

### Buhlmann – Straub Credibility

- Individual losses on a policy are distributed as a Pareto with  $\alpha = 6$  and  $\theta$ . The parameter  $\theta$  is uniformly distributed between 4000 and 6000.

For a group of policyholders, we observe the following two years of experience:

Year	Number of Insureds	Number of Losses	Total Loss
1	300	15	22,500
2	250	12	9,000

Use Buhlman-Straub Credibility to estimate the size of one claim for this group next year.

**Solution:**

$$n = 15 + 12 = 27$$

$$\text{Our Observed Mean} = \frac{22,500 + 9000}{27} = 1166.67$$

$$E[X] = E[E(X | \theta)] = E\left[\frac{\theta}{\alpha - 1}\right] = E\left[\frac{\theta}{6 - 1}\right] = 0.2E[\theta] = 0.2(5000) = 1000$$

$$EPV = E[\text{Var}(X | \theta)] = E\left[\frac{\theta^2 \alpha}{(\alpha - 1)^2 (\alpha - 2)}\right] = E\left[\frac{\theta^2 (6)}{(5)^2 (4)}\right] = 0.06E[\theta^2]$$

$$= 0.06 \int_{4000}^{6000} \theta^2 \left(\frac{1}{2000}\right) d\theta = \frac{0.06}{2000} \left[\frac{\theta^3}{3}\right]_{4000}^{6000} = 1,520,000$$

$$VHM = \text{Var}[E(X | \theta)] = \text{Var}\left[\frac{\theta}{\alpha - 1}\right] = \text{Var}\left[\frac{\theta}{5}\right] = \left(\frac{1}{25}\right)\text{Var}(\theta) = \left(\frac{1}{25}\right)\left(\frac{(6000 - 4000)^2}{12}\right) = 13,333.33$$

$$Z = \frac{N}{N + EPV / VHM} = \frac{27}{27 + (1,520,000 / 13,333.33)} = 0.19149$$

$$\text{Expected Claim} = (Z)(\text{Observed Value}) + (1 - Z)(E(X))$$

$$(0.19149)(1166.67) + (1 - 0.19149)(1000) = 1031.92$$

2. You are given the following data on large employers who are policyholders:
- Losses for each employee of a given policyholder are independent and have a common mean and variance.
  - The overall average loss per employee for all policyholders is 20.
  - The variance of the hypothetical means is 40.
  - The expected value of the process variance is 8000.
  - The following experience is observed for a randomly selected policyholder:

Year	Average Loss per Employee	Number of Employees
1	15	800
2	10	600
3	5	400

Determine the Buhlmann-Straub credibility pure premium for this policyholder for the next year.

**Solution:**

The total number of exposures is  $N = 800 + 600 + 400 = 1800$ .

The average loss per exposure is calculated as the weighted average mean where the weights are the exposures:

$$\bar{x} = \frac{(15)(800) + (10)(600) + (5)(400)}{800 + 600 + 400} = 11.111111$$

$$K = \frac{EPV}{VHM} = \frac{8000}{40} = 200 \implies Z = \frac{N}{N + K} = \frac{1800}{1800 + 200} = 0.9$$

The Buhlmann-Straub credibility pure premium =  $Z(\bar{x}) + (1 - Z)(20)$

$$= (0.9)(11.11111) + (0.1)(20) = 12$$

3. You are given four classes of insureds, each of whom may have zero or one claim, with the following probabilities:

Class	Number of Claims	
	0	1
I	0.9	0.1
II	0.8	0.2
III	0.5	0.5
IV	0.1	0.9

A class is selected at random with a probability of 0.25, and four insureds are selected at random from the class. The total number of claims is 2.

If five insureds are selected at random from the same class, estimate the total number of claims using Buhlmann-Straub credibility.

