

Homework Problems
Stat 479

Chapter 13

80. You are given the following random sample of 3 data points from a population with a Pareto distribution with $\theta = 70$:

X: 15 27 43

Calculate the maximum likelihood estimate for α .

Solution:

$$\begin{aligned} L(\alpha) &= f(15)f(27)f(43) = \frac{\alpha\theta^\alpha}{(15+\theta)^{\alpha+1}} \frac{\alpha\theta^\alpha}{(27+\theta)^{\alpha+1}} \frac{\alpha\theta^\alpha}{(43+\theta)^{\alpha+1}} = \frac{\alpha^3\theta^{3\alpha}}{[(15+\theta)(27+\theta)(43+\theta)]^{\alpha+1}} \\ &= \frac{\alpha^3(70)^{3\alpha}}{[(15+70)(27+70)(43+70)]^{\alpha+1}} = \frac{\alpha^3(70)^{3\alpha}}{[(15+70)(27+70)(43+70)]^{\alpha+1}} = \frac{\alpha^3(70)^{3\alpha}}{[931,685]^{\alpha+1}} \end{aligned}$$

$$l(\alpha) = 3 \ln(\alpha) + 3\alpha [\ln(70)] - (\alpha + 1) \ln(931,685)$$

$$l'(\alpha) = \frac{3}{\alpha} + 3[\ln(70)] - \ln(931,685) = 0 \implies \alpha = \frac{3}{\ln(931,685) - 3[\ln(70)]} = 3.0022$$

(81)

$$L(\theta) = [F(1000)]^7 [F(2000) - F(1000)]^6 [1 - F(2000)]^7$$

$$(1 - e^{-\frac{1000}{\theta}})^7 (e^{-\frac{1000}{\theta}} - e^{-\frac{2000}{\theta}})^6 (e^{-\frac{2000}{\theta}})^7$$

$$= (1 - e^{-\frac{1000}{\theta}})^7 (e^{-\frac{1000}{\theta}})^6 (1 - e^{-\frac{1000}{\theta}})^6 (e^{-\frac{2000}{\theta}})^7$$

$$= (1 - e^{-\frac{1000}{\theta}})^{13} (e^{-\frac{20000}{\theta}})$$

$$l(\theta) = -\frac{20000}{\theta} + 13 \ln(1 - e^{-\frac{1000}{\theta}})$$

$$l'(\theta) = +20000 \theta^{-2} + 13 \left[\frac{1}{1 - e^{-\frac{1000}{\theta}}} \right] (-e^{-\frac{1000}{\theta}}) 1000 \theta^{-2} = 0$$

$$\Rightarrow 20 = 13 \left[\frac{e^{-\frac{1000}{\theta}}}{1 - e^{-\frac{1000}{\theta}}} \right] \Rightarrow 20 - 20e^{-\frac{1000}{\theta}} = 13e^{-\frac{1000}{\theta}}$$

$$\Rightarrow 20 = 33e^{-\frac{1000}{\theta}} \Rightarrow \ln\left(\frac{20}{33}\right) = -\frac{1000}{\theta}$$

$$\Rightarrow \theta = \frac{-1000}{\ln\left(\frac{20}{33}\right)} = 1996.90$$

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$$L(\theta) = f(1.5) \cdot f(1.9) = \frac{2(\theta-1.5)}{\theta^2} \cdot \frac{2(\theta-1.9)}{\theta^2}$$
$$= (4)(\theta-1.5)(\theta-1.9)(\theta^{-4})$$

$$L(\theta) = \ln 4 + \ln(\theta-1.5) + \ln(\theta-1.9) - 4 \ln \theta$$

$$L'(\theta) = 0 + \frac{1}{\theta-1.5} + \frac{1}{\theta-1.9} - \frac{4}{\theta} = 0$$

$$(\theta)(\theta-1.9) + (\theta)(\theta-1.5) - 4(\theta-1.9)(\theta-1.5) = 0$$

$$-2\theta^2 + 4.2\theta - 1.8 = 0$$

$$\theta = \frac{-4.2 \pm \sqrt{(4.2)^2 - (4)(-2)(-1.8)}}{-4} = 1.5 \text{ or } 0.6$$

but $\theta > 1.9$ because

$0 < x < \theta$ and one $x = 1.9$

so 1.5

$$\textcircled{83} \quad \hat{\theta} = \frac{\text{Total}}{\# \text{ of uncensored}} = \frac{0.3 + 0.55 + 0.80}{3} = 0.55$$

