MA 271 Vector Calculus
Fall 1999, Final Examination

Instructor: Yip

- This test booklet has TEN QUESTIONS, totaling 100 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is closed book and closed notes.
- (Any kind of) calculator is allowed. But you should **not** use it whenever it is possible (from the point of view of this class), i.e. your answers should be as **analytical** as possible.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- You can use both sides of the papers to write your answers. But please indicate so if you do.

Name:____

Question	Score
1.(10 pts)	
2.(10 pts)	
3.(10 pts)	
4.(10 pts)	
5.(10 pts)	
6.(10 pts)	
7.(10 pts)	
8.(10 pts)	
9.(10 pts)	
10.(10 pts)	
Total (100 pts)	

1. Determine (with reasons) the convergence/divergence of the following series:

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{n \ln n}{2^n}$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

2. Determine the following limits:

(a) $\lim_{t \to 0} \left(\frac{1}{2 - 2\cos t} - \frac{1}{t^2} \right)$

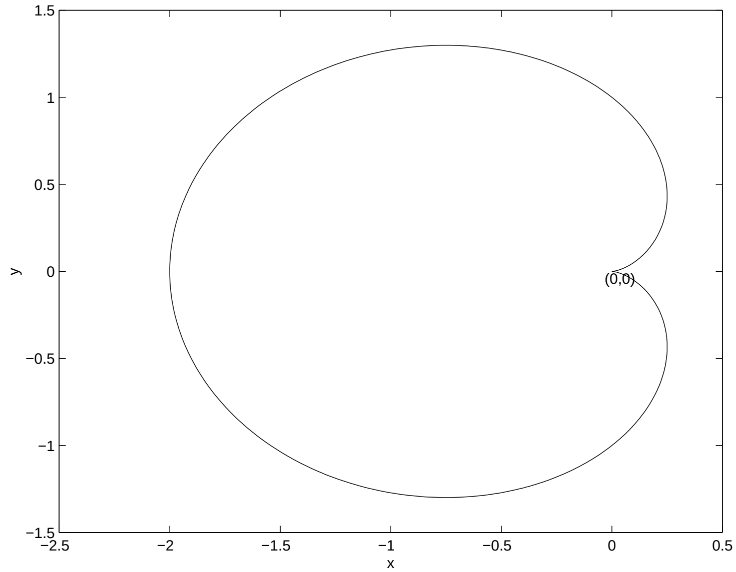
(b) $\lim_{h \to 0} \left(\frac{\frac{\sin h}{h} - \cos h}{h^2} \right)$

3. Given the value of $\sqrt{3} = 1.73205080756888\ldots$ and the following formula:

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\, dt,$$

is it *possible* to write down a series with *fewer than 50 terms* that will give the value of π up to *ten decimal places*? If so, what is the series?

4. Find the arclength of the following curve which is represented in polar coordinates by: $r = 1 - \cos\theta$.



5. Find the *parametric form* of the normal line and the *equation* of the tangent plane to the *level surface* of the following function at $(1, 2, 3)$:

$$w = f(x, y, x) = xyze^{z-3}$$

6. The temperature of the domain $x^2 + y^2 \leq 1$ is given by

$$T(x, y) = 2x^2 - 3xy - 2y^2$$

Find the *absolute maximum* and *absolute minimum* temperatures of the domain.

(This is a scrap paper.)

7. Given the following vector field on R^2:

$$F(x,t) = \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}$$

(a) Find (if possible) the potential function f of F, i.e. $F = \nabla f$.

(b) Compute the circulation of F along the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

8. Given the vector field $F(x, y) = x^2\mathbf{i} - xy\mathbf{j}$. Compute the total flux of F *out of* the triangle with vertices: $A = (0, 0), B = (1, 1), C = (2, 0)$.

9. Find the area of the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ between $z = 0$ and $z = \frac{1}{2}$.

10. Given the following vector field:

$$F(x, y, z) = e^y \mathbf{i} + e^{x+z} \mathbf{j} + e^z \mathbf{k}$$

Compute the total flux of F *out of* the faces of the domain *enclosed by* the following four planes:

$$x = 0, \quad y = 0, \quad z = 0, \quad x + y + z = 1.$$

Taylor Series

$$\begin{aligned}
f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots \frac{f^{(n)}(x-a)^n}{n!} + \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} \\
&\quad \text{(where } c \text{ is in between } x \text{ and } a\text{)} \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}$$

Trigonemetric Formulas

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha; \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta; \\
\sin(2\alpha) &= 2\sin\alpha\cos\alpha; \\
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha;
\end{aligned}$$

Surface Integral in Parametric Form: $S(u,v) = (x(u,v), y(u,v), z(u,v))$

$$\int_S g(x,y,z)\, dA = \int_{(u,v)} g(x(u,v), y(u,v), z(u,v))\, \|S_u \times S_v\|\, dudv$$

$$\int_S F(x,y,z) \cdot N\, dA = \pm \int_{(u,v)} F(x(u,v), y(u,v), z(u,v)) \cdot (S_u \times S_v)\, dudv$$

Surface Integral in Graph Form: $z = f(x,y)$

$$\int g(x,y,z)\, dA = \int g(x,y,f(x,y)) \sqrt{1 + f_x^2 + f_y^2}\, dxdy$$

$$\int F(x,y,z) \cdot N\, dA = \pm \int (-Pf_x - Qf_x + R)\, dxdy, \quad F = (P,Q,R)$$

Green's Theorem in R^2 (Divergence Theorem). D and $F = (P, Q)$ is a domain and vector field in R^2. N is the *outward* normal of ∂D.

$$\int_{\partial D} F \cdot N\, ds = \pm \int_{\partial D} P\, dy - Q\, dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy$$

Green's Theorem in R^3 (Divergence Theorem). D and $F = (P, Q, R)$ is a domain and vector field in R^3. N is the *outward* normal of ∂D.

$$\iint_{\partial D} F \cdot N\, dA = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz$$

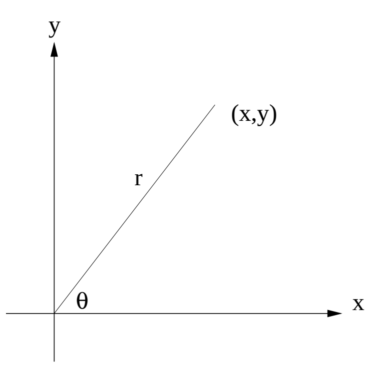
Stoke's Theorem in R^2. D and $F = (P, Q)$ is a domain and vector field in R^2. If the orientation of ∂D is *chosen appropriately*, then

$$\int_{\partial D} F \cdot T\, ds = \int_{\partial D} P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Stoke's Theorem in R^3. S and $F = (P, Q, R)$ is a surface and vector field in R^3. If the orientation of ∂S is *chosen appropriately*, then

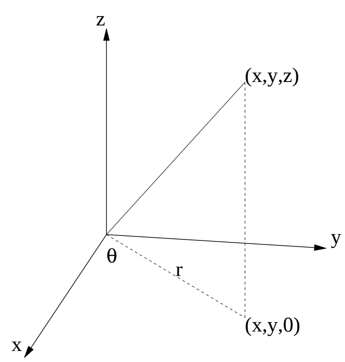
$$\int_{\partial S} F \cdot T\, ds = \iint_S \nabla \times F \cdot N\, dA$$

Polar Coordinates



$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Cylindrical Coordinates



Spherical Coordinates