Chapter 9 - Level 3 - Course FM
Solutions

ONLY CERTAIN PROBLEMS HAVE SOLUTIONS. THE REMAINING WILL BE ADDED OVER TIME.

1. (F11HW) Rivera Insurance Company has committed to paying 10,000 at the end of one year and 40,000 at the end of two years. Its Chief Financial Officer, Miguel, wants to exactly match this obligation using the following two bonds:

   Bond A is a one year bond which matures at par of 1000 and pays and annual dividend at a rate of 6%. This bond can be bought to yield 6% annually.

   Bond B is a two year bond which matures at par off 1000 and pays an annual dividend at a rate of 10%. This bond can be bought to yield 7% annually.

   Calculate the amount of each bond the Rivera should purchase.

   Calculate the cost of Rivera to exactly match this obligation.

   a. 6914.40
   b. 42,692.63
   c. 42,403.77
   d. 44,339.45
   e. 48,169.60

   10,000 = 1060a + 100b
   40,000 = 1100b

   From the last equation,
   b = 40,000/1100 = 36.363636

   Then,
   $a = \dfrac{10,000 - 100(36.363636)}{1060} = 6.0034305$

   Price Bond 1: N=1 I/Y=6% PMT=60 FV=1000 PVCPT=1000
   (We also know intuitively that FV equals PV since the coupon rate is the same as the yield)

   Price Bond 2: N=2 I/Y=7% PMT=100 FV=1000 PVCPT=1054.240545

   Total Price = (6.0034305)(1000) + (36.363636)(1054.24054) = 44,339.45

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2. (S12HW) Wenda owns an 8 year bond with a par value of 1000. The bond matures for par and pays semi-annual coupons at a rate of 6% convertible semi-annually.

   Calculate the Modified duration of this bond at an annual effective interest rate of 8.16%.

   a. 6.3524
   b. 5.8732
   c. 6.108
   d. 5.983
   e. 6.472

   Use calculator to find price of bond:
   PMT =30 FV-1000 N=16 I/Y=4 CPTPV=883.48

$$\text{Modified Duration} = \frac{\sum C_t t v^t}{\sum C_t v^t} v = \frac{30(0.5)v^{0.5} + 30(1)v + 30(1.5)v^{1.5} + \ldots + 30(8)v^8 + 1000(8)v^8}{883.48(1.0816)}$$

$$= \frac{15a_{\overline{16}|0.04} + \frac{15}{0.04}\left(a_{\overline{16}|0.04} - 16(1.04)^{-16}\right) + 8000(1.0816)^8}{883.48(1.0816)} = 5.8732$$

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3. (F11HW) A five year bond matures for 20,000. The bond pays coupons of:

a) 3000 at the end of the first year,
b) 1500 at the end of the second year,
c) 1000 at the end of the third year,
d) 750 at the end of the fourth year, and
e) 600 at the end of the fifth year.

Calculate the Macaulay Duration of this bond at 5%.

a. 1.3123
b. 3.917
c. 4.1824
d. 3.730

$$\text{Macaulay Duration} = \frac{\sum C_t t v^t}{\sum C_t v^t}$$
$$= \frac{3000v + 1500(2)v^2 + 1000(3)v^3 + 750(4)v^4 + 600(5)v^5 + 20000(5)v^5}{3000v + 1500v^2 + 1000v^3 + 750v^4 + 600v^5 + 20000v^5}$$
$$= \frac{3000\left(v + v^2 + v^3 + v^4 + v^5\right) + 100000v^5}{3000v + 1500v^2 + 1000v^3 + 750v^4 + 600v^5 + 20000v^5}$$
$$= 4.1824$$

4. (F11HW) Tokoly Investments owns a preferred stock which pays a quarterly dividend of $5 per quarter with the next dividend paid in 3 months. Calculate the modified duration of this stock at an annual effective rate of 8.243216%.

a. 11.44
b. 12.13
c. 12.75
d. 11.89
e. 11.78

Use $(1.08243216)^{.25} = 1.02$ for i

$$\frac{\sum C_t t v^t}{\sum C_t v^t} v = \frac{5(.25)v^{.25} + 5(.5)v^{.5} + \cdots}{\frac{5}{.02}(1.08243216)} = \frac{1.25\left(\frac{1}{.02} + \frac{1}{.02^2}\right)}{\frac{5(1.08243216)}{.02}} = 11.78$$

5. (S09T3) A 3 year bond has annual coupons.