3 still alive

$$\textcircled{84} \quad \hat{\theta} = \frac{\text{Total days lived}}{\# \text{ of uncensored}} = \frac{(2)(1) + (3)(2) + (4)(1) + (5)(3) + 5(3)}{7}$$

7
do not include
3 still alive

= 6

$$\textcircled{85} \quad \hat{\theta} = \frac{\text{Total Paid}}{\# \text{ of uncensored}} = \frac{15 + 50 + 170 + 216 + 400 + 620 + 750 + 3(900)}{7}$$

= 703

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86. * Four lives are observed from time $t = 0$ until death. Deaths occur at $t = 1, 2, 3,$ and 4 . The lives are assumed to follow a Weibull distribution with $\tau = 2$.

Determine the maximum likelihood estimator for θ .

Solution:

$$L(\theta) = f(1)f(2)f(3)f(4)$$

$$= \left(\frac{2(1/\theta)^2 e^{-(1/\theta)^2}}{1} \right) \left(\frac{2(2/\theta)^2 e^{-(2/\theta)^2}}{2} \right) \left(\frac{2(3/\theta)^2 e^{-(3/\theta)^2}}{3} \right) \left(\frac{2(4/\theta)^2 e^{-(4/\theta)^2}}{4} \right)$$

$$= \frac{16(24)^2 (\theta)^{-8} e^{-(30)(\theta)^{-2}}}{24} = 384(\theta)^{-8} e^{-(30)(\theta)^{-2}}$$

$$l(\theta) = \ln 384 - 8 \ln(\theta) - 30(\theta)^{-2}$$

$$l'(\theta) = 0 - \frac{8}{\theta} - (30)(-2)(\theta)^{-3} = 0 \implies 8(\theta)^2 = (30)(2) \implies \theta = \left(\frac{60}{8} \right)^{1/2} = 2.73861$$

$$\textcircled{87} \hat{\theta} = \left(\frac{3}{2}\right)(5) = 7.5$$

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88. * A random sample of three claims from a dental insurance plan is given below:

225 525 950

Claims are assumed to follow a Pareto distribution with parameters $\theta = 150$ and α .

Determine the maximum likelihood estimate of α .

Solution:

$$L(\theta) = f(225)f(525)f(950) = \left(\frac{\alpha(150)^\alpha}{(225+150)^{\alpha+1}}\right)\left(\frac{\alpha(150)^\alpha}{(525+150)^{\alpha+1}}\right)\left(\frac{\alpha(150)^\alpha}{(950+150)^{\alpha+1}}\right)$$

$$= \frac{\alpha^3(150)^{3\alpha}}{[(375)(675)(1100)]^{\alpha+1}}$$

$$l(\theta) = 3 \ln[\alpha] + 3\alpha \ln(150) - (\alpha + 1) \ln[(375)(675)(1100)]$$

$$l'(\theta) = \frac{3}{\alpha} + 3 \ln(150) - \ln[(375)(675)(1100)] = 0$$

$$\alpha = \frac{3}{\ln[(375)(675)(1100)] - 3 \ln(150)} = 0.67984$$

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89. * The following claim sizes are experienced on an insurance coverages:

100 500 1,000 5,000 10,000

You fit a normal distribution to this experience using maximum likelihood.

Determine the resulting estimate of σ .

Solution:

$$\hat{\mu} = \bar{X} = \frac{100 + 500 + 1000 + 5000 + 10,000}{5} = 3320$$

$$\hat{\sigma} = \sqrt{\frac{\sum (x - \hat{\mu})^2}{n}}$$

$$= \sqrt{\frac{(100 - 3320)^2 + (500 - 3320)^2 + (1000 - 3320)^2 + (5000 - 3320)^2 + (10,000 - 3320)^2}{5}}$$

$$= 3772.21$$

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90. You have the following four claims:

10 20 30 40

You believe that the claims for this insurance are distributed as an exponential distribution with a mean of θ .

