Supplementary Notes for
Actuarial Mathematics for Life Contingent Risks

Mary R. Hardy PhD FIA FSA CERA

David C. M. Dickson PhD FFA FIAA

Howard R. Waters DPhil FIA FFA

Introduction

This note is provided as an accompaniment to 'Actuarial Mathematics for Life Contingent Risks' by Dickson, Hardy and Waters (2009, Cambridge University Press).

Actuarial Mathematics for Life Contingent Risks (AMLCR) includes almost all of the material required to meet the learning objectives developed by the SOA for exam MLC for implementation in 2012. In this note we aim to provide the additional material required to meet the learning objectives in full. This note is designed to be read in conjunction with AMLCR, and we reference section and equation numbers from that text. We expect that this material will be integrated with the text formally in a second edition.

There are four major topics in this note. Section 1 covers additional material relating to mortality and survival models. This section should be read along with Chapter 3 of AMLCR.

The second topic is policy values and reserves. In Section 2 of this note, we discuss in detail some issues concerning reserving that are covered more briefly in AMLCR. This material can be read after Chapter 7 of AMLCR.

The third topic is Multiple Decrement Tables, discussed in Section 3 of this note. This material relates to Chapter 8, specifically Section 8.8, of AMLCR. It also pertains to the Service Table used in Chapter 9.

The final topic is Universal Life insurance. Basic Universal Life should be analyzed using the methods of Chapter 11 of AMLCR, as it is a variation of a traditional with profits contract, but there are also important similarities with unit-linked contracts, which are covered in Chapter 12.

The survival models referred to throughout this note as the Standard Ultimate Survival Model (SUSM) and the Standard Select Survival Model (SSSM) are detailed in Sections 4.3 and 6.3 respectively, of AMLCR.

Contents

1 Survival models and assumptions

1.1 The Balducci fractional age assumption

This section is related to the fractional age assumption material in AMLCR, Section 3.2.

We use fractional age assumptions to calculate probabilities that apply to non-integer ages and/or durations, when we only have information about integer ages from our mortality table. Making an assumption about $_sp_x$, for $0 \leq s < 1$, and for integer x, allows us to use the mortality table to calculate survival and mortality probabilities for non-integer ages and durations, which will usually be close to the true, underlying probabilities. The two most useful fractional age assumptions are Uniform Distribution of Deaths (UDD) and constant force of mortality (CFM), and these are described fully in Section 3.2 of AMLCR.

A third fractional age assumption is the Balducci assumption, which is also known as harmonic interpolation. For integer x, and for $0 \leq s \leq 1$, we use an approximation based on linear interpolation of the reciprocal survival probabilities – that is

$$\frac{1}{_sp_x} = (1-s)\frac{1}{_0p_x} + s\frac{1}{p_x} = 1 - s + \frac{s}{p_x} = 1 + s\left(\frac{1}{p_x} - 1\right).$$

Inverting this to get a fractional age equation for $_sp_x$ gives

$$_sp_x = \frac{p_x}{p_x + s\, q_x}.$$

The Balducci assumption had some historical value, when actuaries required easy computation of $_sp_x^{-1}$, but in a computer age this is no longer an important consideration. Additionally, the underlying model implies a piecewise decreasing model for the force of mortality (see the exercise below), and thus tends to give a worse estimate of the true probabilities than the UDD or CFM assumptions.

Example SN1.1 Given that $p_{40} = 0.999473$, calculate $_{0.4}q_{40.2}$ under the Balducci assumption.

Solution to Example SN1.1 As in Solution 3.2 in AMLCR, we have $_{0.4}q_{40.2} = 1 - {}_{0.4}p_{40.2}$ and

$$_{0.4}p_{40.2} = \frac{_{0.6}p_{40}}{_{0.2}p_{40}} = \frac{p_{40} + 0.2q_{40}}{p_{40} + 0.6q_{40}} = 2.108 \times 10^{-4}.$$

Note that this solution is the same as the answer using the UDD or CFM assumptions (see Examples 3.2 and 3.6 in AMLCR). It is common for all three assumptions to give similar

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answers at younger ages, when mortality is very low. At older ages, the differences between the three methods will be more apparent.

Exercise Show that the force of mortality implied by the Balducci assumption, μ_{x+s}, is a decreasing function of s for integer x, and for $0 \le s < 1$.

1.2 Some comments on heterogeneity in mortality

This section is related to the discussion of selection and population mortality in Chapter 3 of AMLCR, in particular to Sections 3.4 and 3.5, where we noted that there can be considerable variability in the mortality experience of different populations and population subgroups.

There is also considerable variability in the mortality experience of insurance company customers and pension plan members. Of course, male and female mortality differ significantly, in shape and level. Actuaries will generally use separate survival models for men and women where this does not breach discrimination laws. Smoker and non-smoker mortality differences are very important in whole life and term insurance; smoker mortality is substantially higher at all ages for both sexes, and separate smoker / non-smoker mortality tables are in common use.

In addition, insurers will generally use product-specific mortality tables for different types of contracts. Individuals who purchase immediate or deferred annuities may have different mortality than those purchasing term insurance. Insurance is sometimes purchased under group contracts, for example by an employer to provide death-in-service insurance for employees. The mortality experience from these contracts will generally be different to the experience of policyholders holding individual contracts. The mortality experience of pension plan members may differ from the experience of lives who purchase individual pension policies from an insurance company. Interestingly, the differences in mortality experience between these groups will depend significantly on country. Studies of mortality have shown, though, that the following principles apply quite generally.

  ◇ Wealthier lives experience lighter mortality overall than less wealthy lives.

  ◇ There will be some impact on the mortality experience from self-selection; an individual will only purchase an annuity if he or she is confident of living long enough to benefit. An individual who has some reason to anticipate heavier mortality is more likely to

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purchase term insurance. While underwriting can identify some selective factors, there may be other information that cannot be gleaned from the underwriting process (at least not without excessive cost). So those buying term insurance might be expected to have slightly heavier mortality than those buying whole life insurance, and those buying annuities might be expected to have lighter mortality.

⋄ The more rigorous the underwriting, the lighter the resulting mortality experience. For group insurance, there will be minimal underwriting. Each person hired by the employer will be covered by the insurance policy almost immediately; the insurer does not get to accept or reject the additional employee, and will rarely be given information sufficient for underwriting decisions. However, the employee must be healthy enough to be hired, which gives some selection information.

All of these factors may be confounded by tax or legislative systems that encourage or require certain types of contracts. In the UK, it is very common for retirement savings proceeds to be converted to life annuities. In other countries, including the US, this is much less common. Consequently, the type of person who buys an annuity in the US might be quite a different (and more self-select) customer than the typical individual buying an annuity in the UK.

1.3 Mortality trends

A further challenge in developing and using survival models is that survival probabilities are not constant over time. Commonly, mortality experience gets lighter over time. In most countries, for the period of reliable records, each generation, on average, lives longer than the previous generation. This can be explained by advances in health care and by improved standards of living. Of course, there are exceptions, such as mortality shocks from war or from disease, or declining life expectancy in countries where access to health care worsens, often because of civil upheaval. The changes in mortality over time are sometimes separated into three components: trend, shock and idiosyncratic. The trend describes the gradual reduction in mortality rates over time. The shock describes a short term mortality jump from war or pandemic disease. The idiosyncratic risk describes year to year random variation that does not come from trend or shock, though it is often difficult to distinguish these changes.

While the shock and idiosyncratic risks are inherently unpredictable, we can often identify trends in mortality by examining mortality patterns over a number of years. We can then allow for mortality improvement by using a survival model which depends on both age and

calendar year. A common model for projecting mortality is to assume that mortality rates at each age are decreasing annually by a constant factor, which depends on the age and sex of the individual. That is, suppose $q(x, Y)$ denotes the mortality rate for a life aged x in year Y, so that the $q(x, 0)$ denotes the mortality rate for a baseline year, $Y = 0$. Then, the estimated one-year mortality probability for a life age x at time $Y = s$ is

$$q(x, s) = q(x, 0) R_x^s \qquad \text{where} \quad 0 < R_x \leq 1.$$

The R_x terms are called **Reduction Factors**, and typical values are in the range 0.95 to 1.0 , where the higher values (implying less reduction) tend to apply at older ages. Using $R_x = 1.0$ for the oldest ages reflects the fact that, although many people are living longer than previous generations, there is little or no increase in the maximum age attained; the change is that a greater proportion of lives survive to older ages. In practice, the reduction factors are applied for integer values of s.

Given a baseline survival model, with mortality rates $q(x, 0) = q_x$, say, and a set of age-based reduction factors, R_x, we can calculate the survival probabilities from the baseline year, $_tp(x, 0)$, say, as

$$_tp(x, 0) = p(x, 0) \, p(x + 1, 1) \ldots p(x + t - 1, t - 1)$$

$$= (1 - q_x) \, (1 - q_{x+1} R_{x+1}) \, \left(1 - q_{x+2} R_{x+2}^2\right) \ldots \left(1 - q_{x+t-1} R_{x+t-1}^{t-1}\right).$$

Some survival models developed for actuarial applications implicitly contain some allowance for mortality improvement. When selecting a survival model to use for valuation and risk management, it is important to verify the projection assumptions.

The use of reduction factors allows for predictable improvements in life expectancy. However, if the improvements are underestimated, then mortality experience will be lighter than expected, leading to losses on annuity and pension contracts. This risk, called longevity risk, is of great recent interest, as mortality rates have declined in many countries at a much faster rate than anticipated. As a result, there has been increased interest in stochastic mortality models, where the force of mortality in future years follows a stochastic process which incorporates both predictable and random changes in longevity, as well as pandemic-type shock effects. See, for example, Lee and Carter (1992), Li et al (2010) or Cairns et al (2009) for more detailed information.

References:

Cairns A.J.G., D. Blake, K. Dowd, G.D. Coughlan, D. Epstein, A. Ong and I. Balevich (2009).

A quantitative comparison of stochastic mortality models using data from England and Wales and the United States. *North American Actuarial Journal* 13(1) 1-34.

Lee, R. D., and L. R. Carter (1992) Modeling and forecasting U.S. mortality. *Journal of the American Statistical Association* 87, 659 - 675.

Li, S.H., M.R. Hardy and K. S. Tan (2010) Developing mortality improvement formulae: the Canadian insured lives case study. *North American Actuarial Journal* 14(4), 381-399.