   The coupon at the end of the first year is 100.

   The coupon at the end of the second year is 300.

   The coupon at the end of the third year is 500.

   The bond matures for 700.

   Calculate the modified convexity of this bond at an annual effective rate of interest 6%.

   a. 9.025
   b. 10.141
   c. 7.473
   d. 2.516
   e. 4.613

   First find the Price function and its derivatives:

$$P(i) = 100(1+i)^{-1} + 300(1+i)^{-2} + 1200(1+i)^{-3}$$
$$P'(i) = -100(1+i)^{-2} - 600(1+i)^{-3} - 3600(1+i)^{-4}$$
$$P''(i) = 200(1+i)^{-3} + 1800(1+i)^{-4} + 14400(1+i)^{-5}$$

   Modified Convexity:

$$\frac{P''(i)}{P(i)} = \left(\frac{200(1.06)^{-3} + 1800(1.06)^{-4} + 14400(1.06)^{-5}}{100(1.06)^{-1} + 300(1.06)^{-2} + 1200(1.06)^{-3}}\right)$$
$$= \frac{12354.21}{1368.88} = 9.025$$

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6. (S09T3) Jenna owns the following portfolio.

Asset	Price	Macaulay Duration	Macaulay Convexity
Bond 1	25,000	6.0	40
Bond 2	30,000	4.5	25
Bond 3	45,000	3.0	12

The price, Macaulay Duration, and Macaulay Convexity were calculated at an annual effective rate of 5%.

Estimate the price of the portfolio at an annual effective rate of interest of 7% using both the duration and convexity.

a. 92,058
b. 92,491.61
c. 92,036.19
d. 92,000
e. 91,508.39

First find the Convexity and the Duration of the Portfolio:

$$D^{Portfolio}(i,\infty)=\left(\frac{25,000}{100,000}\right)(6)+\left(\frac{30,000}{100,000}\right)(4.5)+\left(\frac{45,000}{100,000}\right)(3)=4.2$$

$$C^{Portfolio}(i,\infty)=\left(\frac{25,000}{100,000}\right)(40)+\left(\frac{30,000}{100,000}\right)(25)+\left(\frac{45,000}{100,000}\right)(12)=22.9$$

Then find the Modified Convexity:

$$C(i,m)=\frac{C(i,\infty)+D(i,\infty)}{(1+i)^2}=\frac{22.9+4.2}{(1.05)^2}=24.5805$$

Now we can find the change in price:

$$\Delta P=-\left(\frac{D(i_0,\infty)}{1+i_0}\right)P(i_0)\Delta i+C(i,m)\left(\frac{P(i_0)(\Delta i)^2}{2}\right)$$

$$=-\left(\frac{4.2}{1.05}\right)100,000(.02)+24.5805\left(\frac{100,000(.02)^2}{2}\right)=-7508.39$$

$$P(7\%)=P(5\%)+\Delta P=100,000-7,508.39=92,491.61$$

7. (F11PP) Sun wants to fully immunize a future payment of 100,000 at time 10 using the following two bonds:
a. A zero coupon bond maturing in 5 years; and
b. A zero coupon bond maturing in 20 years.

Determine the amount that Sun should spend on each bond at an annual effective interest rate of 10%.

a. A=12,851.44, B=25, 702.89
b. A=66,666.67, B=33,333.33
c. A=33,333.33, B=66,666.67
d. A=25,702.89, B=12,851.44
e. A=19,277.16, B=77,108.66

Present Value Matching:
Let A= Present value of 5 year bond
Let B= Present value of 20 year bond
PV(Assets)=PV(Liabilities)

$$A+B=100,000(1.1)^{-10}$$

Duration Matching:
Duration(Assets)=Duration(Liabilities)

$$5A+20B=10(100,000(1.1)^{-10})$$

Now we have two equations with two unknowns. We can solve for A and B.

$$A=100,000(1.1)^{-10}-B$$

$$A=38,554.32894-B$$

$$5(38,554.32894-B)+20B=10(38,554.32894)$$

$$15B=385,543.2894-192,771.6447$$

$$B=12,851.44$$

$$A=38,554.32894-12,851.44$$

$$A=25,702.89$$

8. (F11PP) Lauren wants to fully immunize a future payment of X at time Y using the following two bonds:
a. Bond A is a zero coupon bond maturing in 2 years; and
b. Bond B is a zero coupon bond maturing in 10 years.