You use the maximum likelihood estimate to determine $\hat{\theta}$.

Determine the variance of $\hat{\theta}$.

Solution:

$$\hat{\theta} = \bar{X} = \frac{10+20+30+40}{4} = 25$$

$$\text{Var}(\hat{\theta}) = \frac{\theta^2}{n} = \frac{25^2}{4} = \frac{625}{4} = 156.25$$

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91. * Losses from a group health policy follow an exponential distribution with unknown mean. A sample of losses is:

100 200 400 800 1400 3100

Use the delta method to

- a. Approximate the variance of the maximum likelihood estimator of $S(1500)$.

Solution:

We know that the MLE estimate of θ is \bar{X} . Then

$$\hat{\theta} = \frac{100 + 200 + 400 + 800 + 1400 + 3100}{6} = 1000. \text{ We also know that}$$

$$\text{Var}[\hat{\theta}] = \frac{\theta^2}{n} = \frac{(1000)^2}{6}$$

$S(1500) = 1 - F(1500) = e^{-1500/\theta}$ for an exponential distribution. Therefore, since this is a function of θ , we can use the delta method.

$$g(\theta) = e^{-1500/\theta} \text{ and } g'(\theta) = 1500\theta^{-2}e^{-1500/\theta}$$

$$S(1500) \doteq g(\hat{\theta}) = e^{-1500/1000} = 0.22313$$

$$\text{Var}[S(1500)] = (g'(\theta))^2 \left(\frac{\sigma^2}{n} \right) \doteq (1500\theta^{-2}e^{-1500/\theta})^2 \left(\frac{\theta^2}{n} \right) = \frac{(1500)^2 \theta^{-2} e^{-3000/\theta}}{n}$$

$$= \frac{(1500)^2 (1000)^{-2} e^{-3000/1000}}{6} = 0.01867$$

- b. Determine the approximate 80% confidence interval for $S(1500)$.

Solution:

$$95\% \text{ CI} = 0.22313 \pm 1.282 \sqrt{0.01867} = 0.22313 \pm 0.17517$$

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92. You fit an inverse exponential sample of claims using maximum likelihood to estimate θ . The resulting estimate of θ is 2.07. The variance of the estimate is 0.00264.

$$\text{Let } Y = E[X^{-1}].$$

Use the delta method to estimate the variance of Y .

Solution:

$$E[X^k] = \theta^k \Gamma(1-k)$$

$$g(\theta) = E[X^{-1}] = \theta^{-1} \Gamma(1 - (-1)) = \theta^{-1} \Gamma(2) = \theta^{-1} (2-1)! = \theta^{-1}$$

$$g'(\theta) = \frac{-1}{\theta^2}$$

$$\text{Var}[Y] \doteq (g'(\theta))^2 \left(\frac{\sigma^2}{n} \right) = \left(\frac{-1}{\theta^2} \right)^2 = \frac{0.00264}{(2.07)^2} = 0.000144$$

~~136~~ $\hat{\theta} = \frac{\bar{X}}{m} = \frac{1.72}{6} = 0.28\bar{6}$

~~137~~ Spreadsheet

~~138~~ $\lambda = \bar{X} = 1.72$ $S^2 = \frac{\lambda}{n} = \frac{1.72}{100}$

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Confidence Interval = $1.72 \pm 1.645 \sqrt{\frac{1.72}{100}}$

= $1.72 \pm 1.645 (.13114877)$

= $(1.50426, 1.93574)$

~~139~~ $D(200) = F_n(200) - F^*(200)$

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$F_n(200) = 0.50$ $F^*(200) = 1 - e^{-\frac{200}{427.50}} = 0.3736$

$D(200) = 0.50 - 0.3736 = 0.1264$

~~140~~ coordinates of $x_7 = \left(\frac{i}{n+1}, F^*(x_i) \right)$

95

= $\left(\frac{7}{21}, 1 - e^{-\frac{120}{427.50}} \right) = \left(\frac{1}{3}, .2447 \right)$

~~141~~ All False

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X	$F_n(x^-)$	$F_n(x)$	$F^*(x)$	(MAX DIFF)
100	0	.25	.1812	.1812
350	.25	.50	.5034	.2534
550	.50	.75	.6671	.1671
1000	.75	1.00	.8647	.1353