2 Policy values and reserves

2.1 When are retrospective policy values useful?

In Section 7.7 of AMLCR we introduce the concept of the retrospective policy value, which measures, under certain assumptions, the expected accumulated premium less the cost of insurance, per surviving policyholder, while a policy is in force. We explain why the retrospective policy value is not given much emphasis in the text, the main reason being that the policy value should take into consideration the most up to date assumptions for future interest and mortality rates, and it is unlikely that these will be equal to the original assumptions. The asset share is a measure of the accumulated contribution of each surviving policy to the insurer's funds. The prospective policy value measures the funds required, on average, to meet future obligations. The retrospective policy value, which is a theoretical asset share based on a different set of assumptions (the asset share by definition uses experience, not assumptions), does not appear necessary.

However, there is one application where the retrospective policy value is sometimes useful, and that is where the insurer uses the net premium policy value for determination of the appropriate capital requirement for a portfolio. Recall (from Definition 7.2 in AMLCR) that under the net premium policy value calculation, the premium used is always calculated using the valuation basis (regardless of the true or original premium). If, in addition, the premium is calculated using the equivalence principle, then the retrospective and prospective net premium policy values will be the same. This can be useful if the premium or benefit structure is complicated, so that it may be simpler to take the accumulated value of past premiums less accumulated value of benefits, per surviving policyholder (the retrospective policy value), than to use the prospective policy value. It is worth noting that many policies in the US are still valued using net premium policy values, often using a retrospective formula. In this section we discuss the retrospective policy value in more detail, in the context of the net premium approach to valuation.

2.2 Defining the retrospective net premium policy value

Consider an insurance sold to (x) at time $t = 0$ with term n (which may be ∞ for a whole life contract). For a policy in force at age $x + t$, let L_t denote the present value at time t of all the future benefits less net premiums, under the terms of the contract. The prospective policy

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value, $_tV^P$, say, was defined for policies in force at $t < n$ as

$$_tV^P = E[L_t].$$

If (x) does not survive to time t then L_t is undefined.

The value at issue of all future benefits less premiums payable from time $t < n$ onwards is the random variable

$$I(T_x > t)\, v^t\, L_t$$

where $I()$ is the indicator function.

We define, further, $L_{0,t}$, $t \le n$:

$$
\begin{aligned}
L_{0,t} \;=\; & \text{Present value at issue of future benefits payable up to time } t \\
& -\text{Present value at issue, of future net premiums payable up to } t
\end{aligned}
$$

If premiums and benefits are paid at discrete intervals, and t is a premium or benefit payment date, then the convention is that $L_{0,t}$ would include benefits payable at time t, but not premiums. At issue (time 0) the future net loss random variable L_0 comprises the value of benefits less premiums up to time t, $L_{0,t}$, plus the value of benefits less premiums payable after time t, that is:

$$L_0 = L_{0,t} + I(T_x > t)v^t L_t$$

We now define the retrospective net premium policy value as

$$_tV^R = \frac{-\mathrm{E}[L_{0,t}](1+i)^t}{_tp_x} = \frac{-\mathrm{E}[L_{0,t}]}{_tE_x}$$

and this formula corresponds to the calculation in Section 7.3.1 for the policy from Example 7.1. The term $-\mathrm{E}[L_{0,t}](1+i)^t$ is the expected value of premiums less benefits in the first t years, accumulated to time t. Dividing by $_tp_x$ expresses the expected accumulation per expected surviving policyholder.

Using this definition it is simple to see that under the assumptions

(1) the premium is calculated using the equivalence principle,

(2) the same basis is used for prospective policy values, retrospective policy values and the equivalence principle premium,

the retrospective policy value at time t must equal the prospective policy value $_tV^P$, say. We prove this by first recalling that

$$\mathrm{E}[L_0] = \mathrm{E}\left[L_{0,t} + I(T_x > t)\, v^t\, L_t\right] = 0 \quad \text{by the equivalence principle}$$

$$\Rightarrow -\mathrm{E}[L_{0,t}] = \mathrm{E}\left[I(T_x > t)\, v^t\, L_t\right]$$

$$\Rightarrow -\mathrm{E}[L_{0,t}] = {}_tp_x\, v^t\, {}_tV^P$$

$$\Rightarrow {}_tV^R = {}_tV^P.$$

The same result could easily be derived for gross premium policy values, but the assumptions listed are far less likely to hold when expenses are taken into consideration.

Example SN2.1 An insurer issues a whole life insurance policy to a life aged 40. The death benefit in the first five years of the contract is \$5 000. In subsequent years, the death benefit is \$100 000. The death benefit is payable at the end of the year of death. Premiums are paid annually for a maximum of 20 years. Premiums are level for the first five years, then increase by 50%.

(a) Write down the equation of value for calculating the net premium, using standard actuarial functions.

(b) Write down equations for the net premium policy value at time $t = 4$ using (i) the retrospective policy value approach and (ii) the prospective policy value approach.

(c) Write down equations for the net premium policy value at time $t = 20$ using (i) the retrospective policy value approach and (ii) the prospective policy value approach.

Solution to Example SN2.1

For convenience, we work in \$000s:

(a) The equivalence principle premium is P for the first 5 years, and $1.5\,P$ thereafter, where,

$$P = \frac{5A^{\,1}_{40:\overline{5}|} + 100\,{}_5E_{40}\,A_{45}}{\ddot{a}_{40:\overline{5}|} + 1.5\,{}_5E_{40}\,\ddot{a}_{45:\overline{15}|}} \tag{1}$$

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(b) The retrospective and prospective policy value equations at time $t = 4$ are

$$_4V^R = \frac{P\ddot{a}_{40:\overline{4}|} - 5A^1_{40:\overline{4}|}}{_4E_{40}}, \tag{2}$$

$$_4V^P = 5A^1_{44:\overline{1}|} + 100\,_1E_{44}\,A_{45} - P\left(\ddot{a}_{44:\overline{1}|} + 1.5\,_1E_{44}\,\ddot{a}_{45:\overline{15}|}\right). \tag{3}$$

(c) The retrospective and prospective policy value equations at time $t = 20$ are

$$_{20}V^R = \frac{P\left(\ddot{a}_{40:\overline{5}|} + 1.5\,_5E_{40}\,\ddot{a}_{45:\overline{15}|}\right) - 5\,A^1_{40:\overline{5}|} - 100\,_5E_{40}\,A_{45:\overline{15}|}}{_{20}E_{40}}, \tag{4}$$

$$_{20}V^P = 100A_{60}. \tag{5}$$

From these equations, we see that the retrospective policy value offers an efficient calculation method at the start of the contract, when the premium and benefit changes are ahead, and the prospective is more efficient at later durations, when the changes are past.

Example SN2.2 For Example SN2.1 above, show that the prospective and retrospective policy values at time $t = 4$, given in equations (2) and (3), are equal under the standard assumptions (premium and policy values all use the same basis, and the equivalence principle premium).

Solution to Example SN2.2

Note that, assuming all calculations use the same basis:

$$A^1_{40:\overline{5}|} = A^1_{40:\overline{4}|} + {}_4E_{40}\,A^1_{44:\overline{1}|}$$

$$\ddot{a}_{40:\overline{5}|} = \ddot{a}_{40:\overline{4}|} + {}_4E_{40}\,\ddot{a}_{44:\overline{1}|}$$

and

$$_5E_{40} = {}_4E_{40}\,{}_1E_{44}.$$

Now we use these to re-write the equivalence principle premium equation (1),

$$P\left(\ddot{a}_{40:\overline{5}|} + 1.5\,_5E_{40}\,\ddot{a}_{45:\overline{15}|}\right) = 5A^1_{40:\overline{5}|} + 100\,_5E_{40}\,A_{45}$$

$$\Rightarrow P\left(\ddot{a}_{40:\overline{4}|} + {}_4E_{40}\,\ddot{a}_{44:\overline{1}|} + 1.5\,_4E_{40}\,{}_1E_{44}\,\ddot{a}_{45:\overline{15}|}\right)$$

$$= 5\left(A^1_{40:\overline{4}|} + {}_4E_{40}\,A^1_{44:\overline{1}|}\right) + 100\,_4E_{40}\,{}_1E_{44}\,A_{45}.$$

Rearranging gives

$$P\ddot{a}_{40:\overline{4}|} - 5A^1_{40:\overline{4}|} = {}_4E_{40}\left(5\,A^1_{44:\overline{1}|} + 100\,_1E_{44}\,A_{45} - P\left(\ddot{a}_{44:\overline{1}|} + 1.5\,_1E_{44}\,\ddot{a}_{45:\overline{15}|}\right)\right).$$

Dividing both sides by $_4E_{40}$ gives $_4V^R = {}_4V^P$ as required.

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2.3 Deferred Acquisition Expenses and Modified Premium Reserves

The policy value calculations described in AMLCR Chapter 7, and in the sections above, may be used to determine the appropriate provision for the insurer to make to allow for the uncertain future liabilities. These provisions are called reserves in insurance. The principles of reserve calculation, such as whether to use a gross or net premium policy value, and how to determine the appropriate basis, are established by insurance regulators. While most jurisdictions use a gross premium policy value approach, as mentioned above, the net premium policy value is still used, notably in the US.

In some circumstances, the reserve is not calculated directly as the net premium policy value, but is modified, to approximate a gross premium policy value approach. In this section we will motivate this approach by considering the impact of acquisition expenses on the policy value calculations.

Let $_tV^n$ denote the net premium policy value for a contract which is still in force t years after issue and let $_tV^g$ denote the gross premium policy value for the same contract, using the equivalence principle and using the original premium interest and mortality basis. This point is worth emphasizing as, in most jurisdictions, the basis would evolve over time to differ from the premium basis. Then we have

$$_tV^n = \text{EPV future benefits} - \text{EPV future net premiums}$$

$$_tV^g = \text{EPV future benefits} + \text{EPV future expenses} - \text{EPV future gross premiums}$$

$$_0V^n = {_0V^g} = 0.$$

So we have

$$_tV^g = \text{EPV future benefits} + \text{EPV future expenses}$$

$$- (\text{EPV future net premiums} + \text{EPV future expense loadings})$$

$$\Rightarrow {_tV^g} = {_tV^n} + \text{EPV future expenses} - \text{EPV future expense loadings}$$

That is $\quad _tV^g = {_tV^n} + {_tV^e}, \quad$ say, where

$$_tV^e = \text{EPV future expenses} - \text{EPV future expense loadings}$$

What is important about this relationship is that, generally, $_tV^e$ is negative, meaning that the net premium policy value is greater than the gross premium policy value, assuming the

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same interest and mortality assumptions for both. This may appear counterintuitive – the reserve which takes expenses into consideration is smaller than the reserve which does not – but remember that the gross premium approach offsets the higher future outgo with higher future premiums. If expenses were incurred as a level annual amount, and assuming premiums are level and payable throughout the policy term, then the net premium and gross premium policy values would be the same, as the extra expenses valued in the gross premium case would be exactly offset by the extra premium. In practice though, expenses are not incurred as a flat amount. The initial (or acquisition) expenses (commission, underwriting and administrative) are large relative to the renewal and claims expenses. This results in negative values for $_tV^e$, in general.

