l_{61+}^{(\tau)} = 510.6 - 45 = 465.6$$

$$\Rightarrow d_{60}^{(1)} = l_{60}^{(\tau)} - d_{60}^{(2)} - d_{60}^{(3)} - l_{61+}^{(\tau)} = 1000 - 60 - 45 - 466 = 429$$

Comments:
There are different ways of doing this part; full credit was awarded for any correct method.

Quite a few candidates gave the answer as 451, which is obtained by ignoring decrement (2); i.e.

$$p_{60}^{(1)} = e^{-\int_0^1 1.2t\ dt} = 0.5488 \Rightarrow 1000(1 - 0.5488) = 451.2$$

WA.13. Continued

(c) (i) If decrement 2 occurs at the start of the year, there are fewer lives exposed to force of decrement 1, so $q_{60}^{(1)}$ would be smaller.

(ii) If decrement 2 occurs at the start of the year, there are more lives exposed to force of decrement 2, so $q_{60}^{(2)}$ would be bigger.

(iii) Because all the decrement 3 exits happen at the end of the year, we have

$$q_{60}^{(3)} = \frac{d_{60}^{(3)}}{l_{60}^{(\tau)}} \qquad q_{60}'^{(3)} = \frac{d_{60}^{(3)}}{l_{61^-}^{(\tau)}}$$

Where $l_{61^-}^{(\tau)}$ is the expected number of in-force immediately before the decrement 3 exits at the end of the year. Also $l_{61^-}^{(\tau)} = l_{60}^{(\tau)}\, p_{60}'^{(1)}\, p_{60}'^{(2)}$ and since the independent rates are unchanged, $l_{61^-}^{(\tau)}$ is unchanged, which means that $d_{60}^{(3)}$ is unchanged, which means that $q_{60}^{(3)}$ is unchanged.

Comment:
Quite a few candidates simply gave an answer (i.e. decrease, increase etc) without justification. Partial credit was given if all the answers were correct.

WA.14

(a) The EPV of the annuity payments and expenses, given a single premium G, is

$$50{,}100a_{65}+3000+0.1G \quad \text{where } a_{65}=\ddot{a}_{65}-1=12.5498$$

$$\Rightarrow G=1.1\left(50{,}100a_{65}+3000+0.1G\right)=\frac{694{,}919.48}{0.89}=780{,}808$$

Comments:

This part was done reasonably well. Many candidates achieved full marks, and most others received significant partial credit.

Applying the 1.1 factor created some confusion. Calculating a premium and then multi- plying by 1.1 was a common approach, but is incorrect because the commission valued is based on the wrong premium.

Some candidates used $\ddot{a}_{65}$ in place of a_{65} for the annuity. Others omitted expenses or commission.

(b) Let K denote the curtate future lifetime random variable. The loss at issue random variable is L where

$$L=(50{,}100)(\ddot{a}_{\overline{K+1}|}-1)+3000+0.1G-G=50{,}100\ddot{a}_{\overline{K+1}|}-749{,}827$$

The probability of profit is

$$\Pr[L<0]=\Pr\left[50{,}100\ddot{a}_{\overline{K+1}|}-749{,}827<0\right]=\Pr\left[\ddot{a}_{\overline{K+1}|}<14.9666\right]$$

$$\Pr\left[\frac{1-v^{K+1}}{d}<14.9666\right] \quad \text{where } d=0.04762$$

$$=\Pr\left[1-v^{K+1}<0.7127\right]=\Pr\left[v^{K+1}>0.2873\right]$$

$$=\Pr\left[K+1<\frac{\ln(0.2873)}{\ln[(1.05)^{-1}]}\right]=\Pr\left[K+1<25.56\right]$$

$$=\Pr\left[K\le 24\right]={}_{25}q_{65}=1-\frac{l_{90}}{l_{65}}=1-\frac{41{,}841.1}{94{,}579.7}=0.5576$$

WA.14. Continued

Comments:

Few candidates did this part completely correctly, but many candidates achieved partial credit.

Some candidates did not reverse the inequality, ending with an answer of ${}_{25}p_{65}$.

Some candidates found the correct probability statement for K, *but used* ${}_{24}q_{65}$ *instead of* ${}_{25}q_{65}$.

Some candidates used $a_{\overline{K+1}|}$ *instead of* $\ddot{a}_{\overline{K+1}|}-1$ *(or equivalently* $a_{\overline{K}|}$*) in the loss function.*

Some candidates did not adjust the answer for discrete payment periods – i.e. solved for non-integer mortality period.
Some candidates set up the loss random variable using expectations instead of ran- dom variables. This is a more serious error and received little credit.
Some candidates tried to apply a normal approximation. This is not appropriate for an individual policy, and no credit was given.

WA.14. Continued

(c) Now let L_j denote the loss from the *j*th policy, given a premium of G^P. Let K_{x_j} denote the curtate future lifetime for the *j*th life.

$$L_j = (50{,}100)(\ddot{a}_{\overline{K_{x_j}+1}|} - 1) + 3000 - 0.9G^P$$

$$\Rightarrow E[L_j] = (50{,}100)(12.5498) + 3000 - 0.9G^P = 631{,}745 - 0.9G^P$$

$$\text{and } V[L_j] = (50{,}100)^2 V\left[\ddot{a}_{\overline{K_{x_j}+1}|}\right] = (50{,}100)^2 V\left[\frac{1 - v^{K_{x_j}+1}}{d}\right]$$

$$= \left(\frac{50{,}100}{d}\right)^2 V\left[v^{K_{x_j}+1}\right] = \left(\frac{50{,}100}{d}\right)^2 \left({}^2A_{65} - (A_{65})^2\right)$$

$$= \left(\frac{50{,}100}{d}\right)^2 \left(0.15420 - (0.35477)^2\right) = (177{,}100)^2$$

Let L denote the total loss on 8000 independent and identical contracts. Using the normal approximation and

$$E[L] = 8000E[L_j] \qquad V[L] = 8000V[L_j]$$

$$\Rightarrow \Pr[L \le 0] \approx \Phi\left(\frac{-8000(631{,}745 - 0.9G^P)}{\sqrt{8000}(177{,}100)}\right)$$

Set the probability to 0.9, noting that $\Phi(1.282) = 0.90$

$$\frac{\sqrt{8000}(0.9G^P - 631{,}745)}{177{,}100} = 1.282 \Rightarrow G^P = 704{,}759$$

Comment:
Few candidates completed this part correctly. The main problem was the calculation of the variance, but candidates were also challenged by the immediate annuity part.

WA.14. Continued

(d) Under the portfolio approach, on average, each policy makes a profit. Although the probability of profit for each policy is less than 0.9, the large number of diversified policies means that, with probability 0.9, the gains on the policies where the profit is positive will be greater than the losses on the other policies.

Comments:
This part was answered well.

WA.15.

General comments:
Parts (a) and (b) were answered well by most candidates; those who continued did well on parts (c) and (d). Relatively few candidates could explain the reasoning behind the mechanics of modified premium reserves, as required for part (e).