$1 - e^{-\frac{x}{500}}$ \nearrow
 MAX = .2534

K-S test statistic = D = .2534

critical value = $\frac{1.22}{\sqrt{n}} = \frac{1.22}{\sqrt{4}} = 0.61$

Since $D = 0.2534 < 0.61$ we cannot reject H_0

10% \Rightarrow reject if $A^2 > 1.933$	} definitions
5% \Rightarrow reject if $A^2 > 2.492$	
1% \Rightarrow reject if $A^2 > 3.857$	

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X	$F_n(x^-)$	$F_n(x)$	$F^*(x)$	Max Difference
1.6	0	.5	.64	.64
1.7	.5	.75	.7225	.225
1.9	.75	1.00	.9025	.1525

$F^*(x) = \int_0^x f(u) du = \int_0^x \frac{u}{2} du = \frac{u^2}{4} \Big|_0^x = \frac{x^2}{4}$

critical value

10% $\Rightarrow 1.22/\sqrt{n} = 1.22/2 = .61$

since $D < .61$, REJECT

1% $\Rightarrow 1.63/\sqrt{n} = 1.63/2 = .815$

since $D > .815$ DO NOT REJECT

D = max = .64

(B)

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X	$F_2(x^-)$	$F_2(x)$	$F^*(x)$	Max difference
5	0	.5	.5	.5
9	.5	1	.9	.4

↑
 $1 - S(x) = \frac{7}{10}$

↑
MAX = D = .50

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100

X	$F_9(x^-)$	$F_9(x)$	$F^*(x)$	MAX DIFFERENCE
1	0	1/9	1/11	9/99
2	1/9	2/9	2/11	7/99
4	2/9	3/9	4/11	14/99
5	3/9	5/9	5/11	14/99
7	5/9	6/9	7/11	8/99
8	6/9	7/9	8/11	6/99
9	7/9	9/9	9/11	18/99

D = MAX = $\frac{18}{99} =$

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observed	$E_j = n[F^*(c_j) - F^*(c_{j-1})]$	$\chi^2 = \frac{(E_j - O_j)^2}{E_j}$
0-100	27.398	.2472
100-200	18.625	2.1821
200-500	29.150	2.8722
500-1000	15.975	0.0596
1000+	8.8519	0.1489

$\chi^2 = 5.5100$

Critical Values
 degrees of freedom = $5 - 1 - 2 = 2$ ← Two parameters were estimated
 critical value = 4.605
 Conclusion: Reject H_0 since $5.51 > 4.605$

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Range	Observed	Expected	$\chi^2 = \frac{(E_j - O_j)^2}{E_j}$
0	220	$365 e^{-0.625} = 195.37$	3.105
1	100	$365(0.625)e^{-0.625} = 122.11$	4.003
2	30	$365(0.10454) = 38.16$	1.745
3	10	7.95	.529
4+	5	1.41	9.14

$\chi^2 = 18.522$

Critical values

degree of freedom = $5 - 1 = 4$
critical value = 7.779

Note: Unless the problem tells you do not regroup

Conclusion

Reject because $\chi^2 > 7.779$

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We want expected claims to be at least 5
We have 30 claims so our probability must be at least $\frac{1}{6}$ to have 5. Therefore combine first two groups & last 3 groups

Group	observed	Expected	$\chi^2 = \frac{(E_j - O_j)^2}{E_j}$
0 - 500	3	$(30)(.27) = 8.1$	3.2111
500 - 2498	8	$(30)(.55 - .27) = 8.4$	0.0190
2498 - 4876	9	$(30)(.81 - .55) = 7.8$	0.1846
4876 +	10	$(30)(1 - .81) = 5.4$	3.2439