Suppose the gross premium for a level premium contract is P^g, and the net premium is P^n. The difference, P^e, say, is the expense loading (or expense premium) for the contract. This is the level annual amount paid by the policyholder to cover the policy expenses. If expenses are incurred as a level sum at each premium date, then P^e would equal those incurred expenses (assuming premiums are paid throughout the policy term). If expenses are weighted to the start of the contract, as is normally the case, then P^e will be greater than the renewal expense as it must fund both the renewal and initial expenses. We illustrate with an example.

Example SN2.3 An insurer issues a whole life insurance policy to a life aged 50. The sum insured of \$100 000 is payable at the end of the year of death. Level premiums are payable annually in advance throughout the term of the contract. All premiums and policy values are calculated using the SSSM, and an interest rate of 4% per year effective. Initial expenses are 50% of the gross premium plus \$250. Renewal expenses are 3% of the gross premium plus \$25 at each premium date after the first.

Calculate

(a) the expense loading, P^e and

(b) $_{10}V^e$, $_{10}V^n$ and $_{10}V^g$.

Solution to Example SN2.3

(a) The expense premium P^e depends on the gross premium P^g which we calculate first:

$$P^g = \frac{100\,000\,A_{[50]} + 25\ddot{a}_{[50]} + 225}{0.97\,\ddot{a}_{[50]} - 0.47} = 1435.89$$

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Now P^e can be calculated by valuing the expected present value of future expenses, and calculating the level premium to fund those expenses – that is

$$P^e \ddot{a}_{[50]} = 25\ddot{a}_{[50]} + 225 + 0.03P^g\ddot{a}_{[50]} + 0.47P^g.$$

Alternatively, we can calculate the net premium, $P^n = 100\,000A_{[50]}/\ddot{a}_{[50]} = 1321.31$, and use $P^e = P^g - P^n$. Either method gives $P^e = 114.58$.

Compare the expense premium with the incurred expenses. The annual renewal expenses, payable at each premium date after the first, are \$68.08. The rest of the expense loading, \$46.50 at each premium date, reimburses the acquisition expenses, which total \$967.94 at inception. Thus, at any premium date after the first, the value of the future expenses will be smaller than the value of the future expense loadings.

(b) The expense reserve at time $t = 10$, for a contract still in force, is

$$_{10}V^e = 25\ddot{a}_{60} + 0.03P^g\ddot{a}_{60} - P^e\ddot{a}_{60} = -46.50\ddot{a}_{60} = -770.14.$$

The net premium policy value is

$$_{10}V^n = 100\,000A_{60} - P^n\,\ddot{a}_{60} = 14\,416.12.$$

The gross premium policy value is

$$_{10}V^g = 100\,000A_{60} + 25\ddot{a}_{60} - 0.97P^g\,\ddot{a}_{60} = 13\,645.98.$$

We note that, as expected, the expense reserve is negative, and that

$$_{10}V^g = {}_{10}V^n + {}_{10}V^e.$$

The negative expense reserve is referred to as the deferred acquisition cost, or DAC. The use of the net premium reserve can be viewed as being overly conservative, as it does not allow for the DAC reimbursement. An insurer should not be required to hold the full net premium policy value as capital, when the true future liability value is smaller because of the DAC. One solution would be to use gross premium reserves. But to do so would lose some of the numerical advantage offered by the net premium approach, including simple formulas for standard contracts, and including the ability to use either a retrospective or prospective formula to calculate the valuation. An alternative method, which maintains most of the numerical

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simplicity of the net premium approach, is to modify the net premium method to allow for the DAC, in a way that is at least approximately correct. Modified premium reserves use a net premium policy value approach to reserve calculation, but instead of assuming a level annual premium, assume a lower initial premium to allow implicitly for the DAC. We note briefly that it is only appropriate to modify the reserve to allow for the DAC to the extent that the DAC will be recovered in the event that the policyholder surrenders the contract. The cash values for surrendering policyholders will be determined to recover the DAC as far as is possible. If the DAC cannot be fully recovered from surrendering policyholders, then it would be inappropriate to take full credit for it.

The most common method of adjusting the net premium policy value is the **Full Preliminary Term** (FPT) approach. Before we define the FPT method, we need some notation. Consider a life insurance contract with level annual premiums. Let $P^n_{[x]+s}$ denote the net premium for a contract issued to a life age $x + s$, who was selected at age x. Let $_1P^n_{[x]}$ denote the single premium to fund the benefits payable during the first year of the contract (this is called the first year Cost of Insurance). Then the FPT reserve for a contract issued to a select life aged x is the net premium policy value assuming that the net premium in the first year is $_1P^n_{[x]}$ and in all subsequent years is $P^n_{[x]+1}$. This is equivalent to considering the policy as two policies, a 1-year term, and a separate contract issued to the same life 1 year later, if the life survives.

Example SN2.4

(a) Calculate the modified premiums for the policy in Example SN2.3.

(b) Compare the net premium policy value, the gross premium policy value and the FPT reserve for the contract in Example SN2.3 at durations 0, 1, 2 and 10.

Solution to Example SN2.4

(a) The modified net premium assumed at time $t = 0$ is

$$_1P^n_{[50]} = 100\,000 A^{\,1}_{[50]:\overline{1}|} = 100\,000\, v\, q_{[50]} = 99.36.$$

The modified net premium assumed paid at all subsequent premium dates is

$$P^n_{[50]+1} = \frac{100\,000 A_{[50]+1}}{\ddot{a}_{[50]+1}} = 1387.90$$

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(b) At time 0:

$$_0V^n = 100\,000 A_{[50]} - P^n_{[50]}\ddot{a}_{[50]} = 0,$$

$$_0V^g = 100\,000 A_{[50]} + 225 + 25\,\ddot{a}_{[50]} + 0.47 P^g_{[50]} - 0.97 P^g_{[50]}\ddot{a}_{[50]} = 0,$$

$$_0V^{FPT} = 100\,000 A_{[50]} - {_1}P^n_{[50]} - P^n_{[50]+1} v p_{[50]}\ddot{a}_{[50]+1}$$

$$= 100\,000\left(A^{\,1}_{[50]:\overline{1}|} + v p_{[50]} A_{[50]+1}\right) - 100\,000 A^{\,1}_{[50]:\overline{1}|}$$

$$- \left(\frac{100\,000 A_{[50]+1}}{\ddot{a}_{[50]+1}}\right) v p_{[50]}\ddot{a}_{[50]+1}$$

$$\Rightarrow {_0}V^{FPT} = 0.$$

At time 1:

$$_1V^n = 100\,000 A_{[50]+1} - P^n_{[50]}\ddot{a}_{[50]+1} = 1272.15,$$

$$_1V^g = 100\,000 A_{[50]+1} + 25\,\ddot{a}_{[50]+1} - 0.97 P^g_{[50]}\ddot{a}_{[50]+1} = 383.73,$$

$$_1V^{FPT} = 100\,000 A_{[50]+1} - P^n_{[50]+1}\ddot{a}_{[50]+1} = 0.$$

At time 2:

$$_2V^n = 100\,000 A_{52} - P^n_{[50]}\ddot{a}_{52} = 2574.01,$$

$$_2V^g = 100\,000 A_{[50]+1} + 25\,\ddot{a}_{[50]+1} - 0.97 P^g_{[50]}\ddot{a}_{[50]+1} = 1697.30,$$

$$_2V^{FPT} = 100\,000 A_{[50]+1} - P^n_{[50]+1}\ddot{a}_{[50]+1} = 1318.63.$$

At time 10, we have the net premium and gross premium policy values from Example SN2.3,

$$_{10}V^n = 14\,416.12 \qquad _{10}V^g = 13\,645.98$$

and

$$_{10}V^{FPT} = 100\,000 A_{60} - P^n_{[50]+1}\ddot{a}_{60} = 13\,313.34.$$

The FPT reserve is intended to approximate the gross premium reserve, particularly in the first few years of the contract. We see that the insurer would benefit significantly in the first year from using the FPT approach rather than the net premium policy value.

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As the policy matures, all the policy values converge (though perhaps not until extremely advanced ages).

The FPT method implicitly assumes that the whole first year premium is spent on the cost of insurance and the acquisition expenses. In this case, that assumption overstates the acquisition expenses slightly, with the result that the FPT reserve is actually a little lower than the gross premium policy value. Modifications of the method (partial preliminary term) would allow for a net premium after the first year that lies somewhere between the FPT premium and the level net premium. □

2.4 Exercises

1. An insurer issues a deferred annuity with a single premium. The annuity is payable continuously at a level rate of $50 000 per year after the 20-year deferred period, if the policyholder survives. On death during the deferred period the single premium is returned with interest at rate i per year effective.

   (a) Write down an equation for the prospective net premium policy value (i) during the deferred period and (ii) after the deferred period, using standard actuarial functions. Assume an interest rate of i per year effective, the same as the accumulation rate for the return of premium benefit.

   (b) Repeat (a) for the retrospective net premium policy value.

   (c) Show that under certain conditions, which you should state, the retrospective and prospective policy values are equal.

2. Repeat Example SN2.4, assuming now that the premium term is limited to a maximum of 20 years.

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3 Multiple decrement tables

3.1 Introduction

This section relates to Section 8.8 of AMLCR. Throughout this section we assume a multiple decrement model with $n + 1$ states. The starting state is labelled 0, and is referred to as the 'in-force' state (as we often use the model for insurance and pension movements), and there are n possible modes of exit. The model is described in Figure 8.7 of AMLCR. The objectives of this section are (i) to introduce multiple decrement tables and (ii) to demonstrate how to construct multiple decrement tables from tabulated independent rates of decrement, and *vice versa*. In order to do this, we extend the fractional age assumptions for mortality used in the alive-dead model, as described in Section 3.3 of AMLCR, to fractional age assumptions for multiple decrements.