**Solution:**

Since each class has a Bernoulli distribution, the variance for each class is just  $p(1-p)$ .

$$EPV = \frac{(0.9)(0.1) + (0.8)(0.2) + (0.5)(0.5) + (0.1)(0.9)}{4} = 0.1475$$

Now we calculate VHM:

$$E[N] = \frac{0.1 + 0.2 + 0.5 + 0.9}{4} = 0.425$$

$$E[N^2] = \frac{0.1^2 + 0.2^2 + 0.5^2 + 0.9^2}{4} = 0.2775$$

$$VHM = E[N^2] - (E[N])^2 = 0.2775 - (0.425)^2 = 0.096875$$

$$Z = \frac{N}{N + K} = \frac{4}{4 + (0.1475 / 0.096875)} = 0.7243$$

$$Answer = 5[(Z)(Observed) + (1 - Z)(E(N))] = 5[(0.7243)(2 / 4) + (1 - 0.7243)(0.425)] = 2.3966$$

4. Members of three classes of insureds can have 0, 1, or 2 claims. The following table lists the probabilities for each class:

Class	Number of Claims		
	0	1	2
I	0.9	0	0.1
II	0.8	0.1	0.1
III	0.7	0.2	0.1

A class is chosen at random and varying numbers of insureds from that class are observed over 2 years as shown in the table below:

Year	Number of Insureds	Number of Claims
1	20	7
2	30	10

Determine the Buhlmann-Straub Credibility estimate for the number of claims in year 3 for 35 insureds from the same class.

**Solution:**

$$E[N | I] = (0.9)(0) + (0)(1) + (0.1)(2) = 0.2$$

$$E[N | II] = (0.8)(0) + (0.1)(1) + (0.1)(2) = 0.3$$

$$E[N | III] = (0.7)(0) + (0.2)(1) + (0.1)(2) = 0.4$$

$$E[N^2 | I] = (0.9)(0)^2 + (0)(1)^2 + (0.1)(2)^2 = 0.4$$

$$E[N^2 | II] = (0.8)(0)^2 + (0.1)(1)^2 + (0.1)(2)^2 = 0.5$$

$$E[N^2 | III] = (0.7)(0)^2 + (0.2)(1)^2 + (0.1)(2)^2 = 0.6$$

$$\text{Var}[N | I] = 0.4 - (0.2)^2 = 0.36$$

$$\text{Var}[N | II] = 0.5 - (0.3)^2 = 0.41$$

$$\text{Var}[N | III] = 0.6 - (0.4)^2 = 0.44$$

$$EPV = \frac{0.36 + 0.41 + 0.44}{3} = 0.403333$$

$$E(N) = \frac{0.2 + 0.3 + 0.4}{3} = 0.3$$

$$E(N^2) = \frac{0.2^2 + 0.3^2 + 0.4^2}{3} = 0.096667$$

$$VHM = E(N^2) - (E(N))^2 = 0.096667 - (0.3)^2 = 0.006667$$

We have 50 observations so  $Z = \frac{50}{50 + (0.403333 / 0.006667)} = 0.4525$

$$\text{Number of Claims in Year 3 for 35 insureds} = 35[(Z)(\text{Observed}) + (1 - Z)(E(N))]$$

$$= (0.4525) \left( \frac{17}{50} \right) + (1 - 0.4525)(0.3) = 11.1335$$

### Nonparametric Bayes Procedure for Buhlmann Credibility

5. Two risks are selected at random from a population and observed for 3 years. The risks had the following number of claims over those three years:

Risk	Number of Claims		
	Year 1	Year 2	Year 3
Risk 1	0	2	1
Risk 2	2	2	2

Use the nonparametric empirical Bayes procedure to calculate the credibility weighted estimates of expected number of claims each year for each risk in year 4.

**Solution:**

$$R = 2 \quad N = 3 \quad \hat{\mu}_1 = E(N|1) = \bar{X}_1 = \frac{3}{3} = 1 \quad \hat{\mu}_2 = E(N|2) = \bar{X}_2 = \frac{6}{3} = 2$$

$$\hat{\mu} = \bar{X} = \frac{\hat{\mu}_1 + \hat{\mu}_2}{2} = \frac{1+2}{2} = 1.5$$

$$\hat{\sigma}_1^2 = \frac{(0-1)^2 + (2-1)^2 + (1-1)^2}{N-1=3-1=2} = 1 \quad \hat{\sigma}_2^2 = \frac{(2-2)^2 + (2-2)^2 + (2-2)^2}{N-1=3-1=2} = 0$$