$\chi^2 = 6.6586$

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$$\text{Test statistic} = 2 [\ln(L(\theta_1)) - \ln(L(\theta_0))]$$

$$= 2 (\ln(1.3) - \ln(-1)) = -1.204 - (-2.303)$$

$$= 2.197$$

Critical value

$$\text{degrees of freedom} = 2 - 0 = 2$$

$$\text{critical value} = 4.605$$

← free parameters in H_1
← free parameters in H_0

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False

True

True

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a) $u^{**} = .6$ Pareto $\Rightarrow F_X(x^{**}) = 1 - \left(\frac{\theta}{x^{**} + \theta}\right)^2$

$$.6 = 1 - \left(\frac{2000}{2000 + x^{**}}\right)^3 \Rightarrow \frac{2000}{2000 + x^{**}} = \sqrt[3]{.4}$$

$$\Rightarrow x^{**} = \frac{2000}{\sqrt[3]{.4}} - 2000 = 714.42$$

b) Since $F(x^{**}) = .6$ for $1 \leq x < 2.4$
you use the greatest value = 2.4

c) $F(15^-) = .5$ therefore for $u^{**} = .6$ $x^{**} = 15$
 $F(15) = .75$

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$$F(x) = \int_0^x f(x) dx = \int_0^x \frac{x^2}{9} dx = \frac{x^3}{27} \Big|_0^x = \frac{x^3}{27}$$

$$u^{**} = \frac{x^{**3}}{27} \Rightarrow x^{**} = 3 \sqrt[3]{u^{**}}$$

$$u^{**} = .008 \Rightarrow x^{**} = 3(.2) = .6$$

$$u^{**} = .729 \Rightarrow x^{**} = 3(.9) = 2.7$$

$$u^{**} = .125 \Rightarrow x^{**} = 3(.5) = 1.5$$

$$\bar{x}^{**} = \frac{.6 + 2.7 + 1.5}{3} = 1.6$$

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$$m = 2 \quad q = .3$$

$$p_0 = (1-q)^m = (.7)^2 = .49$$

Therefore all simulated values less than or equal to .49 are mapped to zero

so answer = 2 (.21 and .48)

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$p_0 = e^{-2} = .135335$	Σ .135335
$p_1 = 2e^{-2} = .270671$.406005
$p_2 = \frac{4e^{-2}}{2} = .270671$.6766764
$p_3 = \frac{8e^{-2}}{6} = .180447$.857123

since the first random number = 0.7
we have 3 claims

For CLAIM AMOUNTS

$$u^{**} = 1 - e^{-\frac{x^{**}}{1000}} \Rightarrow x^{**} = -1000 \ln(1 - u^{**})$$

$$u^{**} = 0.1 \Rightarrow x^{**} = 105.36$$

$$u^{**} = 0.5 \Rightarrow x^{**} = 693.15$$

$$u^{**} = 0.8 \Rightarrow x^{**} = 1609.43$$

Kyle pays
105.36

500.00

500.00

1105.36

TOTAL

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Simulated cost = $15000 \pm z(2000)$

- $u^{**} = .5398 \Rightarrow \chi^{**} = 15000 + 2000(.1) = 15,200$
- $u^{**} = .1151 \Rightarrow \chi^{**} = 15000 - 2000(1.2) = 12,600$
- $u^{**} = .0013 \Rightarrow \chi^{**} = 15000 - 2000(3) = 9,000$
- $u^{**} = .7881 \Rightarrow \chi^{**} = 15000 + 2000(.8) = 16,600$

claims

- $15,200 - 10,000 = 5,200$
- $12,600 - 10,000 = 2,600$
- $9000 - 9000 = -0 -$
- $16,600 - 10,000 = 6,600$
- 14,400

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See #8159

	<u>Σ</u>
$p_0 = 0.135335$.135
$p_1 = 0.270671$.406
$p_2 = 0.270671$.676
$p_3 = 0.180447$.857

$u^{**} = .8 \Rightarrow 3 \text{ claims}$

Amount of claims

$$u^{**} = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha = 1 - \left(\frac{500}{x+500}\right)^2 \Rightarrow \chi^{**} = \frac{500}{\sqrt{1-u^{**}}} - 500$$

- $u^{**} = .60 \Rightarrow \chi^{**} = 290.57$
- $u^{**} = .25 \Rightarrow \chi^{**} = 77.35$
- $u^{**} = .70 \Rightarrow \chi^{**} = 412.87$

TOTAL = $290.57 + 77.35$
 $+ 412.87 = 780.79$
 Amt Pd. = $(.8)(79) + 30.79$
 $= \underline{\underline{630.79}}$