(a)

$({}_0V + P)(1+i) = q_x(180,000) + p_x(8147.08)$

where ${}_0V = 0$; $(1+i) = 1/(1-d) = 1/0.9$; $q_x = 0.1$; $p_x = 0.9$

$\Rightarrow P = 22,799$

Comment:
The recurrence method is the natural way to do this problem. Most candidates recognized this and correctly calculated the premium.

(b) Let S_t denote the death benefit in the tth year.

$({}_1V + P)(1+i) = q_{x+1}(S_2) + p_{x+1}(12,480.86)$

where ${}_1V = 8147.08$; $(1+i) = 1/0.9$; $q_x = 0.2$; $p_x = 0.8$

$\Rightarrow S_2 = 122,000$ $\quad$ $(123,116$ using $P = 23,000)$

$({}_2V + P)(1+i) = q_{x+2}(S_3)$

where ${}_2V = 12,480.86$; $(1+i) = 1/0.9$; $q_x = 0.4$; $p_x = 0.6$

$\Rightarrow S_3 = 98,000$ $\quad$ $(98,600$ using $P = 23,000)$

Comment:
Most candidates received full credit for this part.

(c) (i)

The first year net premium is the cost of insurance, i.e. P_1, where

$P_1 = (180,000)vq_x = (180,000)(0.9)(0.1) = 16,200$

WA.15. **Continued**

(ii)
Premiums in year 2 and 3 are level and equal to P_2 ,where

$$P_2 = \frac{122,000vq_{x+1} + 98,000v^2 {}_{1|}q_{x+2}}{1 + vp_{x+1}}$$

$$= \frac{(122,000)(0.9)(0.2) + (98,000)(0.9)^2(0.8)(0.4)}{1 + (0.9)(0.8)} = 27,536$$

(d) Let ${}_2V^{FPT}$ denote the FPT reserve at time 2.

$${}_2V^{FPT} = (98,000)vq_{x+2} - 27,536 = 7744$$

Alternative Solution

$${}_2V^{FPT} = \frac{({}_1V^{FPT} + P_2)(1+i) - 122,000q_{x+1}}{p_{x+1}}$$

$$= \frac{(0 + 27,536)/0.9 - 122,000(0.2)}{0.8} = 7744$$

Comments:
Many candidates stopped after part (b). Those who did not typically earned full credit for parts(c) and (d).

(e)

Modified premium reserves use an adjusted net premium schedule. The net premium is not assumed to be level (even if the gross premium is level). Instead, premiums in the first year are assumed to be lower, so that the excess of gross premium over the net premium is implicitly assumed to be available for the acquisition expenses.

The method gives lower reserves, more consistent with a gross premium policy value approach, whilst maintaining the net premium reserve principle.

Comments:
For full credit, candidates were required to give a coherent explanation of how and why premiums are modified for modified net premium reserves. Relatively few candidates earned full credit for this part.

WA.16.

General Comments:
The candidates who attempted this question did very well, with most earning full credit for the parts attempted. However, a very large number of candidates omitted the question.

The most common error involved using the wrong number of years for final salary calculations.

(a) RR denotes the replacement ratio, FAS denotes the final average salary, S_x denotes salary in the year of age x to $x+1$. Then

$$RR = \frac{(9.5)(900)+(15.5)(FAS)(0.03)}{S_{64}}$$

$$S_{64} = (30,000)(1.02)^{24} = 48,253$$

$$FAS = (30,000)\left(\frac{1.02^{22}+1.02^{23}+1.02^{24}}{3}\right) = 47,313$$

$$\Rightarrow RR = \frac{30,551}{48,253} = 63.3\%$$

(b)

$$RR = \frac{900n+(25-n)(47,313)(0.03)}{48,253} \geq 0.65$$

$$\Rightarrow \frac{35,485-519.4n}{48,253} \geq 0.65 \Rightarrow n \leq 7.9 \text{ years}$$

WA.16. Continued

(c) *Note: The question does not specify whether the accrued benefits of XYZ are based on FAS at retirement or FAS at age 55. Either interpretation was acceptable.*

Version A: using FAS at retirement

The total accrued benefit based on the first 15 years of employment

$$(7)(900)+(8)(0.03)(FAS)=17,655$$

The required benefit is $(0.65)(S_{64})=31,364$

So the annuity payments are 13,709 per year, requiring premium of P where

$$P\ddot{a}_{55:\overline{10}|}=({}_{10}E_{55})(13,709)(\ddot{a}_{65})$$

$$\Rightarrow P(8.0192)=(0.59342)(13,709)(13.5498)$$

$$\Rightarrow P=13,746$$

Version B: using FAS at time 15

The total accrued benefit based on the first 15 years of employment

$$(7)(900)+(8)(0.03)\left(\frac{S_{52}+S_{53}+S_{54}}{3}\right)=15,615$$

The required benefit is $(0.65)(S_{64})=31,364$

So the annuity payments are 15,749 per year, requiring premium of P where

$$P\ddot{a}_{55:\overline{10}|}=({}_{10}E_{55})(15,749)(\ddot{a}_{65})$$

$$\Rightarrow P(8.0192)=(0.59342)(15,749)(13.5498)$$

$$\Rightarrow P=15,791$$

WA.17.

General Comment:
Overall, this question was done reasonably well, although few candidates received maximum credit.

(a) (i) For any age x, the survival function $S_x(t)$ must satisfy the following

1. $S_x(0) = 1$
2. $\lim_{t\to\infty} S_x(t) = 0$
3. $S_x(t)$ must be a non-increasing function of t

(ii)

For b = -0.2

1. $S_{40}(0) = 1$
2. $\lim_{t\to\infty} S_{40}(t) = \lim_{t\to\infty} 0.75e^{-0.2(t-25)} = 0$
3. For the third criterion, we show that $S_x(t)$ is non-increasing before age 65, after age 65, and at age 65:

$$\frac{d}{dt}S_{40}(t) = -0.0008t < 0 \quad \text{for} \quad 0 \le t < 25$$

$$\frac{d}{dt}S_{40}(t) = (-0.2)*(0.75)(e^{-0.2(t-25)}) < 0 \quad \text{for} \quad t > 25$$

and $\lim_{t\uparrow 25-} S_{40}(t) = 0.75 = \lim_{t\uparrow 25+} S_{40}(t)$

Hence $b = -0.2$ is a valid parameter.

For b = 0.0

$$\lim_{t\to\infty} S_{40}(t) = \lim_{t\to\infty} 0.75 \neq 0$$

Hence, $b = 0.0$ is not a valid parameter.

For b = 0.2

$$\lim_{t\to\infty} S_{40}(t) = \lim_{t\to\infty} 0.75e^{0.2(t-25)} = \infty$$

Hence, $b = 0.2$ is not a valid parameter.

WA.17. Continued

Comments:
For full credit, candidates were expected to verify all three criteria from (a)(i) for b=-0.2. Few candidates verified, for example, that the function is non-increasing at t=25. For the invalid values of b, candidates could justify the conclusion by indicating any criterion that is not satisfied.