$$EPV = \frac{\hat{\sigma}_1^2 + \hat{\sigma}_2^2}{2} = \frac{1+0}{2} = 0.5$$

$$\overline{Var[\bar{X}_i]} = \frac{(\bar{X}_1 - \bar{X})^2 + (\bar{X}_2 - \bar{X})^2}{R-1} = \frac{(1-1.5)^2 + (2-1.5)^2}{2-1} = 0.5$$

$$\overline{VHM} = \overline{Var[\bar{X}_i]} - \frac{EPV}{N} = 0.5 - \frac{0.5}{3} = 0.33333$$

$$\hat{Z} = \frac{N}{N - (EPV / VHM)} = \frac{3}{3 + 1.5} = 0.66667$$

$$E[N|1] = (\hat{Z})(\bar{X}_1) + (1 - \hat{Z})(\bar{X}) = (0.66667)(1) + (1 - 0.66667)(1.5) = 1.167$$

$$E[N|2] = (\hat{Z})(\bar{X}_2) + (1 - \hat{Z})(\bar{X}) = (0.66667)(2) + (1 - 0.66667)(1.5) = 1.833$$

6. Three individual policyholders have the following claim amounts over four years:

Policyholder	Year 1	Year 2	Year 3	Year 4
A	2	3	3	4
B	5	5	4	6
C	5	5	3	3

Use the nonparametric empirical Bayes procedure to calculate the estimated variance of the hypothetical means.

**Solutions:**

$$R=3 \quad N=4 \quad \hat{\mu}_A = E(N|A) = \bar{X}_A = \frac{2+3+3+4}{4} = 3 \quad \hat{\mu}_B = E(N|B) = \bar{X}_B = \frac{5+5+4+6}{4} = 5$$

$$\hat{\mu}_C = E(N|C) = \bar{X}_C = \frac{5+5+3+3}{4} = 4 \quad \hat{\mu} = \bar{X} = \frac{\hat{\mu}_A + \hat{\mu}_B + \hat{\mu}_C}{3} = \frac{3+5+4}{3} = 4$$

$$\hat{\sigma}_A^2 = \frac{(2-3)^2 + (3-3)^2 + (3-3)^2 + (4-3)^2}{N-1=3} = \frac{2}{3} \quad \hat{\sigma}_B^2 = \frac{(5-5)^2 + (5-5)^2 + (4-5)^2 + (6-5)^2}{N-1=3} = \frac{2}{3}$$

$$\hat{\sigma}_C^2 = \frac{(5-4)^2 + (5-4)^2 + (3-4)^2 + (3-4)^2}{N-1=3} = \frac{4}{3} \quad \overline{EPV} = \frac{\hat{\sigma}_A^2 + \hat{\sigma}_B^2 + \hat{\sigma}_C^2}{3} = \frac{\frac{2}{3} + \frac{2}{3} + \frac{4}{3}}{3} = \frac{8}{9}$$

$$\overline{Var[\bar{X}_i]} = \frac{(\bar{X}_A - \bar{X})^2 + (\bar{X}_B - \bar{X})^2 + (\bar{X}_C - \bar{X})^2}{R-1} = \frac{(3-4)^2 + (5-4)^2 + (4-4)^2}{3-1} = 1$$

$$\overline{VHM} = \overline{Var[\bar{X}_i]} - \frac{\overline{EPV}}{N} = 1 - \frac{8/9}{4} = \frac{7}{9}$$



7. Three policyholders have the following claims experience over three months:

Policyholder	Month 1	Month 2	Month 3	Mean	Variance
I	4	6	5	5	1
II	8	11	8	9	3
III	5	7	6	6	1

Use the nonparametric empirical Bayes procedure to determine the credibility factor of  $Z$  that would be used to estimate premium for the fourth month.