Lauren pays 13,622.79 for Bond A and 6,192.18 for Bond B. Determine X and Y if the annual effective interest rate is 5%.

a. X=28,570.19, y=7.5
b. X=24,680, y= 4.5
c. X=15,908.95, y=4.5
d. X=13,742.75, y=7.5

Present Value Matching:

$x(1.05)^{-y} = 13622.79 + 6192.18$

Duration Matching:

$yx(1.05)^{-y} = 2(13622.79) + 10(6192.18)$

Now we have two equations with two unknowns, so we can solve for x and y:

$x = 19814.97(1.05)^{y}$

$y(19814.97(1.05)^{y})(1.05)^{-y} = 89167.38$

$y = \dfrac{89167.38}{19814.97}$

$y = 4.5$

$x = 19814.97(1.05)^{4.5}$

$x = 24680.00$

9. (FMP07) Calculate the Macaulay duration of a perpetuity immediate of 1 less the Macaulay duration of a perpetuity due of 1 at an interest rate of 10%.

a. 0.0
b. 0.9
c. 1.0
d. 1.1
e. 1.2

10. (S07PQ) Calculate the Macaulay duration of a perpetuity immediate less the Macaulay duration of a perpetuity due at an interest rate of 10%.

    a. 0.0
    b. 0.9
    c. 1.0
    d. 1.1
    e. 1.2

$$\frac{\frac{1}{.1}+\frac{1}{.1^2}}{\frac{1}{.1}}-\frac{\frac{1}{.1}+\frac{1}{.1^2}}{1+\frac{1}{.1}}=\frac{110}{10}-\frac{110}{11}=1$$

11. (S12PQ) A perpetuity pays 10 at the end of each year for the first 10 years and 20 at the end of each year thereafter.

    Calculate the Macaulay duration of this perpetuity at an annual effective interest rate of 5%.

$$\frac{\frac{20}{.05}+\frac{20}{.05^2}-\left[10\left(\frac{1-1.05^{-10}}{.05}\right)+\frac{10}{.05}\left(\frac{1-1.05^{-10}}{.05}-10(1.05)^{-10}\right)\right]}{\frac{20}{.05}-10\left(\frac{1-1.05^{-10}}{.05}\right)}$$

$$=\frac{400+8000-(77.217+316.52)}{400-77.217}=24.80$$

12. (S12PP) Calculate the Modified convexity at 7% of an annuity which has increasing payments. There are four payments as follows:

a. Payment of 1000 at the end of one year;
b. Payment of 2000 at the end of two years;
c. Payment of 4000 at the end of three years; and
d. Payment of 8000 at the end of four years.

$$\frac{v^2\sum C_t(t)(t+1)v^t}{\sum C_t v^t}$$

$$=1.07^{-2}\left[\frac{1000(1)(2)(1.07^{-1})+2000(2)(3)(1.07^{-2})+4000(3)(4)(1.07^{-3})+8000(4)(5)(1.07^{-5})}{1000(1.07^{-1})+2000(1.07^{-2})+4000(1.07^{-3})+8000(1.07^{-4})}\right]$$

$$=1.07^{-2}\left[\frac{1869.158879+10481.26474+39182.29809+122063.2339}{934.5794393+1746.877457+3265.191508+6103.161696}\right]$$

$$=1.07^{-2}\left[\frac{173595.9556}{12049.8101}\right]=12.58$$

13. (S12PP) Purdue Life Insurance Company has agreed to make the following payments to Nancy:

    a. 40,000 at the end of one year;
    b. 20,000 at the end of two years; and
    c. 10,000 at the end of three years.