(b) (i)

$$\mu_{40+t} = -\left(\frac{1}{S_{40}(t)}\right)\frac{d}{dt}S_{40}(t)$$
$$\Rightarrow \mu_{60} = \left(\frac{1}{0.84}\right)(2)(0.02)^2(20) = 0.01905$$

Alternative

$$\mu_{40+t} = -\frac{d}{dt}\ln\left[S_{40}(t)\right] = -\frac{d}{dt}\ln\left[1-(0.02t)^2\right] \quad \text{for} \quad t \le 25$$

$$\mu_{60} = \frac{(2)(0.02)^2(20)}{1-((0.02)(20))^2} = 0.01905$$

(ii)

$$\mu_{40+t} = -\left(\frac{1}{S_{40}(t)}\right)\frac{d}{dt}S_{40}(t) = \frac{(0.75)(0.1)e^{-0.1(t-25)}}{0.75e^{-0.1(t-25)}} = 0.1 \quad \text{for} \quad t > 25$$
$$\Rightarrow \mu_{70} = 0.1$$

Alternative

$$\mu_{40+t} = -\frac{d}{dt}\ln\left[S_{40}(t)\right] = -\frac{d}{dt}\left(\ln\left[0.75e^{2.5}\right] + \ln\left[e^{-0.1t}\right]\right) \quad \text{for} \quad t > 25$$

$$= -\frac{d}{dt}\ln\left[0.75e^{2.5}\right] + -\frac{d}{dt}\ln\left[e^{-0.1t}\right] = 0 - \frac{d}{dt}(-0.1t) = 0.1$$

$$\mu_{70} = 0.1$$

WA.17. Continued

(iii)

$$\overset{\circ}{e}_{40:\overline{35}|} = \int_0^{25} S_{40}(t)\ dt + \int_{25}^{35} S_{40}(t)\ dt = \int_0^{25} (1-0.0004t^2)\ dt + \int_{25}^{35} 0.75e^{-0.1(t-25)}\ dt$$

$$\int_0^{25} (1-0.0004t^2)\ dt = \left[t - \frac{0.0004t^3}{3} \right]_0^{25} = 22.917$$

$$\int_{25}^{35} 0.75e^{-0.1(t-25)}\ dt = 4.741$$

$$\Rightarrow \overset{\circ}{e}_{40:\overline{35}|} = 22.917 + 4.741 = 27.658$$

Comments:

Most candidates who attempted this part correctly calculated the first integral on the right hand side above. Only the stronger candidates successfully setup and evaluated the second integral.

WA.18.

General Comment:
This proved to be one of the most challenging questions on the paper.

(a) The future lifetimes of (x) and (y) are dependent, because the force of mortalityfor each is different depending on whether the other is alive or not, as $\mu^{01}_{x+t:y+t} \neq \mu^{23}_{x+t}$ and $\mu^{02}_{x+t:y+t} \neq \mu^{13}_{y+t}$. That means that the distribution of the time to death of (x) is different if (y) dies early than if (y) dies later (and vice versa), which means the future lifetime random variables T_x and T_y are dependent.

Comment:
The key point, which around one-third of the candidates identified, is that the force of mortality for (x) *from state 2 is different than the force from state 0, and the force of mortality for* (y) *is different from state 1 than from state 0, which indicates dependence. The fact that there is no common shock transition does not imply independence. When justifying dependence, candidates were expected to compare appropriate pairs (i.e.* $\mu^{01}_{x+t:y+t}$ *and* $\mu^{23}_{x+t:y+t}$*) which both concern the death of* (x), and $\mu^{02}_{x+t:y+t}$ *and* $\mu^{13}_{x+t:y+t}$*) which both concern the death of* (y)*)*

(b) (i)

$$\frac{d}{dt}\,{}_tp^{00}_{xy} = -{}_tp^{00}_{xy}\left(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t}\right) \text{ with boundary condition of } {}_0p^{00}_{xy} = 1$$

WA.18. Continued

(ii)

$$\frac{d}{dr}\ {}_r p_{xy}^{00} = -{}_r p_{xy}^{00}\left(\mu_{x+r:y+r}^{01} + \mu_{x+r:y+r}^{02}\right)$$

$$\Rightarrow \frac{1}{{}_r p_{xy}^{00}}\frac{d}{dr}\ {}_r p_{xy}^{00} = -\left(\mu_{x+r:y+r}^{01} + \mu_{x+r:y+r}^{02}\right)$$

$$\Rightarrow \frac{d}{dr}\ \ln[{}_r p_{xy}^{00}] = -\left(\mu_{x+r:y+r}^{01} + \mu_{x+r:y+r}^{02}\right)$$

Integrate both sides from 0 to t

$$\int_0^t \frac{d}{dr}\ \ln[{}_r p_{xy}^{00}]dr = \int_0^t -\left(\mu_{x+r:y+r}^{01} + \mu_{x+r:y+r}^{02}\right)dr$$

$$\Rightarrow \ln[{}_t p_{xy}^{00}] - \ln[{}_0 p_{xy}^{00}] = \int_0^t -\left(\mu_{x+r:y+r}^{01} + \mu_{x+r:y+r}^{02}\right)dr$$

from the boundary condition, $\ln[{}_0 p_{xy}^{00}] = 0,$ so

$$\ln[{}_t p_{xy}^{00}] = \int_0^t -\left(\mu_{x+r:y+r}^{01} + \mu_{x+r:y+r}^{02}\right)dr$$

$$\Rightarrow {}_t\ p_{xy}^{00} = \exp\left(-\left(\mu_{x+r:y+r}^{01} + \mu_{x+r:y+r}^{02}\right)dr\right)$$

Comment:

Most candidates scored partial credit for this part. Some candidates wrote down a few steps, but missed the key parts of the proof. An acceptable alternative approach was to plug the given solution into the Kolmogorov equation and demonstrate that the integral equation for ${}_t p_{xy}^{00}$ *satisfies the Kolmogorov differential equation and the boundary condition. This approach was awarded full points if done correctly.*

WA.18. Continued

(c) (i) At time 10, (x) is age 60 and (y) is age 65. The value at time 10 of the joint and reversionary annuities is

$$50,000\bar{a}^{00}_{60:65} + 30,000\bar{a}^{01}_{60:65} + 30,000\bar{a}^{02}_{60:65}$$

$$= (50,000)(8.8219) + (30,000)(1.3768) + (30,000)(3.0175) = 572,924$$

(ii) The EPV of the deferred annuity for the case where both lives survive 10 years uses the results from (i):

$${}_{10}p^{00}_{50:55} \cdot v^{10} \cdot 572,924 = (0.86041)(0.61391)(572,924) = 302,627$$

The EPV of the deferred annuity for the case where (y) survives 10 years but (x) does not, is

$${}_{10}p^{01}_{50:55} \cdot v^{10}(30,000\bar{a}^{11}_{65}) = (0.04835)(0.61391)(30,000)(10.1948) = 9078$$