**Solution:**

$$EPV = \frac{1+3+1}{3} = \frac{5}{3}$$

$$\bar{X} = \frac{5+9+6}{3} = \frac{20}{3}$$

$$Var[\bar{X}_i] = \frac{(\bar{X}_I - \bar{X})^2 + (\bar{X}_{II} - \bar{X})^2 + (\bar{X}_{III} - \bar{X})^2}{R-1} = \frac{(5-20/3)^2 + (9-20/3)^2 + (6-20/3)^2}{3-1} = \frac{13}{3}$$

$$VHM = Var[\bar{X}_i] - \frac{EPV}{3} = \frac{13}{3} - \frac{5}{9} = \frac{34}{9}$$

$$\hat{Z} = \frac{N}{N + \frac{EPV}{VHM}} = \frac{3}{3 + \frac{5/3}{34/9}} = 0.8718$$

## Nonparametric Bayes Procedure for Buhlmann-Straub Credibility

8. You are given:

	Group	Year 1	Year 2	Year 3	Total
Total Claims	1		10,000	15,000	25,000
Number in Group			50	60	110
Average			200	250	227.27
Total Claims	2	16,000	18,000		34,000
Number in Group		100	90		190
Average		160	200		178.95
Total Claims					59,000
Number in Group					300
Average					196.67

You are also given that  $VHM = 651.03$ .

Use the nonparametric empirical Bayes method to estimate the credibility factor for Group 1.

**Solution:**

$$EPV = \frac{(50)(200 - 227.27)^2 + (60)(250 - 227.27)^2 + (100)(160 - 178.95)^2 + (90)(200 - 178.95)^2}{4 - 2}$$

$$= 71,985.65$$

$$\hat{Z} = \frac{N}{N + \hat{K}} = \frac{110}{110 + \frac{71,985.65}{651.03}} = 0.4987$$

9. You are given the following data for three groups of policyholder. This is the number of claims submitted by members of each group over a one year period:

Number of Claims	Number of Members		
	Group A	Group B	Group C
0	15	13	8
1	7	1	2
2	3	1	0
<b>Total</b>	25	15	10

Use the nonparametric Bayes method to estimate the credibility factor to apply to predicting claim counts for Group 1.

**Solution:**

$$R = 3 \quad N = 3 \quad \hat{\mu}_A = E(N | A) = \bar{X}_A = \frac{7+3(2)}{25} = 0.52 \quad \hat{\mu}_B = E(N | B) = \bar{X}_B = \frac{1+(2)1}{15} = 0.2$$

$$\hat{\mu}_C = E(N | C) = \bar{X}_C = \frac{2}{10} = 0.2 \quad \hat{\mu} = \bar{X} = \frac{(25)\hat{\mu}_A + (15)\hat{\mu}_B + (10)\hat{\mu}_C}{50} = \frac{13+3+2}{50} = 0.36$$

$$\hat{\sigma}_A^2 = \frac{(15)(0-0.52)^2 + (7)(1-0.52)^2 + (3)(2-0.52)^2}{N-1=24} = 0.51$$

$$\hat{\sigma}_B^2 = \frac{(13)(0-0.2)^2 + (1)(1-0.2)^2 + (1)(2-0.2)^2}{N-1=14} = \frac{11}{35} \quad \hat{\sigma}_C^2 = \frac{(8)(0-0.2)^2 + (2)(1-0.2)^2}{N-1=9} = \frac{8}{45}$$

$$\overline{EPV} = \frac{(25-1)\hat{\sigma}_A^2 + (15-1)\hat{\sigma}_B^2 + (10-1)\hat{\sigma}_C^2}{(25-1) + (15-1) + (10-1)} = \frac{(24)(0.51) + (14)(11/35) + (9)(8/45)}{(25-1) + (15-1) + (10-1)} = 0.388085$$

$$\overline{VHM} = \frac{m_A(\bar{X}_A - \bar{X})^2 + m_B(\bar{X}_B - \bar{X})^2 + m_C(\bar{X}_C - \bar{X})^2 - (R-1)\overline{EPV}}{m - \frac{m_A^2 + m_B^2 + m_C^2}{m}}$$

$$= \frac{(25)(0.52 - 0.36)^2 + (15)(0.2 - 0.36)^2 + (10)(0.2 - 0.36)^2 - (3-1)0.388085}{50 - \frac{25^2 + 15^2 + 10^2}{50}} = \frac{0.503830}{31} = 0.016253$$

$$\text{For Group 1 with 25 members, } \hat{Z} = \frac{25}{25 + (0.388085 / 0.016253)} = 0.5115$$