3.2 Multiple decrement tables

In discussing the multiple decrement models in Section 8.8 of AMLCR, we assumed that the transition forces are known, and that all probabilities required can be constructed using the numerical methods described in Chapter 8. It is sometimes convenient to express a multiple decrement model in table form, similarly to the use of the life table for the alive-dead model. Recall that in Chapter 3 of AMLCR we describe how a survival model for the future lifetime random variable is often summarized in a table of l_x, for integer values of x. We showed that the table can be used to calculate survival and mortality probabilities for integer ages and durations. We also showed that if the table was the only information available, we could use a fractional age assumption to derive estimates for probabilities for non-integer ages and durations.

The multiple decrement table is analogous to the life table. The table is used to calculate survival probabilities and exit probabilities, by mode of exit, for integer ages and durations. With the addition of a fractional age assumption for decrements between integer ages, the table can be used to calculate all survival and exit probabilities for ages within the range of the table.

We expand the life table notation of Section 3.2 of AMLCR as follows.

Let l_{x_0} be the radix of the table (an arbitrary positive number) at the initial age x_0. Define

$$l_{x+t} = l_{x_0} \, _t p_{x_0}^{00}$$

and for $j = 1, 2, ..., n$, and $x \geq x_0$,

$$d_x^{(j)} = (l_x) \, p_x^{0j}.$$

Given integer age values for l_x and for $d_x^{(j)}$, all integer age and duration probabilities can be calculated. We interpret l_x, $x > x_0$ as the expected number of survivors in the starting state 0 at age x out of l_{x_0} in state 0 at age x_0; $d_x^{(j)}$ is the expected number of lives exiting by mode of decrement j in the year of age x to $x + 1$, out of l_x lives in the starting state at age x.

Note that the pension plan service table in Chapter 9 of AMLCR is a multiple decrement table, though with the slightly different notation that has evolved from pension practice.

Example SN3.1 The following is an excerpt from a multiple decrement table for an insurance policy offering benefits on death or diagnosis of critical illness. The insurance expires on the earliest event of death ($j = 1$), surrender ($j = 2$) and critical illness diagnosis ($j = 3$).

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
40	100 000	51	4 784	44
41	95 121	52	4 526	47
42	90 496	53	4 268	50
43	86 125	54	4 010	53
44	82 008	55	3 753	56
45	78 144	56	3 496	59
46	74 533	57	3 239	62
47	71 175	57	2 983	65
48	68 070	58	2 729	67
49	65 216	58	2 476	69
50	62 613	58	2 226	70

Table 1: Excerpt from a critical illness multiple decrement table.

(a) Calculate (i) $_3p_{45}^{00}$, (ii) p_{40}^{01}, (iii) $_5p_{41}^{03}$.

(b) Calculate the probability that a policy issued to a life aged 45 generates a claim for death or critical illness before age 47.

(c) Calculate the probability that a policy issued to a life age 40 is surrendered between ages 45 and 47.

Solution to Example SN3.1

(a) (i) $\quad {}_3p_{45}^{00} = \dfrac{l_{48}}{l_{45}} = 0.87108$

(ii) $\quad p_{40}^{01} = \dfrac{d_{40}^{(1)}}{l_{40}} = 0.00051$

(iii) $\quad {}_5p_{41}^{03} = \dfrac{d_{41}^{(3)} + d_{42}^{(3)} + ... + d_{45}^{(3)}}{l_{41}} = 0.00279$

(b) $\quad {}_2p_{45}^{01} + {}_2p_{45}^{03} = \dfrac{d_{45}^{(1)} + d_{46}^{(1)} + d_{45}^{(3)} + d_{46}^{(3)}}{l_{45}} = 0.00299$

(c) $\quad {}_5p_{40}^{00}\,{}_2p_{45}^{02} = \dfrac{d_{45}^{(2)} + d_{46}^{(2)}}{l_{40}} = 0.06735.$

3.3 Fractional age assumptions for decrements

Suppose the only information that we have about a multiple decrement model are the integer age values of l_x and $d_x^{(j)}$. To calculate non-integer age or duration probabilities, we need to make an assumption about the decrement probabilities or forces between integer ages.

UDD in the Multiple Decrement Table Here UDD stands for uniform distribution of decrements. For $0 \le t \le 1$, and integer x, and for each exit mode j, assume that

$$ {}_tp_x^{0j} = t\,(p_x^{0j}) \tag{6} $$

The assumption of UDD in the multiple decrement model can be interpreted as assuming that for each decrement, the exits from the starting state are uniformly spread over each year.

Constant transition forces For $0 \le t \le 1$, and integer x, assume that for each exit mode j, μ_{x+t}^{0j} is a constant for each age x, equal to $\mu^{0j}(x)$, say. Let

$$ \mu^{0\bullet}(x) = \sum_{k=1}^{n} \mu^{0k}(x) $$

so $\mu^{0\bullet}(x)$ represents the total force of transition out of state 0 at age $x + t$ for $0 \le t < 1$. It is convenient also to denote the total exit probability from state 0 for the year of age x to $x + 1$ as $p_x^{0\bullet}$. That is

$$ p_x^{0\bullet} = 1 - p_x^{00} = \sum_{k=1}^{n} p_x^{0k} = 1 - e^{-\mu^{0\bullet}(x)}. $$

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Assuming constant transition forces between integer ages for all decrements,

$$_t p_x^{0j} = \frac{p_x^{0j}}{p_x^{0\bullet}} \left(1 - \left(p_x^{00}\right)^t\right). \tag{7}$$

We prove this as follows:

$$
\begin{aligned}
_t p_x^{0j} &= \int_0^t {}_r p_x^{00} \, \mu_{x+r}^{0j} dr \tag{8} \\
&= \int_0^t e^{-r\,\mu^{0\bullet}(x)} \, \mu^{0j}(x) dr \qquad \text{by the constant force assumption} \\
&= \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - e^{-t\,\mu^{0\bullet}(x)}\right) \\
&= \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - \left(p_x^{00}\right)^t\right). \tag{9}
\end{aligned}
$$

Now let $t \to 1$, and rearrange, giving

$$\frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} = \frac{p_x^{0j}}{p_x^{0\bullet}} \tag{10}$$

where the left hand side is the ratio of the mode j force of exit to the total force of exit, and the right hand side is the ratio of the mode j probability of exit to the total probability of exit. Substitute from equation (10) back into (9) to complete the proof.

The intuition here is that the term $1 - \left(p_x^{00}\right)^t$ represents the total probability of exit under the constant transition force assumption, and the term $p_x^{0j}/p_x^{0\bullet}$ divides this exit probability into the different decrements in proportion to the full 1-year exit probabilities.

Example SN3.2

Calculate $_{0.2} p_{50}^{0j}$ for $j = 1, 2, 3$ using the model summarized in Table 1, and assuming (a) UDD in all the decrements between integer ages, and (b) constant transition forces in all decrements between integer ages.

Solution to Example SN3.2

(a) $_{0.2} p_{50}^{0j} = 0.2 \, p_{50}^{0j}$ which gives

$$_{0.2} p_{50}^{01} = 0.000185, \qquad _{0.2} p_{50}^{02} = 0.007110, \qquad _{0.2} p_{50}^{03} = 0.000224.$$

(b) Now

$$_{0.2} p_{50}^{0j} = \frac{p_{50}^{0j}}{p_{50}^{0\bullet}} \left(1 - (p_{50}^{00})^{0.2}\right)$$

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which gives

$$_{0.2}p_{50}^{01} = 0.000188 \qquad _{0.2}p_{50}^{02} = 0.007220 \qquad _{0.2}p_{50}^{03} = 0.000227$$

3.4 Independent and Dependent Probabilities

In AMLCR Section 8.8 we introduce the concept of dependent and independent probabilities of decrement. Recall that the **independent rates** are those that would apply if no other modes of decrement were present. The **dependent rates** are those determined with all modes of decrement included. In the following subsections of this note, we discuss in much more detail how the dependent and independent probabilities are related, under different fractional age assumptions for decrements, and how these relationships can be used to adjust or construct multiple decrement tables when only integer age rates are available. In order to do this, we introduce some notation for the independent transition probabilities associated with each decrement in a multiple decrement model. Let

$$_{t}p_{x}^{j} = e^{-\int_{0}^{t} \mu_{x+r}^{0j} dr} \qquad \text{and} \quad _{t}q_{x}^{j} = 1 - {_{t}p_{x}^{j}}.$$

This means that $_{t}q_{x}^{j}$ represents the t-year probability that a life aged x moves to state j from state 0, and $_{t}p_{x}^{j}$ represents the probability that (x) does not move, under the hypothetical 2-state model where j is the only decrement. The force of transition from state 0 to state j, μ_{x}^{0j}, is assumed to be the same in both the dependent and independent cases. The independent transition probabilities, and the associated transition forces, have all the same relationships as the associated life table probabilities from Chapter 2 of AMLCR, because the structure of the independent model is the same 2-state, one transition model as the alive-dead model. It is illustrated in Figure 8.9 of AMLCR. The independent model is also called the associated single decrement model.

3.5 Constructing a multiple decrement table from dependent and independent decrement probabilities

When constructing or adjusting multiple decrement tables without knowledge of the underlying transition forces, we need to assume (approximate) relationships between the dependent and independent decrement probabilities. For example, suppose an insurer is using a double decrement table of deaths and lapses to model the liabilities for a product. When a new mortality table is issued, the insurer may want to adjust the dependent rates to allow for the more

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up-to-date mortality probabilities. However, the mortality table is an independent table – the probabilities are the pure mortality probabilities. In the double decrement table, what we are interested in is the probability that death occurs from the 'in-force' state – so deaths after lapsation do not count here. We can combine independent probabilities of lapse and mortality to construct the dependent multiple decrement table, but first we may need to extract the independent lapse probabilities from the original table, which generates dependent rates, not independent rates.