The EPV of the deferred annuity for the case where (x) survives 10 years but (y) does not, is

$${}_{10}p^{02}_{50:55} \cdot v^{10}(30,000\bar{a}^{22}_{60}) = (0.08628)(0.61391)(30,000)(11.8302) = 18,799$$

Let P denote the premium. Then the EPV of the benefit paid on the second death during the deferred period is

$$3P\bar{A}^{03}_{\overset{\;\;1}{50:55}:\overline{10}|} = 3P(0.003421) = (0.010263)P$$

Then by the equivalence principle, $P = 302,627 + 9078 + 18,799 + 0.010263P$

$$\Rightarrow P = \frac{330,504}{0.989737} = 333,931$$

Comments: *Only the top few candidates achieved full credit for this part. Many candidates did not allow for the possibility that only one life would survive the deferred period. The lower scoring candidates used combinations of probabilities and annuities that indicated less than adequate understanding of multiple state models and notation. For example, the expression* $({}_{10}p^{01}_{50:55}\bar{a}^{01}_{60:65})$ *is meaningless as it requires the lives to be in state 0 and also in state 1at time 10. The second superscript of* ${}_t p^{ij}$ *in this time of combination must be the same as the first superscript of* $\bar{a}^{jk}$ *.*

WA.18. Continued

(d) (i)

$${}_{10}V^{(0)} = 572{,}924 \quad \text{from (c)(i)}$$
$${}_{10}V^{(1)} = 30{,}000\bar{a}^{11}_{65} = (30{,}000)(10.1948) = 305{,}844$$
$${}_{10}V^{(2)} = 30{,}000\bar{a}^{22}_{60} = (30{,}000)(11.8302) = 354{,}906$$

(ii) For $t \geq 10$ and $\delta = \ln(1.05)$:

$$\frac{d}{dt}\,{}_tV^{(0)} = \delta\,{}_tV^{(0)} - 50{,}000 - \mu^{01}_{50+t:55+t}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{50+t:55+t}\left({}_tV^{(2)} - {}_tV^{(0)}\right)$$

(iii) We need $\mu^{01}_{60:65} = A + Bc^{60} = 0.009076$ and $\mu^{02}_{60:65} = A + Bc^{65} = 0.015919$. Then

$${}_{t+h}V^{(0)} \approx {}_tV^{(0)} + h\left(\delta\,{}_tV^{(0)} - 50{,}000 - \mu^{01}_{50+t:55+t}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{50+t:55+t}\left({}_tV^{(2)} - {}_tV^{(0)}\right)\right)$$

$$\approx 572{,}924 + 0.5\{(0.048790)(572{,}924) - 50{,}000 - (0.009076)(305{,}844 - 572{,}924) - (0.015919)(354{,}906 - 572{,}924)\}$$

$$= 564{,}848$$

Comment:

As in (c), a few of the very best candidates achieved full credit for this part. Many students correctly calculated the three reserves in (i). Some did not understand that, for example, ${}_tV^{(1)}$ is the reserve assuming the policy is in state 1 at time t, that is that (50) has already died, so the correct annuity factor must be $\bar{a}^{11}_{65}$, not $\bar{a}^{01}_{60:65}$. Common errors in (ii) and (iii) included omitting the release of reserve terms involving ${}_tV^{(0)}$ on the right hand side, omitting the annuity paid in state 0, including annuities paid in the other states (these are implicitly valued in ${}_tV^{(1)}$ and ${}_tV^{(2)}$), and getting one or more signs wrong.

WA.19.

General Comment:
This question was a very high scoring question with a large number of candidates achieving full credit.

(a) <u>Death in Year 1:</u>

$L_0 \mid Event = (1000 + G(1+i))v - G = 1000v = 943.4$

Probability $= 0.06$

<u>Withdrawal in Year 1:</u>

$L_0 \mid Event = -G$

Probability $= 0.04$

<u>Death in Year 2:</u>

$L_0 \mid Event = (1000 + G(1+i) + G(1+i)^2)v^2 - G(1+v) = 1000v^2 = 890.0$

Probability $= (0.90)(0.12) = 0.108$

<u>Survival in force to end of year 2:</u>

$L_0 \mid Event = -G(1+v) = -1.9434G$

Probability $= (0.9)(0.88) = 0.792$

In the table form:

Event	Value of L_0, Given that the Event Occurred	Probability of Event
Death in year 1	943.4	0.06
Withdrawal in year 1	$-G$	0.04
Death in year 2	890.0	0.108
Neither death or withdrawal	$-1.9434G$	0.792

Comments:
Most candidates did well in this part. Some candidates confused dependent and independent rates of mortality and withdrawal. A few calculated an equivalence principle premium, even though it was not required or relevant. Amongst candidates who did not achieve full marks, the most common problem was determining the amount of the return of premiums benefit.

(b) (i)

$E[L_0] = (943.4)(0.06) + (-G)(0.04) + (890.0)(0.108) + (-1.9434G)(0.792)$

$= 152.7 - 1.579G$

$\Rightarrow a = 152.7 \quad \text{and} \quad b = 1.579$

WA.19. Continued

(ii)

$$E[L_0^2] = (943.4)^2(0.06) + (-G)^2(0.04) + (890.0)^2(0.108) + (-1.9434G)^2(0.792)$$
$$= 138,947 + 3.0312G^2$$

$$E[L_0]^2 = (152.7 - 1.579G)^2 = 23,317 - 482.2G + 2.4932G^2$$

$$V[L_0] = E[L_0^2] - E[L_0]^2 = 115,630 + 482.2G + 0.538G^2$$

$$\Rightarrow c = 0.538 \quad d = 482.2 \quad e = 115,630$$

Comments:
The table in part (a) was used by stronger candidates to answer part (b), as the examiners' intended. Standard variance formulas for level benefit term insurance do not work in this case, and candidates who tried to use memorized formulas received little or no credit.

(c) For each policy:
(d)

$$E[L_0] = a - bG = -52.57$$
$$V[L_0] = cG^2 + dG + e = 187,408$$

So for the aggregate loss

$$E[L_{agg}] = (200)(-52.57) = -10,514$$
$$V[L_{agg}] = (200)(187,408) = 37,481,600 = (6122.2)^2$$

$$\Rightarrow \Pr[L_{agg} > 0] = 1 - \Phi\left(\frac{0-(-10,514)}{6122.2}\right) = 1 - \Phi(1.72)$$
$$= 1 - 0.9573 = 0.0427$$

Comments:
Some candidates used the wrong tail of the Normal distribution for this part. Otherwise, this was done well.

WA.20.

General Comments:

This was the lowest scoring question on the test in which it appeared. There was some evidence that candidates were pressed for time, but also evidence that candidates struggled with some of the concepts covered. Generally, passing candidates understood what to do with a force of interest that varies with time (from Exam FM), and, further, understood that conditional variance was needed, and knew how to find it.

