MA 182

Final Exam

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Name in capitals:

1. The integral

$$\int_0^1 \int_x^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

is equivalent to which of the following integrals?

A) $\int_0^{\pi/4} 2\cos^2\theta d\theta$

B) $\int_{\pi/4}^{\pi/2} 8/3\cos^3\theta d\theta$

C) $\int_0^{\pi/2} 2\cos^2\theta d\theta$

D) $\int_{\pi/4}^{\pi/2} 2\cos^2\theta d\theta$

E) $\int_0^{\pi/2} 8/3\cos^3\theta d\theta$

2. The integral

$$\int\int_R e^{(x+y)/(x-y)}\, dA$$

where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$, is equivalent to which of the following integrals?

A) $\int_1^2 \int_0^1 e^{(v/u)} 2\, dv\, du$
B) $\int_1^2 \int_{-u}^{u} e^{(v/u)} 2\, dv\, du$
C) $\int_1^2 \int_{u-2}^{u-1} e^{(v/u)} 1/2\, dv\, du$
D) $\int_1^2 \int_{-u}^{u} e^{(v/u)} 1/2\, dv\, du$
E) $\int_1^2 \int_0^1 e^{(v/u)}\, dv\, du$

3. Evaluate

$$\int_0^3 \int_{y^2}^9 y \cos\left(x^2\right)\, dx\, dy$$

by reversing the order of integration.

A) $\dfrac{\sin 81}{8}$ B) $\dfrac{\sin 9}{16}$ C) $\dfrac{\sin 9}{8}$ D) $\dfrac{\sin 81}{4}$ E) $\dfrac{\sin 81}{16}$

4. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$

A) $\dfrac{8\sqrt{2}\,\pi}{3}$ B) $\dfrac{16\sqrt{2}\,\pi}{3}$ C) $\dfrac{16\sqrt{2}\,\pi}{9}$ D) $\dfrac{8\sqrt{2}\,\pi}{9}$

E) $\dfrac{4\sqrt{2}\,\pi}{3}$

5. Let D be the solid region bounded by the planes $x + 2y + z = 2$, $y = 2x$, $x = 0$, and $z = 0$ with density function $\delta(x, y, z) = 2x^2 + 1$. Use a triple integral to find the moment of inertia of D about the y-axis. Write your answer as an iterated integral in the order z first , then y, and finally x **but do not evalute it.**

Find the volume of the region D inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$ using cylindrical coordinates. Express your answer as an iterated integral.

6. Rewrite the integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z) dz\, dy\, dx$ as an equivalent iterated integral using the order x, y, and z.

7. Evaluate the integral $\int_0^1 \int_x^1 e^{x/y} dy\, dx$ by reversing the order of integration.

8. Let D be the region inside the sphere with radius $1/2$ and center $(0, 0, 1/2)$ given by $x^2 + y^2 + z^2 = z$, and above the cone $z = \sqrt{x^2 + y^2}$. Express the volume of D as an iterated integral in spherical coordinates. Do not compute the integral.

9. Compute the volume of the region D inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$. Choose any coordinate system you want.

10. Use spherical coordinates to determine the volume of the solid over the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Express your answer in terms of iterated integrals. Do not compute these integrals.

11. Use a change of variable $u = x + y$, $v = x - y$ to compute the integral

$$\int_R e^{\frac{x+y}{x-y}}\, dA,$$

where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$. As part of your answer you must sketch the region of integration in the uv-plane.

12. Express the moment of inertia about the y-axis of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x + 2y$, $x = 0$, $z = 0$ as an iterated triple integral. Do not evaluate the integral.

13. Let f be a continuous function on $[0, 1]$ and R the triangular region with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$. Show that $\iint_R f(x + y)\, dA = \int_0^1 u f(u)\, du$.

14. Sketch the curve C: $\vec{r}(t) = \langle \cos t, \sin t, \sin t\rangle$, $0 \le t \le 2\pi$. Find $\oint_C 2xe^{2y}\, dx + (2x^2 e^{2y} + 2y \cot z)\, dy - y^2 \csc^2 z\, dz$.

15. Compute the integral $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy$ after converting to polar coordinates.

16. Evaluate the integral $\int_0^1 \int_x^1 e^{x/y} dy\, dx$ by reversing the order of integration.

17. Let D be the region inside the sphere $x^2 + y^2 + z^2 = 4$, outside the sphere $x^2 + y^2 + z^2 = 2$ and above the xy-plane. Express $\int\int\int_D yz\, dV$ as an iterated integral in spherical coordinates. Do not compute the integral.