In order to deconstruct a multiple decrement table into the independent models, we need to find values for the independent decrement probabilities $_tq_x^j$ from the dependent decrement probabilities $_tp_x^{0j}$, $j = 1, 2, ...n$. Then to re-construct the table, we need to reverse the process. The difference between the dependent and independent rates for each cause of decrement depends on the pattern of exits from all causes of decrement between integer ages. For example, suppose we have dependent rates of mortality and withdrawal for some age x in the double decrement table, of $p_x^{01} = 0.01$ and $p_x^{02} = 0.05$ respectively. This means that, given 100 lives in force at age x, we expect 1 to die (before withdrawing) and 5 to withdraw. Suppose we know that withdrawals all happen right at the end of the year. Then from 100 lives in force, none can after withdrawal, and the independent mortality rate must also be $1/100$ – as we expect 1 person to die from 100 lives observed for 1 year. But if, instead, all the withdrawals occur right at the beginning of the year, then we have 1 expected death from 95 lives observed for 1 year, so the independent mortality rate is $1/95$.

If we do not have such specific information, we use fractional age assumptions to derive the relationships between the dependent and independent probabilities.

UDD in the MDT

Assume, as above, that each decrement is uniformly distributed in the multiple decrement model. Then we know that for integer x, and for $0 \leq t < 1$,

$$_tp_x^{0k} = t\,p_x^{0k}, \qquad _tp_x^{00} = 1 - tp_x^{0\bullet} \quad \text{and} \quad _tp_x^{00}\,\mu_{x+t}^{0j} = p_x^{0j} \tag{11}$$

where the last equation, is derived exactly analogously to Equation (3.9) in AMLCR. Notice that the right hand side of the last equation does not depend on t. Then from (11) above

$$\mu_{x+t}^{0j} = \frac{p_x^{0j}}{1 - t\,p_x^{0\bullet}}$$

and integrating both sides gives

$$\int_0^1 \mu_{x+t}^{0j}\,dt = \frac{p_x^{0j}}{p_x^{0\bullet}}\left(-\log(1 - p_x^{0\bullet})\right) = \frac{p_x^{0j}}{p_x^{0\bullet}}\left(-\log(p_x^{00})\right)$$

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Note that the decrement j independent survival probability is

$$p_x^j = e^{-\int_0^1 \mu_{x+t}^{0j} dt}$$

and substituting for the exponent, we have

$$p_x^j = \left(p_x^{00}\right)^{\left(p_x^{0j}/p_x^{0\bullet}\right)} \tag{12}$$

So, given the table of dependent rates of exit, p_x^{0j}, we can calculate the associated independent rates, under the assumption of UDD in the MDT.

Constant forces of transition

Interestingly, the relationship between dependent and independent rates under the constant force fractional age assumption is exactly that in equation (12). From Equation (10) we have

$$\mu^{0j}(x) = \mu^{0\bullet}(x)\frac{p_x^{0j}}{p_x^{0\bullet}},$$

so

$$p_x^j = e^{-\mu^{0j}(x)} = \left(e^{-\mu^{0\bullet}(x)}\right)^{\left(p_x^{0j}/p_x^{0\bullet}\right)} = \left(p_x^{00}\right)^{\left(p_x^{0j}/p_x^{0\bullet}\right)}.$$

Example SN3.3

Calculate the independent 1-year exit probabilities for each decrement for ages 40-50, using Table 1 above. Assume uniform distribution of decrements in the multiple decrement model.

Solution to Example SN3.3

The results are given in Table 2.

You might notice that the independent rates are greater than the dependent rates in all cases. This will always be true, as the effect of exposure to multiple forces of decrement must reduce the probability of exit by each individual mode, compared with the probability when only a single force of exit is present.

Now suppose we know the independent rates, and wish to construct the table of dependent rates.

UDD in the MDT

We can rearrange equation (12) to give

$$p_x^{0j} = \frac{\log p_x^j}{\log p_x^{00}} \, p_x^{0\bullet} \tag{13}$$

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x	q_x^1	q_x^2	q_x^3
40	0.000523	0.047863	0.000451
41	0.000560	0.047607	0.000506
42	0.000600	0.047190	0.000566
43	0.000642	0.046590	0.000630
44	0.000687	0.045795	0.000699
45	0.000733	0.044771	0.000773
46	0.000782	0.043493	0.000851
47	0.000819	0.041947	0.000933
48	0.000870	0.040128	0.001005
49	0.000907	0.038004	0.001079
50	0.000944	0.035589	0.001139

Table 2: Independent rates of exit for the MDT in Table 1, assuming UDD in the multiple decrement table.

In order to apply this, we use the fact that the product of the independent survival probabilities gives the dependent survival probability, as

$$\prod_{j=1}^n {}_tp_x^j = \prod_{j=1}^n \exp\left(-\int_0^t \mu_{x+r}^{0j}dr\right) = \exp\left(-\int_0^t \sum_{j=1}^n \mu_{x+r}^{0j}dr\right) = {}_tp_x^{00}$$

Constant transition forces

Equation (13) also applies under the constant force assumption.

UDD in the independent models

If we assume a uniform distribution of decrement in each of the independent models, the result will be slightly different from the assumption of UDD in the multiple decrement model.

The assumption now is that for each decrement j, and for integer x, $0 \le t \le 1$,

$$ {}_tq_x^j = t\, q_x^j \Rightarrow {}_tp_x^j\, \mu_{x+t}^{0j} = q_x^j.$$

Then

$$ p_x^{0j} = \int_0^1 {}_tp_x^{00}\mu_{x+t}^{0j}dt = \int_0^1 {}_tp_x^1\, {}_tp_x^2 \cdots {}_tp_x^n\, \mu_{x+t}^{0j}dt.$$

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Extract $_tp_x^j\,\mu_{x+t}^{0j} = q_x^j$ to give

$$p_x^{0j} = q_x^j \int_0^1 \prod_{k=1,k\neq j}^n {}_tp_x^j dt$$

$$= q_x^j \int_0^1 \prod_{k=1,k\neq j}^n \left(1 - t\,q_x^k\right) dt.$$

The integrand here is just a polynomial in t, so for example, if there are two decrements, we have

$$p_x^{01} = q_x^1 \int_0^1 \left(1 - t\,q_x^2\right) dt$$

$$= = q_x^1 \left(1 - \frac{1}{2}q_x^2\right)$$

and similarly for p_x^{02}.

Exercise: Show that with three decrements, under the assumption of UDD in each of the single decrement models, we have

$$p_x^{01} = q_x^1 \left\{ 1 - \frac{1}{2}\left(q_x^2 + q_x^3\right) + \frac{1}{3}\left(q_x^2\,q_x^3\right) \right\},$$

and similarly for p_x^{02} and p_x^{03}.

Generally it will make little difference whether the assumption used is UDD in the multiple decrement model or UDD in the single decrement models. The differences may be noticeable though where the transition forces are large.

Example SN3.4

The insurer using Table 1 above wishes to revise the underlying assumptions. The independent annual surrender probabilities are to be decreased by 10% and the independent annual critical illness diagnosis probabilities are to be increased by 30%. The independent mortality probabilities are unchanged.

Construct the revised multiple decrement table for ages 40 to 50 assuming UDD in the multiple decrement model and comment on the impact of the changes on the dependent mortality probabilities.

Solution to Example SN3.4

This is a straightforward application of Equation (13). The results are given in Table 3. We note

the increase in the mortality ($j = 1$) probabilities, even though the underlying (independent) mortality rates were not changed. This arises because fewer lives are withdrawing, so more lives die before withdrawal, on average.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
40	100 000.00	51.12	4 305.31	57.34
41	95 586.22	52.38	4 093.01	61.55
42	91 379.28	53.64	3 878.36	65.80
43	87 381.48	54.92	3 661.31	70.07
44	83 595.18	56.19	3 442.71	74.39
45	80 021.90	57.47	3 221.64	78.73
46	76 664.06	58.75	2 998.07	83.09
47	73 524.15	59.00	2 772.92	87.48
48	70 604.74	60.28	2 547.17	90.53
49	67 906.76	60.50	2 319.97	93.58
50	65 432.71	60.71	2 093.26	95.27

Table 3: Revised Multiple Decrement Table for Example SN3.4.

3.6 Comment on Notation

Multiple decrement models have been used by actuaries for many years, but the associated notation is not standard. We have retained the more general multiple state notation for multiple decrement (dependent) probabilities. The introduction of the hypothetical independent models is not easily incorporated into our multiple state model notation, which is why we revert to something similar to the 2-state alive-dead notation from earlier chapters for the single decrement probabilities. In the table below we have summarized the multiple decrement notation that has evolved in North America (see, for example, Bowers *et al* (1997)), and in the UK and Australia (Neill, 1977). In the first column we show the notation used in this note (which we call MS notation, for Multiple State), in the second we show the North American notation, and in the third we show the notation that has been used commonly in the UK and Australia.

	Multiple State	US and Canada	UK and Australia
Dependent survival probability	$_tp_x^{00}$	$_tp_x^{(\tau)}$	$_t(ap)_x$
Dependent transition probability	$_tp_x^{0j}$	$_tq_x^{(j)}$	$_t(aq)_x^j$
Dependent total transition probability	$_tp_x^{0\bullet}$	$_tq_x^{(\tau)}$	$_t(aq)_x$
Independent transition probability	$_tq_x^j$	$_tq_x^{\prime(j)}$	$_tq_x^j$
Independent survival probability	$_tp_x^j$	$_tp_x^{\prime(j)}$	$_tp_x^j$
Transition forces	μ_{x+t}^{0j}	$\mu_x^{(j)}(t)$	μ_{x+t}^j
Total transition force	$\mu_{x+t}^{0\bullet}$	$\mu_x^{(\tau)}(t)$	$(a\mu)_{x+t}$

3.7 Exercises

1. You are given the following three-decrement service table for modelling employment.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
60	10 000	350	150	25
61	9 475	360	125	45
62	8 945	380	110	70

(a) Calculate $_3p_{60}^{01}$.

(b) Calculate $_2p_{61}^{00}$.

(c) Calculate the expected present value of a benefit of \$10 000 payable at the end of the year of exit, if a life aged 60 leaves by decrement 3 before age 63. Use a rate of interest of 5% per year.

(d) Calculate the expected present value of an annuity of \$1 000 per year payable at the start of each of the next 3 years if a life currently aged 60 remains in service. Use a rate of interest of 5% per year.

(e) By calculating the value to 5 decimal places, show that $q_{62}^1 = 0.0429$ assuming a constant force of decrement for each decrement.

(f) Calculate the revised service table for age 62 if q_{62}^1 is increased to 0.1, with the other independent rates remaining unchanged. Use (a) the constant force assumption and (b) the UDD in the single decrement models assumption.

2. Employees of a certain company enter service at exact age 20, and, after a period in Canada, may be transferred to an overseas office. While in Canada, the only causes of

decrement, apart from transfer to the overseas office, are death and resignation from the company.