(a) Let Y_j denote the PV of benefits for the jth life, $Y = \sum_{j=1}^{100} Y_j$

Let $v(5)$ denote the discount factor for time 5. That is

$$v(5) = \begin{cases} \exp(-\int_0^5 0.03t^{0.5}\,dt) & \text{with a probability of } 0.6 \\ \exp(-\int_0^5 0.02\,dt) & \text{with a probability of } 0.4 \end{cases}$$

$$v(5) = \begin{cases} \exp(-\left[0.02t^{3/2}\right]_0^5 = \exp(-0.22361) = 0.79963 & \text{with a probability of } 0.6 \\ \exp(-0.1) = 0.90484 & \text{with a probability of } 0.4 \end{cases}$$

Also ${}_5p_{85} = \dfrac{41{,}841.1}{61{,}184.9} = 0.68385$

$$\Rightarrow E[Y_j \mid v(5)] = 0.68385v(5)$$

$$\Rightarrow E[Y_j \mid v(5)] = \begin{cases} 0.54683 & \text{with a probability of } 0.6 \\ 0.61877 & \text{with a probability of } 0.4 \end{cases}$$

$$\Rightarrow E[Y \mid v(5)] = \begin{cases} 54.683 & \text{with a probability of } 0.6 \\ 61.877 & \text{with a probability of } 0.4 \end{cases}$$

$$\Rightarrow E[Y] = E[E[Y \mid v(5)]] = 57.561$$

Comments:

The most common error was incorrect calculation of $v(5)$.

WA.20. Continued

(b)

$$V[Y] = E[V[Y \mid v(5)]] + V[E[Y \mid v(5)]]$$
$$E[Y \mid v(5)] = 100v(5)\,{}_5p_{85}$$
$$V[Y \mid v(5)] = 100[v(5)]^2({}_5p_{85})(1-{}_5p_{85})$$

so we have:

$$E[Y \mid v(5)] = \begin{cases} 54.683 & \text{with a probability of } 0.6 \\ 61.877 & \text{with a probability of } 0.4 \end{cases}$$

$$V[Y \mid v(5)] = \begin{cases} 13.824 & \text{with a probability of } 0.6 \\ 17.701 & \text{with a probability of } 0.4 \end{cases}$$

so

$$E[V[Y \mid v(5)]] = (0.6)(13.824) + (0.4)(17.701) = 15.375$$
$$V[E[Y \mid v(5)]] = (54.683)^2(0.6) + (61.877)^2(0.4) - (57.561)^2 = 12.375$$

$$\Rightarrow V[Y] = 15.375 + 12.375 = 27.750 = 5.268^2$$

So the required probability is

$$\Pr[Y<50] = \Phi\left(\frac{50-57.561}{5.268}\right) = \Phi(-1.43) = 1-\Phi(1.43) = 1-0.9236 = 0.0764$$

Comments:

The most common error for this part was omitting the second part of the conditional variance calculation.

WA.21.

(a) Nancy has a Defined Benefit plan. A Defined Contribution pension plan specifies how much an employer will contribute, usually as a percentage of salary, into a plan. The plan may allow employees to also contribute. The contributions are accumulated in a notional account which is available to the employee when he or she leaves the company. The contributions may be set to meet a target benefit level, but the actual retirement income may be well below or above the target, depending on investment experience.

To complete Part (b) and (c), we note that Nancy will retire at age 60 or 61. Therefore we will need to know how much benefit has been accrued for both 60 and 61. We will also need to know the monthly annuity values at age 60 and 61. Using the 2-term Woolhouse approximation we have

$$\ddot{a}_{60}^{(12)} = \ddot{a}_{60} - \frac{11}{24} = 14.4458 \quad \text{and} \quad \ddot{a}_{61}^{(12)} = \ddot{a}_{61} - \frac{11}{24} = 14.1908$$

(b) (i) Under the Projected Unit Credit cost method, the actuarial accrued liability is the actuarial present value of the projected benefit. The projected benefit is equal to the final average salary at the decrement date multiplied by service as of the valuation date and by the accrual rate.

We have the following information.

Projected Final Average Salary at 60	$50,000[(1.03)^3 + (1.03)^4 + (1.03)^5]/3 = 56,292$
Projected Final Average Salary at 61	$50,000[(1.03)^4 + (1.03)^5 + (1.03)^6]/3$ $= (56,292)(1.03) = 57,981$
Service at valuation date	25
Accrual Rate	0.016
Projected Benefit for retirement at 60 (PB_{60})	(56,292)(0.016)(25) = 22,517
Projected Benefit for retirement at *61* (PB_{61})	(57,981)(0.016)(25) = 23,192

The actuarial accrued liability is the actuarial present value (as of the valuation date) of the projected benefit and is given by

$$\begin{aligned}\text{AAL}_{55} &= PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{55}^{(\tau)}} \cdot v^5 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{During} + i_{60} + l_{61}^{(\tau)}}{l_{55}^{(\tau)}} v^6 \\ &= (22,517)(14.4458)\left(\frac{27,925.6}{104,687.7}\right)(1.05)^{-5} + (23,192)(14.1908)\left(\frac{6187.6 + 61.9 + 58,699.9}{104,687.7}\right)(1.05)^{-6} \\ &= 220,351\end{aligned}$$

WA.21. **Continued**

(ii) We now need the accrued benefit at age 56

Projected Benefit for retirement at 60 (PB_{60})	(56,292)(0.016)(26) = 23,417
Projected Benefit for retirement at 61 (PB_{61})	(57,981)(0.016)(26) = 24,120

The actuarial accrued liability at December 31, 2016 is given by

$$\text{AAL}_{56} = PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{56}^{(\tau)}} \cdot v^4 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{During} + i_{60} + l_{61}^{(\tau)}}{l_{56}^{(\tau)}} v^5$$

$$= (23,417)(14.4458)\left(\frac{27,925.6}{102,307.9}\right)(1.05)^{-4} + (24,120)(14.1908)\left(\frac{6187.6 + 61.9 + 58,699.9}{102,307.9}\right)(1.05)^{-5}$$

$$= 246,221$$

Then

$${}_tV + C_t = EPV \text{ of benefits for mid-year exits} + v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V \quad \text{where:}$$

C_t = Normal Cost for year t to $t+1$ and ${}_tV$ is the Actuarial Accrued Liability at time t
Note that EPV of benefits for mid-year exits is zero. Then:

$${}_tV + C_t = EPV \text{ of benefits for mid-year exits} + v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V$$

$$220,351 + C_t = 0 + (1.05)^{-1}\left(\frac{102,307.9}{104,687.7}\right)(246,221)$$

$$C_t = 8815$$

WA.21. **Continued**

(c) (i)

Under the Traditional Unit Credit cost method the actuarial accrued liability (AAL) is the actuarial present value of the accrued benefit on the valuation date.
The formula for the accrued benefit, *B*, is

$$B_{55} = 0.016 \cdot 25 \cdot 50{,}000 \cdot \left(\frac{1 + (1.03)^{-1} + (1.03)^{-2}}{3} \right) = 19{,}423$$

and thus

$$\begin{aligned} \text{AAL}_{55} &= B_{55} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{55}^{(\tau)}} \cdot v^5 + B_{55} \cdot \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{during} + i_{60} + l_{61}^{(\tau)}}{l_{55}^{(\tau)}} v^6 \\ &= B_{55} \left(\ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{55}^{(\tau)}} \cdot v^5 + \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{during} + i_{60} + l_{61}^{(\tau)}}{l_{55}^{(\tau)}} v^6 \right) \end{aligned}$$

$$= 19{,}423 \left[(14.4458) \left(\frac{27{,}925.6}{104{,}687.7} \right) (1.05)^{-5} + (14.1908) \left(\frac{6{,}187.6 + 61.9 + 58{,}699.9}{104{,}687.7} \right) (1.05)^{-6} \right]$$

$$= 186{,}248$$

WA.21. **Continued**

(ii)