18. The critical points of $f(x, y) = x^2y - 6y^2 - 3x^2$ are $(0, 0)$, $(6, 3)$, and $(-6, 3)$. Classify them as local maxima, local minima, or saddle points.

19. Find the critical points of $f(x, y) = (1 + xy)(x + y)$.

20. The volume of a pyramid of height h and square base l is given by $V(l,h) = \frac{1}{3}l^2h$. Use differentials to estimate the change in the volume if the l is decreased from 1m to 0.9m and h is increased from 2m to 2.2m.

21. The temperature of a point in space is $T(x,y,z) = 5x^2 - 3xy + xyz$.
(a) In what direction does the temperature increase fastest at $P(3,4,5)$
(b) Find the rate of change of T at P in the direction $\langle 1, 1, -1\rangle$.

22. Show that any function of the form $z = f(xy)$ satisfies the partial differential equation $\partial z/\partial x + \partial z/\partial y = 0$.

23. Use differentials to estimate how much $f(x, y, z) = e^x \cos yz$ changes as the point $P(x, y, z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction $\langle 2, 2, -2 \rangle$

24. Find all points on $z = 9x^2 + 4y^2$ where the normal line is parallel to the line through $P(4, -2, 5)$ and $Q(-2, -6, 4)$.

25. Use Lagrange multipliers to find the extreme values of $f(x, y, z) = x - y + z$ on the unit sphere $x^2 + y^2 + z^2 = 1$.

26. Find the speed of the particle with position function $\vec{r}(t) = \langle e^{3t}, e^{-3t}, te^{3t} \rangle$ when $t = 0$.

a) $\langle 3, -3, 1 \rangle$
b) 1
c) $\sqrt{2}$
d) $\sqrt{17}$
e) $\sqrt{19}$

27. The plane S passes through the point $P(1, 2, 3)$ and contains the line $x = 3t$, $y = 1 + t$, and $z = 2 - t$. Which of the following vectors are normal to S?

a) $\langle 1, 2, 1 \rangle$
b) $\langle 1, -2, 1 \rangle$
c) $\langle 1, 0, 1 \rangle$
d) $\langle 1, -2, 0 \rangle$
e) $\langle 1, 2, 0 \rangle$

28. A particle moves so that the magnitude of its velocity is constant with value 2 and the magnitude of its acceleration is constant with value 3. Find the curvature of the particle's path.

a) 0
b) 1/4
c) 1/2
d) 2/3
e) 3/4

29. Find the points of intersection of the sphere $x^2 + y^2 + z^2 = 10$ and the space curve $\vec{r}(t) = \langle \cos(\pi t), \sin(\pi t), t \rangle$.

a) $(\sqrt{10}, 0, 0)$, $(-\sqrt{10}, 0, 0)$
b) $(\sqrt{10}/2, -\sqrt{10}/2, \sqrt{5})$, $(\sqrt{10}/2, -\sqrt{10}/2, -\sqrt{5})$
c) $(1, 0, 3)$, $(1, 0, -3)$
d) $(-1, 0, 3)$, $(-1, 0, -3)$
e) $(1/\sqrt{2}, 1/\sqrt{2}, 3)$, $(1/\sqrt{2}, 1/\sqrt{2}, -3)$

30. Let $f(x, y) = ye^x$. Find an equation for the level curve that passes through $(\ln 2, 1)$.

31. Sketch the domain of the function $f(x, y) = \sqrt{y - x}\ \ln(y + x)$? Is it closed, open, or neither? Is it bounded or unbounded?

32. Compute $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$, if it exists. Is f continuous at $(0, 0)$? Explain your answer.

33. Given a position function $\vec{r}(t)$ define the unit tangent vector, $\vec{T}(t)$, the unit normal vector, $\vec{N}(t)$, the binormal vector, $\vec{B}(t)$, and the curvature $\kappa(t)$. Let $\vec{r}(t) = \langle t, t^2, 0 \rangle$. Find $\kappa(0)$.