(a) Using a radix of 100 000 at exact age 39, construct a service table covering service in Canada for ages 39, 40 and 41 last birthday, given the following information about (independent) probabilities:

Mortality $(j = 1)$**:** Standard Ultimate Survival Model.

Transfer $(j = 2)$**:** $q_{39}^2 = 0.09$, $q_{40}^2 = 0.10$, $q_{41}^2 = 0.11$.

Resignation $(j = 3)$**:** 20% of those reaching age 40 resign on their 40th birthday. No other resignations take place.

Assume uniform distribution of deaths and transfers between integer ages in the single decrement tables.

(b) Calculate the probability that an employee in service in Canada at exact age 39 will still be in service in Canada at exact age 42.

(c) Calculate the probability that an employee in service in Canada at exact age 39 will transfer to the overseas office between exact ages 41 and 42.

(d) The company has decided to set up a scheme to give each employee transferring to the overseas office between exact ages 39 and 42 a grant of $10,000 at the date of transfer. To pay for these grants the company will deposit a fixed sum in a special account on the 39th, 40th and 41st birthday of each employee still working in Canada (excluding resignations on the 40th birthday). The special account is invested to produce interest of 8% per year.

Calculate the annual deposit required.

3. The following table is an extract from a multiple decrement table modelling withdrawals from life insurance contracts. Decrement (1) represents withdrawals, and decrement (2) represents deaths.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
40	15490	2400	51
41	13039	2102	58
42	10879	1507	60

(a) Stating clearly any assumptions, calculate q_{40}^2.

(b) What difference would it make to your calculation in (a) if you were given the additional information that all withdrawals occurred on the policyholders' birthdays?

4 Universal Life Insurance

This section should be read as an addition to Chapter 11 of AMLCR, although Chapter 12 might also offer some useful background.

4.1 Introduction to Universal Life Insurance

Universal Life (UL) was described briefly in Section 1.3.3 of AMLCR, as a policy which is popular in Canada and the US, and which is similar to the European unit-linked policy. The UL contract offers a mixture of term life insurance and an investment product in a transparent, flexible combination format. The policyholder may vary the amount and timing of premiums, within some constraints. The premium is first used to pay for the death benefit cover, and an expense charge is deducted to cover the insurer's costs. The remainder of the premium is invested, earning a rate of interest at the discretion of the insurer (the **credited interest**) which is used to increase the death or maturity benefit. Typically, the insurer will declare the credited interest rate based on the overall investment performance of some underlying funds, with a margin, but the policy will also carry a minimum credited interest rate guarantee. The accumulated premiums (after deductions) are tracked through the UL policy **account balance** or **account value**.

The account value represents the the insurer's liability, analogously to the reserve under a traditional contract. Under the basic UL design, the account value is a notional amount. Policyholder's funds are merged with the other assets of the insurer. The policyholder's (notional) account balance is not associated with specific assets. The credited interest declared need not reflect the interest earned on funds. Variable Universal Life (VUL), on the other hand, is essentially the same as the European unit-linked contract. The VUL policyholder's funds are held in an identifiable **separate account**; interest credited is directly generated by the yield on the separate account assets, with no annual minimum interest rate guarantee. VUL and Variable Annuities (described in Chapter 12 of AMLCR), are often referred to, collectively, as separate account policies.

In this note we will consider only the basic UL policy, which can be analyzed using the profit test techniques from Chapter 11 of AMLCR. The VUL policy is an equity-linked policy, and would be analyzed using the techniques of Chapter 12 of AMLCR.

The key design features of a UL policy are:

1. **Death Benefit:** On the policyholder's death the benefit amount is the account value of the policy, plus an **additional death benefit** (ADB).

   The ADB is required to be a significant proportion of the total death benefit, except at very old ages, to justify the policy being considered an insurance contract for tax purposes. The proportions are set through the **corridor factor requirement** which sets the minimum value for ratio of the total death benefit to the account value at death. This ratio is called the corridor factor. In the US the corridor factor is around 2.5 up to age 40 , decreasing to 1.05 at age 90, and to 1.0 at age 95 and above.

   There are two types of death benefit, which we will describe here.

   Type A offers a level total benefit, which comprises the account value plus the additional death benefit. As the account value increases, the ADB decreases. However the ADB cannot decline to zero, except at very old ages, because of the corridor factor requirement. For a type A UL policy the level death benefit is the **Face Amount** of the policy.

   Type B offers a level ADB. The amount paid on death would be the policyholder's fund plus the additional death benefit selected by the policyholder, provided this satisfies the corridor factor requirement.

   The policyholder may have the option to adjust the ADB to allow for inflation. Other death benefit increases may require evidence of health to avoid adverse selection.

2. **Premiums:** These may be subject to some minimum level, but otherwise are highly flexible. Any amount not required to support the death benefit or the expense charge is credited to the policyholder's account balance.

3. **Expense Charges:** These are deducted from the premium. The rates will be variable at the insurer's discretion, subject to a maximum specified in the original contract.

4. **Credited Interest:** Usually the credited interest rate will be declared by the insurer, but it may be based on a published exogenous rate, such as yields on government bonds. A minimum guaranteed credited interest rate is generally specified in the policy document.

5. **Mortality Charge:** Each premium is subject to a charge to cover the cost of the selected death benefit cover, up to the following premium date. The charge is called the **Cost of Insurance**, or CoI. Usually, the CoI is calculated using an estimate (perhaps conservative) of the mortality rate for that period, so that, as the policyholder ages, the mortality charge (per \$1 of ADB) increases. However, there are (in Canada) some level cost of insurance

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UL policies, under which the death benefit cover is treated as traditional term insurance, with a level risk premium through the term of the contract deducted from the premium deposited.

If the premium is insufficient to pay the total charge for expenses and mortality, the balance will be deducted from the policyholder's account value.

6. **Surrender Charge:** If the policyholder chooses to surrender the policy early, the surrender value paid will be the policyholder's account balance reduced by a surrender charge. The main purpose of the charge is to ensure that the insurer receives enough to pay the acquisition costs. The total cash available to the policyholder on surrender is the account value minus the surrender charge (or zero if greater), and is referred to as the **cash value** of the contract at each duration.

7. **Secondary Guarantees:** There may be additional benefits or guarantees attached to the policy. A common feature is the **no lapse guarantee** under which the death benefit cover continues even if the policyholder fund is exhausted, provided that the policyholder pays a pre-specified minimum premium at each premium date. This could come into the money if expense and mortality charges increase sufficiently to exceed the minimum premium. The policyholder's account would support the balance until it is exhausted, at which time the no lapse guarantee would come into effect.

8. **Policy Loans:** A common feature of UL policies is the option for the policyholder to take out a loan using the policy account or cash value as collateral. The interest rate on the loan could be fixed in the policy document, or could depend on prevailing rates at the time the loan is taken, or might be variable. Pre-specified fixed interest rates add substantial risk to the contract; if interest rates rise, it could benefit the policyholder to take out the maximum loan at the fixed, lower rate, and re-invest at the prevailing, higher rate.

As mentioned above, the UL policy should be treated similarly to a traditional insurance policy, except that the schedule of death benefits and surrender values depends on the accumulation of the policyholder's funds, which depends on the interest credited by the insurer, as well as on the variable premium flow arising from the ability of the policyholder to pay additional premiums, or to skip premiums.

In the basic UL contract, the insurer expects to earn more interest than will be credited to the policyholder account value. The difference between the interest earned and the interest

credited is the **interest spread**, and this is the major source of profit for the insurer. This is, in fact, no different to any traditional whole life or endowment insurance. For a traditional insurance, the premium might be set assuming interest of 5% per year, even though the insurer expects to earn 7% per year. The difference generates profit for the insurer. The difference for UL, perhaps, is that the interest spread is more transparent.

4.2 Universal Life examples

In this section we illustrate a simple UL contract through some examples.

Example SN4.1 A universal life policy with a 20-year term is sold to a 45 year old man. The initial premium is $2250 and the ADB is a fixed $100 000 (which means this is a type B death benefit). The initial policy charges are:

Cost of Insurance: 120% of the mortality of the Standard Select Mortality Model, (AMLCR, Section 6.3), 5% per year interest.

Expense Charges: $48+1% of premium at the start of each year.

Surrender penalties at each year end are the lesser of the full account value and the following surrender penalty schedule:

Year of surrender	1	2	3-4	5-7	8-10	> 10
Penalty	$4500	$4100	$3500	$2500	$1200	$0

Assume (i) the policy remains in force for the whole term, (ii) interest is credited to the account at 5% per year, (iii) a no lapse guarantee applies to all policies provided full premiums are paid for at least 6 years and (iv) all cash flows occur at policy anniversaries. Project the account value and the cash value at each year end for the full 20-year term, given

(a) the policyholder pays the full premium of $2250 each year;

(b) the policyholder pays the full premium of $2250 for 6 years, and then pays no further premiums.

Solution to Example SN4.1 The account value projection is similar to the method used for unit-linked contracts in Chapter 12 of AMLCR. The purpose of projecting the account value is that it is needed to determine the death and surrender benefit amounts, as well as the policy

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reserves. These values are required for a profit test of the contract. The profit test for this policy is described in Example SN4.2, below.

Projecting the account value also shows how the policy works under the idealised assumptions – level premiums, level credited interest. Each year, the insurer deducts from the account value the expense charge and the cost of insurance (which is the price for a 1-year term insurance with sum insured equal to the Additional Death Benefit), and adds to the account value any new premiums paid, and the credited interest for the year.

The spreadsheet calculation for (a) is given in Table 4 and for (b) is given in Table 5. The columns are calculated as follows:

(1) denotes the term at the end of the policy year;

(2) is the premium assumed paid at $t - 1$;

(3) is the expense deduction at $t - 1$, $(3)_t = 48 + 0.01\,(2)_t$;

(4) is the cost of insurance for the year from $t-1$ to t, assumed to be deducted at the start of the year. The mortality rate assumed is $1.2q^d_{[45]+t-1}$ where $q^d_{[x]+t}$ is taken from the Standard Select Survival Model from AMLCR. Multiply by the Additional Death Benefit, and discount from the year end payment date, to get the CoI, $(4)_t = 100\,000(1.2)q^d_{[45]+t-1}\,v_{5\%}$.