    Purdue wants to exactly match its liability to Nancy using the following three bonds:

    a. A one year bond maturing for 1000 with annual coupons of 60. The price of the bond is 990.
    b. A two year bond maturing for 2000 with annual coupons of 100. This bond has a price based on a annual yield of 8%.
    c. A three year bond with a price of 1000. The bond matures for 1000 and has annual coupons of 90.

    Calculate the cost of the portfolio of bonds the Purdue would purchase.

$$C(1090) = 1000$$
$$C = 9.174311927$$

$$9.174311927(90) + B(2100) = 20000$$
$$B = 9.130624727$$

$$9.174311927(90) + 9.130624727(100) + A(1060) = 40000$$
$$A = 36.09551835$$
$$P_B : N = 2, I/Y = 8, FV = 2000, PMT = 100,$$
$$(CPT)PV = 1893.004115$$

$$36.09551835(990) + 9.130624747(1893.004115) + 9.174311927(1000)$$
$$= 62193.19$$

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14. (S12PP) You are given the following three annuities:

   a. A one year annuity immediate with annual payments of 950. The annuity has a present value of 900.

   b. A two year annuity immediate with annual payments of 500 which has a present value of 900.

   c. A three year annuity immediate with annual payments of 360. The present value of the annuity is 900.

   Determine $f_{[2,3]}$.

$$900(1+r_1)=950$$
$$(1+r_1)=1.0555555$$

$$900=500(1.0555555^{-1})+500(1+r_2)^{-2}$$
$$(1+r_2)^{-2}=.8563158$$
$$(1+r_2)=1.082977149$$

$$900=360(1.0555555)^{-1}+360(1.082977149)^{-2}+360(1+r_3)^{-3}$$
$$(1+r_3)^{-3}=.961144469$$
$$(1+r_3)=1.126247881$$

$$(1.082977149)^2(1+f_{2,3})=1.126247881^3$$
$$(1+f_{2,3})=1.218045116$$
$$f_{2,3}=21.80\%$$

15. (S12PQ) An annuity immediate has geometrically increasing payments. The payments are made annually. The first payment is P and each subsequent payment is 106% of the prior payment.

    The Macaulay duration of this annuity when calculated at an interest rate of 6% is 3.

    Calculate the Modified convexity of this annuity at an interest rate of 6%.

$$3 = \frac{\sum C_t (t) v^t}{\sum C_t v^t}$$

$$= \frac{P(1)(\frac{1}{1.06}) + P(2)(1.06)(\frac{1}{1.06})^2 + P(3)(1.06)^2(\frac{1}{1.06})^3 + \cdots + P(n)(1.06)^{n-1}(\frac{1}{1.06})^n}{P(\frac{1}{1.06}) + P(1.06)(\frac{1}{1.06})^2 + P(1.06)^2(\frac{1}{1.06})^3 + \cdots + P(1.06)^{n-1}(\frac{1}{1.06})^n}$$

$$= \frac{P[(1)(\frac{1}{1.06}) + (2)(1.06)(\frac{1}{1.06})^2 + (3)(1.06)^2(\frac{1}{1.06})^3 + \cdots + (n)(1.06)^{n-1}(\frac{1}{1.06})^n]}{P[(\frac{1}{1.06}) + (1.06)(\frac{1}{1.06})^2 + (1.06)^2(\frac{1}{1.06})^3 + \cdots + (1.06)^{n-1}(\frac{1}{1.06})^n]}$$

$$= \frac{(\frac{1}{1.06})[1+(2)(\frac{1.06}{1.06})+(3)(\frac{1.06}{1.06})^2+\cdots+(n)(\frac{1.06}{1.06})^{n-1}]}{(\frac{1}{1.06})[(1+(\frac{1.06}{1.06})+(\frac{1.06}{1.06})^2+\cdots+(\frac{1.06}{1.06})^{n-1}]}$$

