We saw in class how the least upper bound property of the real numbers implies:

$(*)$ *every bounded-above increasing sequence has a limit.*

Now let's do the converse, i.e., assuming the truth of $(*)$, we'll deduce the least upper bound axiom.

First we deduce **Archimedes axiom:**[1]

The sequence $1, 2, 3, \ldots$ *of natural numbers has no upper bound.*

Indeed, if there were a bound then $(*)$ would give that this sequence has a limit, say L; and then for all large enough n we would have $|n - L| < \frac{1}{2}$, which can't be since $|n - L| < \frac{1}{2}$ and $|(n+1) - L| < \frac{1}{2}$ can't both hold.

It follows that $\lim_{n \to 0} 1/n = 0$, because for any $\epsilon > 0$, $1/\epsilon$ is not an upper bound for the integers, so for all large enough integers n we have $1/\epsilon < n$, i.e., $1/n < \epsilon$, i.e., $|1/n - 0| < \epsilon$. And since $2^n > n$ for all integers (proved by induction, see Appendix A1, or by showing that $2^x - x$ is an increasing function of x for $x \geq 1$), therefore $\lim_{n \to 0} 1/2^n = 0$. Hence (†): for any constant C, $\lim_{n \to 0} C/2^n = 0$.

Now for the least upper bound axiom. Suppose that S is a nonempty bounded-above set. To show it has a least upper bound, we will construct two sequences

(#) $$s_0 \leq s_1 \leq s_2 \leq s_3 \leq \cdots \leq t_3 \leq t_2 \leq t_1 \leq t_0$$

such that

(i) No s_i is an upper bound for S.

(ii) Every t_i is an upper bound for S,

(iii) For all i, $t_i - s_i = (t_0 - s_0)/2^i$.

Once we've done this, $(*)$ gives that the increasing sequence (s_i) has a limit, say ℓ. *This* ℓ *must be an upper bound of* S: if S had a member $s > \ell$, then for any n large enough that $(t_0 - s_0)/2^n < s - \ell$ (see (†)), since $\ell \geq s_n$ (why?) therefore

$$(t_n - \ell) \leq (t_n - s_n) = (t_0 - s_0)/2^n < (s - \ell),$$

and therefore $t_n < s$, contradicting that t_n is an upper bound for S.

Moreover, if $\ell > k$ then there is an m such that $\ell - s_m < \ell - k$ (because $\ell = \lim s_i$); and since s_m is not an upper bound for S, neither is the smaller number k. Thus ℓ is the *least* upper bound of S, whose existence we had to show.

So let's construct the sequences (#). Let t_0 be an upper bound for S. Choose an $s \in S$ and set $s_0 = s - 1$. Then s_0 is not an upper bound for S. (Notice how we've just used the two hypotheses that S is non empty and that S is bounded above.)

We proceed recursively, that is, one step at a time. Suppose we've constructed $s_0 \leq s_1 \leq \cdots \leq s_n < t_n \leq t_{n-1} \leq \cdots t_1 \leq t_0$ satisfying (i), (ii) and (iii), above. (We just did that for $n = 0$.) Then if $u_n = (s_n + t_n)/2$ $\big($the midpoint of $[s_n, t_n]\big)$ is not an upper bound for S, set $s_{n+1} = u_n,\ t_{n+1} = t_n$; otherwise set $s_{n+1} = s_n,\ t_{n+1} = u_n$.

Then (check), $s_0 \leq s_1 \leq \cdots \leq s_n \leq s_{n+1} < t_{n+1} \leq t_n \leq t_{n-1} \leq \cdots t_1 \leq t_0$ satisfies (i), (ii), and (iii). In this way, the sequences (#) get generated.

This completes the proof.

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[1] Archimedes (287 BC–212 BC) is considered to be the greatest mathematician until Newton. Check out his accomplishments by googling his name. For example, he anticipated some of the techniques of Calculus. His axiom is an explicit statement of a property of real numbers which was apparently taken for granted before him, but can't be deduced from Euclid's axioms, namely, given any two line segments, by adding the smaller one to itself sufficiently many times, you eventually get something longer than the bigger one.