(5) is the credited interest at t, assuming a 5% level crediting rate applied to the account value from the previous year, plus the premium, minus the expense loading and CoI, that is $(0.05)((6)_{t-1} + (2)_t - (3)_t - (4)_t)$

(6) is the year end account value, which comprises the previous year's account value carried forward, plus the premium, minus the expense and CoI deductions, plus the interest earned, $(6)_t = (6)_{t-1} + (2)_t - (3)_t - (4)_t + (5)_t$;

(7) is the year end cash value, which is the account value minus any applicable surrender penalty, with a minimum value of $0.

In more detail, the first two rows are calculated as follows:

First Year

Premium: 2250

Expense Charge: $48 + 0.01 \times 2250 = 70.50$

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CoI:	$100\,000 \times 1.2 \times 0.0006592 \times v_{5\%} = 75.34$
Interest Credited:	$0.05 \times (2250 - 70.50 - 75.34) = 105.21$
Account Value:	$2250 - 70.50 - 75.34 + 105.21 = 2209.37$
Cash Value:	$\max(2209.37 - 4500, 0) = 0$

Second Year

Premium:	2250
Expense Charge:	$48 + 0.01 \times 2250 = 70.50$
CoI:	$100\,000 \times 1.2 \times 0.0007973 \times v_{5\%} = 91.13$
Interest Credited:	$0.05 \times (2209.37 + 2250 - 70.50 - 91.13) = 214.89$
Account Value:	$2209.37 + 2250 - 70.50 - 91.13 + 214.89 = 4512.63$
Cash Value:	4512.63-4100=412.63

We note that the credited interest rate is easily sufficient to support the cost of insurance and expense charge after the first six premiums are paid, so it appears that the no-lapse guarantee is not a significant liability. However, this will not be the case if the interest credited is allowed to fall to very low levels. Note also that the total death benefit is always greater than four times the account value, well above the 2.5 maximum, so the corridor factors will not be significant in this example.

Example SN4.2

For each of the two scenarios described below, calculate the profit signature, the discounted payback period and the net present value, using a hurdle interest rate of 10% per year effective for the UL policy described in Example SN4.1.

For both scenarios, assume:

- Premiums of \$2 250 are paid for six years, and no premiums are paid thereafter.

- The insurer does not change the CoI rates, or expense charges from the initial values given in Example SN4.1 above.

- Interest is credited to the policyholder account value in the tth year using a 2% interest spread, with a minimum credited interest rate of 2%. In other words, if the insurer earns more than 4%, the credited interest will be the earned interest rate less 2%. If the insurer

t^{th} year t	Premium	Expense Charge	Cost of Insurance	Interest Credited	Account Value at year end	Cash Value at year end
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	2 250	70.50	75.34	105.21	2 209.37	0.00
2	2 250	70.50	91.13	214.89	4 512.63	412.63
3	2 250	70.50	104.71	329.37	6 916.79	3 416.79
4	2 250	70.50	114.57	449.09	9 430.80	5 930.80
5	2 250	70.50	125.66	574.23	12 058.87	9 558.87
6	2 250	70.50	138.12	705.01	14 805.27	12 305.27
7	2 250	70.50	152.12	841.63	17 674.28	15 174.28
8	2 250	70.50	167.85	984.30	20 670.23	19 470.23
9	2 250	70.50	185.54	1 133.21	23 797.40	22 597.40
10	2 250	70.50	205.41	1 288.57	27 060.06	25 860.06
11	2 250	70.50	227.75	1 450.59	30 462.40	30 462.40
12	2 250	70.50	252.84	1 619.45	34 008.51	34 008.51
13	2 250	70.50	281.05	1 795.35	37 702.31	37 702.31
14	2 250	70.50	312.74	1 978.45	41 547.53	41 547.53
15	2 250	70.50	348.35	2 168.93	45 547.61	45 547.61
16	2 250	70.50	388.37	2 366.94	49 705.68	49 705.68
17	2 250	70.50	433.33	2 572.59	54 024.44	54 024.44
18	2 250	70.50	483.84	2 786.01	58 506.11	58 506.11
19	2 250	70.50	540.59	3 007.25	63 152.27	63 152.27
20	2 250	70.50	604.34	3 236.37	67 963.80	67 963.80

Table 4: Example SN4.1(a). Projected account values for the UL policy, assuming level premiums throughout the term.

t^{th} year	Premium	Expense	Cost of	Interest	Account Value	Cash Value
t		Charge	Insurance	Credited	at year end	at year end
(1)	(2)	(3)	(4)	(5)	(6)	(7)
0	2 250.00	75.34	70.50	105.21	2 209.37	0.00
1	2 250.00	91.13	70.50	214.89	4 512.63	412.63
2	2 250.00	104.71	70.50	329.37	6 916.79	3 416.79
3	2 250.00	114.57	70.50	449.09	9 430.80	5 930.80
4	2 250.00	125.66	70.50	574.23	12 058.87	9 558.87
5	2 250.00	138.12	70.50	705.01	14 805.27	12 305.27
6	0.00	152.12	48.00	730.26	15 335.41	12 835.41
7	0.00	167.85	48.00	755.98	15 875.53	14 675.53
8	0.00	185.54	48.00	782.10	16 424.09	15 224.09
9	0.00	205.41	48.00	808.53	16 979.22	15 779.22
10	0.00	227.75	48.00	835.17	17 538.64	17 538.64
11	0.00	252.84	48.00	861.89	18 099.69	18 099.69
12	0.00	281.05	48.00	888.53	18 659.17	18 659.17
13	0.00	312.74	48.00	914.92	19 213.35	19 213.35
14	0.00	348.35	48.00	940.85	19 757.85	19 757.85
15	0.00	388.37	48.00	966.07	20 287.56	20 287.56
16	0.00	433.33	48.00	990.31	20 796.54	20 796.54
17	0.00	483.84	48.00	1 013.24	21 277.94	21 277.94
18	0.00	540.59	48.00	1 034.47	21 723.82	21 723.82
19	0.00	604.34	48.00	1 053.57	22 125.05	22 125.05

Table 5: Example SN4.1(b). Projected account values for the UL policy, assuming level premiums for 6 years, no premiums subsequently.

earns less than 4% the credited interest rate will be 2%.

- The ADB remains at $100 000 throughout.

Scenario 1

- Interest earned on all insurer's funds at 7% per year.

- Mortality experience is 100% of the Standard Select Survival Model.

- Incurred expenses are $2000 at inception, $45 plus 1% of premium at renewal, $50 on surrender (even if no cash value is paid), $100 on death.

- Surrenders occur at year ends. The surrender rate given in the following table is the proportion of in-force policyholders surrendering at each year end.

Duration at year end	Surrender Rate q^w_{45+t-1}
1	5%
2-5	2%
6-10	3%
11	10%
12-19	15%
20	0%.

- The insurer holds the full account value as reserve for this contract.

Scenario 2

As Scenario 1, but stress test the interest rate sensitivity by assuming earned interest on insurer's funds follows the schedule below. Recall that the policyholder's account value will accumulate by 2% less than the insurer's earned rate, with a minimum of 2% per year.

Year	Interest rate per year on insurer's funds
1-5	6%
6-10	3%
11-15	2%
16-20	1%

Solution to Example SN4.2

We note here that the insurer incurs expenses that are different in timing and in amount to the expense charge deducted from the policyholder's account value. The **expense charge** (also called the **MER**, for Management Expense Rate) is determined by the insurer, and is set at the outset of the policy. It may be changeable by the insurer, within some constraints. It is a part of the policy terms. There does not need to be any direct relationship with the insurer's actual or anticipated expenses, although, overall, the insurer will want the expense charge to be sufficient to cover the expenses incurred. In this example, the expense charge is $48 plus 1% of the premium paid. This impacts the account value calculation, but otherwise does not directly influence the profit test. The profit test expense assumption is the estimated **incurred expenses** for the insurer, selected for projecting the insurer's cash flows. This is not part of the policy conditions. It does not impact the policyholder's account value. It is included in the profit test table as an outgoing cash flow for the insurer. In this example, the estimated incurred expenses for the contract are $2000 at inception, $45 plus 1% of premium at renewal, as well as contingent expenses on surrender of $50 and contingent expenses on death of $100.

Similarly, we have two different expressions of interest rates. The first is the **credited** interest rate for determining how the account value accumulates. This may be fixed or related to some measure of investment performance. It may be a rate declared by the insurer without any well-defined basis. The **earned** interest rate is the rate that the insurer actually earns on its assets, and this is an important factor in determining the profitability of the contract.

In this example, we assume the credited rate is always 2% less than the earned rate, subject to a minimum of 2%. The minimum credited rate would be established in the policy provisions at inception. For scenario 1, the earned rate is assumed to be 7% throughout, so the credited rate for the policyholder is 5% throughout, which corresponds to the assumption in Example SN4.1. For scenario 2, the earned rates are given in the schedule above. The credited rates will be 4% in the first 5 years, and 2% thereafter.

For the profit test, we note that the income cash flows each year arise from premiums, from the account values brought forward, and from interest. The account value here plays the role of the reserve in the traditional policy profit test. We discuss this in more detail below. The outgo cash flows arise from incurred expenses at the start of each year (associated with policy renewal), from cash value payouts for policyholders who choose to surrender (including expenses of payment), from death benefits for policyholders who do not survive the year, and from the account value established at the year end for continuing policyholders. Because surrender and

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death benefits depend on the account value, we need to project the account values and cash values for the policy first. For scenario 1, this has been done in Example SN4.1. For Scenario 2, we need to re-do the account value projection with the revised credited interest rates.

Scenario 1:

We describe here the calculations for determining the profit vector. The results are given in Table 6. Details of the numerical inputs for the first two rows of the table are also given below.

(1) labels the policy duration at the end of the year, except for the first row which is used to account for the initial expense outgo of $2000.

(2) is the account value brought forward for a policy in force at $t-1$; this is taken from Table 5.

(3) is the gross premium received, $2250 for six years, zero thereafter.

(4) is the assumed incurred expenses at the start of the tth year; the first row covers the initial expenses, subsequent rows allow for the renewal expenses, of $45 plus 1% of the premium.

(5) is the interest assumed earned in the year t to $t+1$, on the net investment. Under scenario 1, this is assumed to be at a rate of 7% each year, and it is paid on the premium and account value brought forward, net of the expenses incurred at the start of the year.