For the NC, we must calculate the expected accrued benefit, B_{61}, one year after the valuation date.

$$B_{56} = 0.016 \cdot 26 \cdot 50,000 \cdot \left(\frac{1.03 + 1 + 1.03^{-1}}{3} \right) = 20,806$$

$$\text{AAL}_{56} = B_{56} \cdot \left(\ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{56}^{(\tau)}} \cdot v^4 + \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{during} + i_{60} + l_{61}^{(\tau)}}{l_{56}^{(\tau)}} v^5 \right)$$

$$= 20,806 \left[(14.4458) \left(\frac{27,925.6}{102,307.9} \right) (1.05)^{-4} + (14.1908) \left(\frac{6,187.6 + 61.9 + 58,699.9}{102,307.9} \right) (1.05)^{-5} \right]$$

$$= 214,358$$

Then:

${}_tV + C_t = EPV$ of benefits for mid-year exits $+ v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V$ where:

$C_t =$ Normal Cost for year t to $t+1$ and ${}_tV$ is the Actuarial Accrued Liability at time t

Note that EPV of benefits for mid-year exits is zero. Then:

${}_tV + C_t = EPV$ of benefits for mid-year exits $+ v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V$

$$186,248 + C_t = 0 + (1.05)^{-1} \left(\frac{102,307.9}{104,687.7} \right) (214,358)$$

$$C_t = 13,262$$

(d)

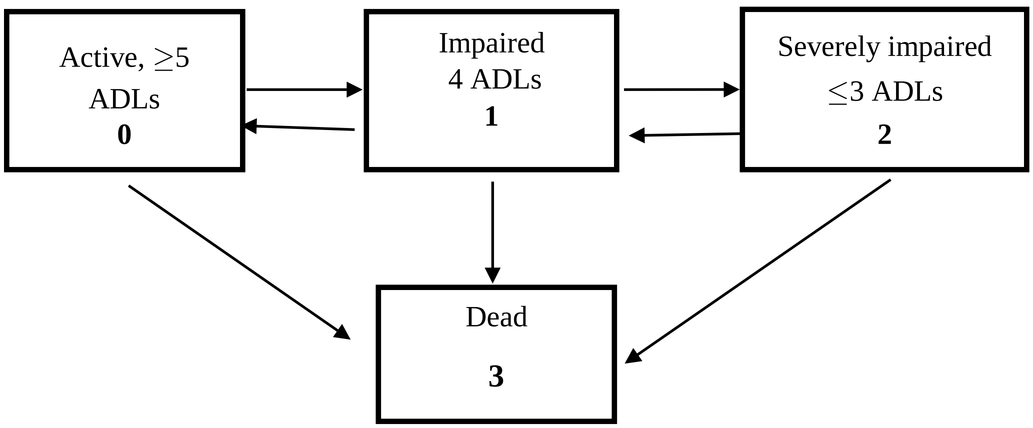
Under both funding approaches, the contribution rate tends to increase as the member acquires more service and gets closer to retirement. The TUC contributions start smaller than the PUC contributions and, rise more steeply, ending at considerably more than the PUC contribution. This is true because the TUC contributions do not project future salary increases or future service credit while PUC contributions are based on salary with projected future salary increases and assuming future service credits will be earned. Therefore, the TUC contributions in any given year must reflect the full increase in the salary and the additional year of service now reflected in the accrued benefit at the end of the year of service which were not reflected in the accrued benefit at the end of the year of service. The PUC contribution will not need to reflect the salary increase as it was already reflected in the accrued liability at the beginning of the year.

WA.22.

(a) The six common ADLs are

- Bathing
- Dressing
- Eating
- Toileting
- Continence
- Transferring

(b) One model is given below. Others may also be appropriate.



(c)

(i) Over the course of the policy, Sheila was active for 12 + 12 + 12 + 6 + 4 + 12 +12 + 12 + 12 + 6 = 100, months, thereby paying a total of 100 × 150 = 15,000 in premium. For her first disability, (6 - 3) × 1000 = 3000 in benefits were paid; for her second disability, (8-3) × 2000 = 10,000 in benefits were paid. Since the sum of the premiums paid exceeds the disability benefits paid out under the policy, under the "return of premium" approach, the remainder of 15,000-13,000 = 2000 is added to the death benefit, for a total death benefit of 102,000.

(ii) As before, over the course of the policy, 13,000 in benefits were paid. This amount is deducted from the sum insured of 100,000, leaving a death benefit payment of 87,000.

WA.22. Continued

(d)

(i) With a 12 month off period, the second disability, which starts 9 months after the end of the first disability period, is not subject to the 3 month waiting period. Hence, benefit payments will commence immediately for this disability, increasing the long term care benefits paid by $3 \times 2000 = 6000$. However, since the total long term care benefits paid (19,000) would now exceed the total premium paid (15,000), there would no longer be a return of premium added to the death benefit, lowering it by 2000. Hence, the total benefits paid would increase by 4000.

(ii) Again, in this scenario, the long term care benefits paid would increase by 6000. However, the death benefit paid would be 6,000 less than before. Hence, there is no net change in total benefits paid by the policy.

WA.23.

(a) $${}_{t+h}p_x^{12} = {}_tp_x^{11}\,{}_hp_{x+t}^{12} + {}_tp_x^{12}\,{}_hp_{x+t}^{22} \quad \text{(Markov property)}$$

$$= {}_tp_x^{11}\,{}_hp_{x+t}^{12} + {}_tp_x^{12}\left(1 - {}_hp_{x+t}^{23} - {}_hp_{x+t}^{24}\right) \quad \text{(complete probability)}$$

$$= {}_tp_x^{12} + {}_tp_x^{11}\,{}_hp_{x+t}^{12} - {}_tp_x^{12}\,{}_hp_{x+t}^{23} - {}_tp_x^{12}\,{}_hp_{x+t}^{24}$$

$$\Rightarrow \frac{{}_{t+h}p_x^{12} - {}_tp_x^{12}}{h} = {}_tp_x^{11}\left(\frac{{}_hp_{x+t}^{12}}{h}\right) - {}_tp_x^{12}\left(\frac{{}_hp_{x+t}^{23}}{h}\right) - {}_tp_x^{12}\left(\frac{{}_hp_{x+t}^{24}}{h}\right)$$

and, where the transition intensity exists, $\mu_{x+t}^{ij} = \lim_{h\to 0}\frac{{}_hp_{x+t}^{ij}}{h}$, and taking limits gives

$$\frac{d}{dt}\,{}_tp_x^{12} = {}_tp_x^{11}\mu_{x+t}^{12} - {}_tp_x^{12}\mu_{x+t}^{23} - {}_tp_x^{12}\mu_{x+t}^{24}$$