34. The position function of a spaceship is $\vec{r}(t) = \langle t^2, -t, 6\rangle$, and the coordinates of a space station are (-8,1,6). The captain wants the spaceship to coast into the space station. At what value of t should the engine of the spaceship be turned off?

a) -4
b) -2
c) 0
d) 2
e) 4

35. Suppose $\langle a, b, c\rangle$ is a unit vector. Find an equation of the plane with normal vector $\langle a, b, c\rangle$ that contains (a, b, c).(d)

a) $x = at$, $y = bt$, $z = ct$, where t is a real number.
b) $x = a$
c) $ax + by + cz = 0$
d) $ax + by + cz = 1$
e) $a^2x + b^2y + c^2z = 1$

36. Let $f(x,y) = x^2 + y^2 + x^2y + 4$. Find all critical points of f and classify them as local maximums, local minimums, or saddle points.

37. True or false. Justify your answer.

a) Any constant vector field is conservative.

b) The binormal vector is $\vec{B}(t) = \vec{N}(t) \times \vec{T}(t)$.

c) There exists a function f with continuous second order partial derivatives such that $f_x(x, y) = x + y^2$ and $f_y(x, y) = x - y^2$.

38. Find an equation of the tangent plane to the surface
$x\cos(z) - y^2\sin(xz) = 2$ at P(2,1,0).

39. Let D be the region inside the sphere with center (0,0,0) and radius 2, outside the cylinder $x^2 + y^2 = 1$, satisfying $x - y \geq 0$, and $y \geq 0$ Express $\int\int\int_D yz dV$ as one or more iterated integrals using spherical coordinates. Do not evaluate the integral.

40. a) Find the rate of change of $f(x, y) = 3x^2y + y^3$ at (1,2) in the direction $\langle -3, 4 \rangle$.

b) In what direction should one move from (1,2) so that f decreases fastest?

c) Use the total differential of f to estimate the change in f in moving from (1,2) to (0.9,2.2).

41. Find the area of the region bounded by $y = x^2$, $y = 2x^2$, $x = y^2$, and $x = 4y^2$. Use the substitution $u = \frac{y}{x^2}$, $v = \frac{x}{y^2}$. State clearly the Jacobian and sketch the region of integration in the uv-plane.

42. Find the circulation of $\vec{F}(x, y, z) = \langle y + x^2, x, 3xz + y \rangle$ around C, the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = x + 2y$ with counterclockwise orientation when viewed from above. Use Stoke's Theorem.

43. Find the outward flux through the sphere with center (0,0,0) and radius 5 of the vector field
$\vec{F}(x,y,z) = \langle x + e^{y^2+z^2}, y + e^{z^2+x^2}, z + e^{x^2+y^2} \rangle$.

44. State the first three Maxwell Equations for an electric field $\vec{E}$ and magnetic field $\vec{B}$, defined on a region D with charge density $\rho(x, y, z)$ and boundary surface S. Given Faraday's Law for a changing magnetic field

$$\oint_C \vec{E} \cdot d\vec{r} = \frac{\partial}{\partial t} \iint_S \vec{B} \cdot \vec{n}\, d\sigma,$$

where C is a closed curve and S is any capping surface of C derive Maxwell's third equation.

45. Use Lagrange multipliers to find the extreme values of $f(x, y, z) = 8x - 4z$ subject to the condition $x^2 + 10y^2 + z^2 = 5$.

46. Find the points on the surface $(y+z)^2 + (z-x)^2 = 16$ where the normal line is parallel to the yz-plane.

47. Given a parametrization $\vec{r}(t)$ of a space curve C, state formulas for $\vec{T}(t)$, $\vec{N}(t)$, $\vec{B}(t)$, arc length from $t = a$ to $t = b$, and curvature $\kappa(t)$.

48. A blue plane $B : x + 3y - 2z = 6$ and a yellow plane $Y : 2x + y + z = 3$ intersect in a grenn line G. a) If you stand at $P(1, -2, -1)$ and look in the direction $\vec{v} = \langle 1, 2, 1 \rangle$, what color do you see?
b) Find a parametrization of G.
c) What directions should you look to see G?

Answers:

49. Let $\vec{F}(x,y,z) = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z \rangle$. Find curl$\vec{F}$.
Compute the work done by $\vec{F}$ in moving a particle from $A(0,0,2)$ to $B(0,3,0)$ along the curve in the diagram:

50. Calculate the outward flux of the vector field $\vec{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ across the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$, $z = 0$, and $z = 2$.

51. Sketch the curve C: $\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$, $0 \le t \le 2\pi$. Find $\oint_C 2xe^{2y}\,dx + (2x^2e^{2y} + 2y\cot z)\,dy - y^2\csc^2 z\,dz$.

52. Express the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x + 2y$, $x = 0$, $z = 0$ as an iterated triple integral. Do not evaluate the integral.