(6) gives the expected cost of the total death benefit payable at t, given that the policy is in force at $t-1$. The death benefit paid at t is AV_t+ADB, where AV_t is the projected end year account value, taken from Table 5, and there are expenses of an additional $100. The probability that the policyholder dies during the year is assumed (from the scenario assumptions) to be $q^d_{[45]+t-1}$. Hence, the expected cost of deaths in the year $t-1$ to t is

$$\text{EDB}_t = q^d_{[45]+t-1}\left(AV_t + \text{ADB} + 100\right).$$

(7) The surrender benefit payable at t for a policy surrendered at that time is the Cash Value, CV_t from Table 5, and there is an additional $50 of expenses. The surrender probability at t for a life in-force at $t-1$ is $(1-q^d_{[45]+t-1})q^w_{45+t-1}$, so the expected cost of the total surrender benefit, including expenses, payable at t, given the policy is in force at $t-1$, is

$$\text{ESB}_t = (1-q^d_{[45]+t-1})q^w_{45+t-1}\left(CV_t + 50\right).$$

(8) For policies which remain in force at the year end, the insurer will set the account value, AV_t, as the reserve. The probability that a policy which is in-force at $t-1$ remains in force at t is $(1 - q^d_{[45]+t-1})(1 - q^w_{45+t-1})$, so the expected cost of maintaining the account value for continuing policyholders at t, per policy in force at $t-1$, is

$$\text{EAV}_t = (1 - q^d_{[45]+t-1})(1 - q^w_{45+t-1})\,(\text{AV}_t).$$

(9) The profit vector in the final column shows the expected profit emerging at t for each policy in force at $t-1$;

$$\text{Pr}_t = (2)_t + (3)_t - (4)_t + (5)_t - (6)_t - (7)_t - (8)_t$$

The profit test details for Scenario 1 are presented in Table 6. The numbers are rounded to the nearest integer for presentation only. EDB denotes the expected cost of death benefits at the end of each year, ESB is the expected cost of surrender benefits, and EAV is the expected cost of the account value carried forward at the year end. As usual, the expectation is with respect to the lapse and survival probabilities, conditional on the policy being in force at the start of the policy year.

To help to understand the derivation of the table, we show here the detailed calculations for the first two years cash flows.

At t=0

Initial Expenses: 2000
Pr_0: -2000

First Year

Account Value brought forward: 0
Premium: 2250
Expenses: 0 (all accounted for in Pr_0)
Interest Earned: $0.07 \times 2250 = 157.50$
Expected Death Costs: $0.0006592 \times (100\,000 + 2209.37 + 100) = 67.44$
Expected Surrender Costs: $0.999341 \times 0.05 \times (0 + 50) = 2.50$
Expected Cost of AV
  for continuing policyholders: $0.999341 \times 0.95 \times 2209.37 = 2097.52$
Pr_1: $2250 + 157.50 - 67.44 - 2.50 - 2097.52 = 240.04$

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Second Year

Account Value brought forward:	2209.37
Premium:	2250
Expenses:	$45 + 0.01 \times 2250 = 67.50$
Interest Earned:	$0.07 \times (2209.37 + 2250 - 67.50) = 307.43$
Expected Death Costs:	$0.0007973 \times (100\,000 + 4512.63 + 100) = 83.41$
Expected Surrender Costs:	$0.9992027 \times 0.02 \times (412.63 + 50) = 9.25$
Expected Cost of AV for continuing policyholders:	$0.9992027 \times 0.98 \times 4512.63 = 4418.85$
Pr_2:	$2209.37 + 2250 - 67.50 + 307.43 - 83.41 - 9.25 - 4418.85$
	$=187.79$

To determine the NPV and discounted payback period, we must apply survival probabilities to the profit vector to generate the profit signature, and discount the profit signature values to calculate the present value of the profit cashflow, at the risk discount rate. The details are shown in Table 7. The profit signature is found by multiplying the elements of the profit vector by the in-force probabilities for the start of each year; that is, let $_tp^{00}_{[45]}$ denote the probability that the policy is in-force at t, then the profit signature $\Pi_k = \text{Pr}_k \; _{k-1}p^{00}_{[45]}$ for $k = 1, 2, ..., 20$. The final column shows the emerging NPV, from the partial sums of the discounted emerging surplus, that is

$$\text{NPV}_t = \sum_{k=0}^{t} \Pi_k \, v^k_{10\%}.$$

NPV_{20} is the total net present value for the profit test. The Π_t column in Table 7 gives the profit signature. From the final column of the table, we see that the NPV of the emerging profit, using the 10% risk discount rate, is \$75.15. The table also shows that the discounted payback period is 17 years.

Scenario 2

Because the interest credited to the policyholder's account value will change under this scenario, we must re-calculate the AV and CV to determine the benefits payable. In Table 8 we show the AV, the profit signature and the emerging NPV for this scenario. We note that as the earned rate decreases, the profits decline. In the final years of this scenario the interest spread

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Year t	AV	Premium	Expenses	Interest	EDB	ESB	EAV	$\Pr_t$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
0	0	0	2 000					−2 000
1	0	2 250	0	158	67	2	2 098	240
2	2 209	2 250	68	307	83	9	4 419	188
3	4 513	2 250	68	469	98	69	6 772	224
4	6 917	2 250	68	637	110	119	9 233	274
5	9 431	2 250	68	813	123	192	11 805	306
6	12 059	2 250	68	997	139	370	14 344	385
7	14 805	0	45	1 033	154	386	14 856	398
8	15 335	0	45	1 070	170	441	15 377	373
9	15 876	0	45	1 108	189	457	15 906	387
10	16 424	0	45	1 147	210	474	16 440	401
11	16 979	0	45	1 185	234	1 755	15 753	377
12	17 539	0	45	1 225	262	2 716	15 351	390
13	18 100	0	45	1 264	292	2 799	15 821	406
14	18 659	0	45	1 303	326	2 882	16 287	422
15	19 213	0	45	1 342	365	2 962	16 743	440
16	19 758	0	45	1 380	409	3 040	17 186	458
17	20 288	0	45	1 417	458	3 115	17 610	476
18	20 797	0	45	1 453	514	3 186	18 010	495
19	21 278	0	45	1 486	576	3 251	18 378	514
20	21 724	0	45	1 518	646	0	22 008	542

Table 6: Scenario 1 profit test part 1 – calculating the profit vector.

t	Pr[in force at start of year]	$\Pr_t$	Π_t	NPV_t
0	1.00000	−2000.00	−2000.00	−2000.00
1	1.00000	240.04	240.04	−1781.78
2	0.94937	187.79	178.28	−1634.44
3	0.92964	224.22	208.45	−1477.83
4	0.91022	274.02	249.41	−1307.48
5	0.89112	306.24	272.90	−1138.03
6	0.87234	385.44	336.23	−948.23
7	0.84514	398.25	336.57	−775.52
8	0.81870	372.64	305.08	−633.20
9	0.79297	386.51	306.49	−503.22
10	0.76793	400.94	307.89	−384.51
11	0.74356	376.50	279.95	−286.39
12	0.66787	389.57	260.18	−203.49
13	0.56643	405.70	229.80	−136.92
14	0.48028	422.41	202.88	−83.50
15	0.40712	439.70	179.01	−40.65
16	0.34500	457.56	157.86	−6.29
17	0.29225	475.98	139.11	21.23
18	0.24747	494.96	122.49	43.26
19	0.20946	514.47	107.76	60.88
20	0.17720	541.96	96.03	75.15

Table 7: Profit signature, NPV and DPP at 10% risk discount rate for Example SN4.2, scenario 1.

t	AV_t	Π_t	NPV_t
0		$-2\,000.00$	$-2\,000.00$
1	$2\,188.33$	237.53	$-1\,784.06$
2	$4\,447.77$	176.99	$-1\,637.79$
3	$6\,783.46$	206.24	$-1\,482.84$
4	$9\,202.32$	245.92	$-1\,314.87$
5	$11\,706.41$	267.68	$-1\,148.67$
6	$14\,022.75$	205.23	$-1\,032.82$
7	$14\,099.08$	201.28	-929.53
8	$14\,160.89$	165.58	-852.29
9	$14\,205.90$	162.88	-783.21
10	$14\,231.54$	160.28	-721.42
11	$14\,234.91$	22.97	-713.37
12	$14\,212.74$	21.38	-706.55
13	$14\,161.37$	20.44	-700.63
14	$14\,076.64$	19.53	-695.49
15	$13\,953.90$	18.64	-691.03
16	$13\,787.88$	-30.20	-697.60
17	$13\,572.68$	-23.20	-702.19
18	$13\,301.66$	-17.31	-705.30
19	$12\,967.33$	-12.37	-707.33
20	$12\,561.29$	-6.92	-708.35

Table 8: Account Values, profit signature and emerging NPV for Example SN4.2, scenario 2.

is negative, and the policy generates losses each year. The initial expenses are never recovered, and the policy earns a significant loss.

4.3   Note on reserving for Universal Life

In the example above, we assume that the insurer holds the full account value as reserve for the contract. There is no need here for an additional reserve for the death benefit, as the CoI always covers the death benefit cost under this policy design. In fact, it might be possible to hold a reserve less than the full account value, to allow for the reduced benefit on surrender,

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and perhaps to anticipate the interest spread.

From a risk management perspective, allowing for the surrender penalty in advance by holding less than the account value is not ideal; surrenders are notoriously difficult to predict. History does not always provide a good model for future surrender patterns, in particular as economic circumstances have a significant impact on policyholder behaviour. In addition, surrenders are not as diversifiable as deaths; that is, the impact of the general economy on surrenders is a systematic risk, impacting the whole portfolio at the same time.

There may be a case for holding more than the account value as reserve, in particular for level CoI policies. In this case, the CoI deduction is calculated as a level premium. The term insurance aspect of the policy is similar to a stand alone level premium term policy, and like those policies, requires a reserve. This is particularly important if the policy is very long term.

The example above is simplified. In particular, we have not addressed the fact that the expense charge, CoI and credited interest are changeable at the discretion of the insurer. However, there will be maximum, guaranteed rates set out at issue for expense and CoI charges, and a minimum guaranteed credited interest rate. The profit test would be conducted using several assumptions for these charges, including the guaranteed rates. However, it may be unwise to set the reserves assuming the future charges and credited interest are at the guaranteed level. Although the insurer has the right to move charges up and interest down, it may be difficult, commercially, to do so unless other firms are moving in the same direction. When there is so much discretion, both for the policyholder and the insurer, it would be usual to conduct a large number of profit tests with different scenarios to assess the full range of potential profits and losses.