(b) $${}_tp_x^{12} = \int_0^t {}_rp_x^{11}\,\mu_{x+r}^{12}\,{}_{t-r}p_{x+r}^{22}\,dr = \int_0^t e^{-0.24r}\times 0.10\times e^{-0.34(t-r)}dr$$

$$= 0.10(e^{-0.34t})\int_0^t e^{0.1r}dr = 0.10(e^{-0.34t})\frac{e^{0.1t}-1}{0.1} = e^{-0.24t} - e^{-0.34t}$$

(c) The EPV is $3000\bar{a}_{90}^{12}$ where

$$\bar{a}_{90}^{12} = \int_0^\infty {}_tp_{90}^{12}\,e^{-\delta t}dt = \int_0^\infty\left(e^{-0.24t} - e^{-0.34t}\right)e^{-0.05t}dt$$

$$= \int_0^\infty e^{-0.29t} - e^{-0.39t}\,dt = \frac{1}{0.29} - \frac{1}{0.39}$$

$$= 0.88417$$

So the EPV is $3000(0.88417) = 2652.5$

(d) $$\bar{a}_{90:\overline{0.5}|}^{\overline{22}} = \int_0^{0.5} {}_tp_{90}^{\overline{22}}\,e^{-\delta t}dt = \int_0^{0.5} e^{-0.34t}\,e^{-0.05t}dt = \frac{1-e^{-0.5\times 0.39}}{0.39} = 0.4543$$

WA.23. **Continued**

(e) The EPV of a benefit of 1 per year payable continuously after the waiting period is

$$\int_0^\infty {}_r p_{90}^{11}\, \mu_{90+r}^{12} \left(\bar{a}_{90+r}^{\overline{22}} - \bar{a}_{90+r:\overline{0.5}|}^{\overline{22}}\right) e^{-\delta r}\, dr \quad \text{where} \quad \bar{a}_{90+r}^{\overline{22}} = \frac{1}{0.39} = 2.5641$$

So the EPV is

$$\int_0^\infty e^{-0.24r} \times 0.10 \times \left(2.5641 - 0.4542\right) e^{-0.05r}\, dr = 0.21099 \int_0^\infty e^{-0.29r}\, dr = \frac{0.21099}{0.29}$$

$=0.7276$

So for a benefit of 3000 per year the EPV is 3000×0.7276=2182.7.

(f) Reason 1: Short term payments involve relatively high expenses. Once the illness has extended beyond the waiting period, it is likely to be more significant and less costly relative to the benefits

Reason 2: In some cases the policyholders will have other sources of income for short term sickness, eg sick pay from employment.

Reason 3: Offer policyholders a choice, for the same premium they will receive higher benefits for long term sickness in return for giving up benefit for shorter bouts.

WA.24.

(a) Using the table below we get an estimate for $S(80)$ of 0.1714.

i	y_i	s_i	b_i	r_i	λ_i	$S(y)$	Range for $S(y)$
0	0					1.0000	[0,12)
1	12	1	0	10	0.1000	0.9000	[12,35)
2	35	1	1	9	0.1111	0.8000	[35,59)
3	59	1	2	7	0.1429	0.6857	[59,73)
4	73	2	0	4	0.5000	0.3429	[73,80)
5	80	1	1	2	0.5000	0.1714	[80,90)

(b)(i) $$Var(S(80)) \approx (S(80))^2 \sum_{i:y_i \le 80} \frac{s_i}{r_i(r_i - s_i)}$$ \

$$= (0.1714)^2 \left(\frac{1}{10\times 9} + \frac{1}{9\times 8} + \frac{1}{7\times 6} + \frac{2}{4\times 2} + \frac{1}{2\times 1} \right) = 0.02347 = 0.1532^2$$

$\Rightarrow$ 95% Confidence Interval is approx. $(0.1714 \pm 1.96 \times 0.1532)$

$= \quad (-0.1289, 0.4717)$

(ii) The log-transformed 95% confidence interval is $\left(\left(\hat{S}(y)\right)^{1/u}, \left(\hat{S}(y)\right)^{u} \right)$, where

$$u = \exp\left(\frac{1.96\sqrt{Var(\hat{S}(y))}}{\hat{S}(y)\ln(\hat{S}(y))} \right) \approx \exp\left(\frac{1.96 \times 0.1532}{0.1714 \times (-1.7638)} \right) = 0.3703$$

giving a confidence interval $\left(0.1714^{2.701}, 0.1714^{0.3703}\right) = (0.0085, 0.5204)$.

(iii) In this case the linear confidence interval for S(y) gives a lower bound which is less than 0. The log-confidence interval method gives a confidence interval that will always generate bounds between 0 and 1.

WA.24. Continued

(c) In both cases the Kaplan-Meier estimate of $S(y)$ for $80 \le y < 90$ is $\hat{S}(y) = 0.1714$. At 90 and above, Efron's tail correction yields an estimate of $\hat{S}^*(y) = 0$, whereas Brown, Hollander and Korwar's tail correction is $\hat{S}\ (y) = \left(\hat{S}(80)\right)^{y/90}$.

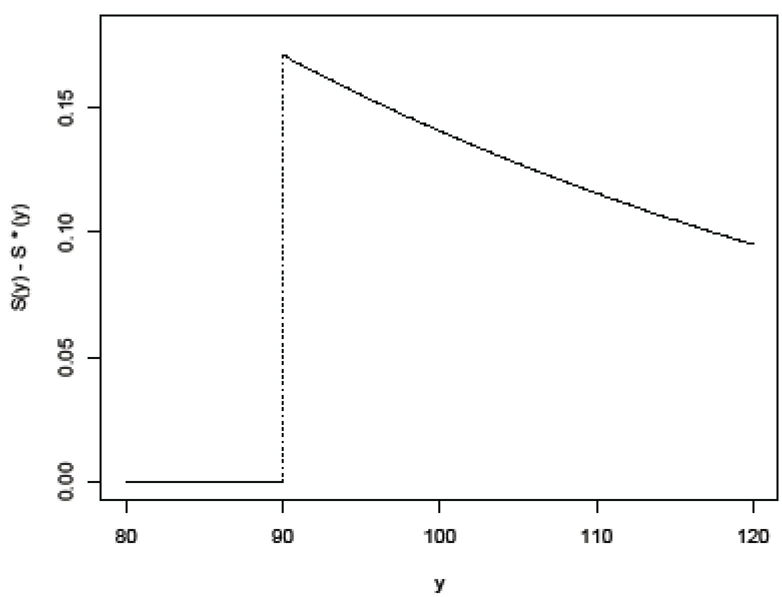
The key values of $\hat{S}(y) - \hat{S}^*(y)$ are:

$$\hat{S}(y) - \hat{S}^*(y) = 0 \qquad 80 \le y < 90$$

$$\hat{S}(y) - \hat{S}^*(y) = 0.1714 \qquad y = 90$$

$$\lim_{y\to\infty} \left[\hat{S}(y) - \hat{S}^*(y)\right] = 0 \qquad y > 90$$

A plot of the difference is shown below.