$$= \frac{1+2+3+\cdots+n}{1+1+1+\cdots+1^n}$$

$$= \frac{\frac{n+1}{2}n}{n}$$

$$3 = \frac{n+1}{2}$$

$$6 = n+1$$

$$n = 5$$

$$\frac{P(1)(2)(\frac{1}{1.06})+P(2)(3)(1.06)(\frac{1}{1.06})^2+P(3)(4)(1.06)^2(\frac{1}{1.06})^3+P(4)(5)(1.06)^3(\frac{1}{1.06})^4+P(5)(6)(1.06)^4(\frac{1}{1.06})^5}{(1.06)^{-2}(P(\frac{1}{1.06})+P(1.06)(\frac{1}{1.06})^2+P(1.06)^2(\frac{1}{1.06})^3+P(1.06)^3(\frac{1}{1.06})^4+P(1.06)^4(\frac{1}{1.06})^5)}$$

$$= (1.06)^{-2}\ \frac{P[(1)(2)(\frac{1}{1.06})+(2)(3)(1.06)(\frac{1}{1.06})^2+(3)(4)(1.06)^2(\frac{1}{1.06})^3+(4)(5)(1.06)^3(\frac{1}{1.06})^4+(5)(6)(1.06)^4(\frac{1}{1.06})^5]}{P[(\frac{1}{1.06})+(1.06)(\frac{1}{1.06})^2+(1.06)^2(\frac{1}{1.06})^3+(1.06)^3(\frac{1}{1.06})^4+(1.06)^4(\frac{1}{1.06})^5]}$$

$$= (1.06)^{-2}\ \frac{(\frac{1}{1.06})[2+6(\frac{1.06}{1.06})+12(\frac{1.06}{1.06})^2+20(\frac{1.06}{1.06})^3+30(\frac{1.06}{1.06})^4]}{(\frac{1}{1.06})[1+(\frac{1.06}{1.06})+(\frac{1.06}{1.06})^2+(\frac{1.06}{1.06})^3+(\frac{1.06}{1.06})^4]}$$

$$= (1.06)^{-2}\ \frac{2+6+12+20+30}{1+1+1+1+1}$$

$$= (1.06)^{-2}\ \frac{70}{5}$$

$$= (1.06)^{-2}(14)$$

$$= 12.46$$

16. (S12PP) An insurance company has a liability where they must pay 100,000 in 8 years. The company wants to immunize this payment at 6% interest using the following two assets:

   a. A zero coupon bond which matures in 4 years; and
   b. A zero coupon bond which matures in 10 years.

   The Company will spend X to purchase the 4 year bond and Y to purchase the 10 year bond.

   Determine Y-X.

$$100000(1.06)^{-8} = 62741.23713 = a + b$$
$$62741.23713 - a = b$$

$$\tfrac{4a+10b}{a+b} = 8$$
$$\tfrac{4a+10(62741.23713-a)}{a+62741.23713-a} = 8$$
$$\tfrac{627412.3713-6a}{62731.23713} = 8$$
$$627412.3713 - 6a = 501929.8971$$
$$6a = 125482.4742$$
$$a = 20913.7457$$

$$b = 627412.23713 - 20913.7457$$
$$b = 41827.49143$$

$$41827.49142 - 20913.7457$$
$$= 20913.75$$

17. (S12PQ) An insurance company has a liability where they must pay 100,000 in 8 years. The company wants to immunize this payment at 6% interest using the following two assets:

    a. A zero coupon bond which matures in 4 years; and
    b. A zero coupon bond which matures in 10 years.

    If X is the maturity value of the 4 year bond and Y is the maturity value of the 10 year bond, determine Y-X.

$$100000(1.06)^{-8} = 62741.23713 = a + b$$
$$62741.23713 - a = b$$

$$\frac{4a+10b}{a+b} = 8$$
$$\frac{4a+10(62741.23713 – a)}{a+62741.23713 – a} = 8$$
$$\frac{627412.3713 – 6a}{62731.23713} = 8$$
$$627412.3713 - 6a = 501929.8971$$
$$6a = 125482.4742$$
$$a = 20913.7457$$

$$41827.49142 - 20913.7457$$
$$= 20913.75$$
$$X = 20913.7457(1.06)^{-4} = 26403.12209$$

$$Y = 41827.49143(1.06)^{-10} = 74906.66667$$

$$74906.66667 - 26403.12209 = 48503.54$$