WA.25.

(a) One advantage of using stochastic mortality improvement models is that they allow us to assess the impact of random variation in the underlying mortality rates in our model. While stochastic models still have parameter uncertainty, they nonetheless allow us to get a sense of how much difference the uncertainty in our mortality model will make to our results. Deterministic mortality improvement models do not allow us to do this.

(b) For this model we have:

$$lq(72,2019) = K_{2019}^{(1)} + K_{2019}^{(2)}(72-75)$$

$$K_{2019}^{(1)} = -1.9 - 0.03 + 0.02\,Z_{2019}^{(1)}$$

$$K_{2019}^{(2)} = 0.3 + 0.001 + 0.02\,Z_{2019}^{(2)}$$

$$lq(72,2019) = -1.9 - 0.03 + 0.02\,Z_{2019}^{(1)} - 3(0.3 + 0.001 + 0.02\,Z_{2019}^{(2)})$$

$$\Rightarrow lq(72,2019) = -2.833 + 0.02Z_{2019}^{(1)} - 0.06Z_{2019}^{(2)}$$

Note that both $Z_{2019}^{(1)}$ and $Z_{2019}^{(2)}$ are standard N(0,1) random variables, so that $lq(72,2019)$ also has a normal distribution.

(i) $$E\left[lq(72,2019)\right] = E\left[-2.833 + 0.02Z_{2019}^{(1)} - 0.06Z_{2019}^{(2)}\right] = -2.833$$

(ii) $$Var\left[lq(72,2019)\right] = Var\left[-2.833 + 0.02Z_{2019}^{(1)} - 0.06Z_{2019}^{(2)}\right]$$

$$= Var\left[0.02Z_{2019}^{(1)} - 0.06Z_{2019}^{(2)}\right]$$

$$= (0.02)^2 Var[Z_{2019}^{(1)}] + 0.06^2 Var[Z_{2019}^{(2)}] - 2(0.02)(0.06)Cov[Z_{2019}^{(1)}, Z_{2019}^{(2)}]$$

$$= 0.02^2 + 0.06^2 - 2(0.02)(0.06)(0.3)$$

$$= 0.00328 = 0.057^2$$

$$\Rightarrow \text{Standard Deviation } = 0.057$$

(c) $$lq(72,2019) \sim \mathrm{N}(-2.833, 0.057^2)$$

$$\Rightarrow e^{lq(72,2019)} = \frac{q(72,2019)}{1-q(72,2019)} \sim \ln N(-2.833, 0.057)$$

$$\Rightarrow E\left[\frac{q(72,2019)}{1-q(72,2019)}\right] = e^{-2.833+0.057^2/2} = 0.0589$$

WA.25. **Continued**

(d) As the logit and logarithm functions are monotonic increasing functions, we have

$$\begin{aligned}
\Pr\left[p(72,2019) \geq 0.94\right] &= \Pr\left[q(72,2019) \leq 0.06\right] \\
&= \Pr\left[\frac{q(72,2019)}{1-q(72,2019)} \leq \frac{0.06}{1-0.06}\right] \\
&= \Pr\left[\ln\left(\frac{q(72,2019)}{1-q(72,2019)}\right) \leq \ln\left(\frac{0.06}{1-0.06}\right)\right] \\
&= \Pr\left[lq(72,2019) \leq -2.7515\right] \\
&= \Phi\left(\frac{-2.7515-(-2.833)}{0.057}\right) = \Phi(1.43) \\
&= 0.9236
\end{aligned}$$

(e) The CBD M7 model introduces a model term G_{t-x} that can be used to describe a cohort effect; modeling a cohort effect is not possible in the original CBD model, so if we want to include such an effect, CBD M7 is a more advantageous choice. Another possible advantage is that the CBD M7 includes a quadratic age difference term, which may result in an improved model.

WA.26.

(a)(i) EPV of future costs is

$$12\times 3000\times \ddot{a}_{65}^{(12)00}+12\times 7500\times \ddot{a}_{65}^{(12)01}+12\times 15000\times \ddot{a}_{65}^{(12)02}=600,321.6$$

So F is 25% of the EPV = 150,080

(a)(ii) The monthly fee is M where

$$12M(\ddot{a}_{65}^{(12)00}+\ddot{a}_{65}^{(12)01}+\ddot{a}_{65}^{(12)02})=0.75(600321.6)\Rightarrow M=2854.95$$

(b)(i) The reserve is EPV future outgo – EPV future fee income, so

$$ {}_5V^{(0)}=12(3000)\ddot{a}_{70}^{(12)00}+12(7500)\ddot{a}_{70}^{(12)01}+12(15000)\ddot{a}_{70}^{(12)02} -12(2854.95)\left(\ddot{a}_{70}^{(12)00}+\ddot{a}_{70}^{(12)01}+\ddot{a}_{70}^{(12)02}\right)$$

$$=183,563$$

(b)(ii) Now we have

$$ {}_5V^{(1)}=12\times 7500\times \ddot{a}_{70}^{(12)11}+12\times 15000\times \ddot{a}_{70}^{(12)12}-12\times 2854.95\times\left(\ddot{a}_{70}^{(12)11}+\ddot{a}_{70}^{(12)12}\right)$$

$$=712,340$$

(c) By recursion, we have

$$\left({}_{4\frac{11}{12}}V^{(0)}+M-3000\right)(1.05)^{\frac{1}{12}}={}_{\frac{1}{12}}p_{69\frac{11}{12}}^{00}\;{}_5V^{(0)}+{}_{\frac{1}{12}}p_{69\frac{11}{12}}^{01}\;{}_5V^{(1)}+{}_{\frac{1}{12}}p_{69\frac{11}{12}}^{02}\;{}_5V^{(2)}$$

$$ {}_5V^{(2)}=12\times(15000-2854.95)\times \ddot{a}_{70}^{(12)22}=1,340,245$$

$$\Rightarrow {}_{4\frac{11}{12}}V^{(0)}=\left(0.94937(183,563)+0.00906(712,340)+0.00003(1,340,245)\right)v^{\frac{1}{12}}+145.05$$

$$=180,175$$

(d)(i) The equation of value for F is now

$$F=0.25\left(600,321.6+0.5FA_{65}^{(12)03}\right)=150,080+0.04501F$$

$$\Rightarrow F=157,153$$

(d)(ii) The equation of value for M is now

$$12M(\ddot{a}_{65}^{(12)00}+\ddot{a}_{65}^{(12)01}+\ddot{a}_{65}^{(12)02})=0.75(600321.6+0.5FA_{65}^{(12)03})$$

$$\Rightarrow M=2989.52$$

(d)(iii) The time 5 reserve in state 0 will increase, as it now comprises the original time 5 reserve, plus the amount required to support 75% of the cost of the refund of fees.