Duality, Residues, Fundamental class

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Outline

1. 1. Algebraic local cohomology.
2. 2. Local duality.
3. 3. Overall motivation: the Residue Theorem.
4. 4. Idempotent pairs in closed categories.
5. 5. Formal foundations—duality setups.

1. Algebraic local cohomology.

All rings are assumed commutative and noetherian.

S: a ring; $\mathcal{M}(S)$: category of S-modules.

$\Sigma \subset \operatorname{Spec} S$: specialization-stable, i.e. $p' \supset p \in \Sigma \implies p' \in \Sigma$.

$\mathfrak{U}$: the topology on S for which finite products of members of Σ form a basis of neighborhoods of 0.

(Any topology on S for which addition and multiplication are continuous and the square of an open ideal is open arises in this way, with $\Sigma := \{\text{all open primes}\}$.)

$\Gamma = \Gamma_{\mathfrak{U}}$: left-exact *torsion* subfunctor of $\mathbf{1}_{\mathcal{M}(S)}$

$$\Gamma M = \{\, x \in M \mid \text{for some open ideal } J,\ Jx = 0 \,\}.$$

$\mathbf{D}(S)$: derived category of S.

$\mathbf{R}\Gamma\colon \mathbf{D}(S) \to \mathbf{D}(S)$: right-derived $\Gamma \Rightarrow$ local (hyper)cohomology.

Deriving the inclusion $\Gamma \hookrightarrow \mathbf{1}$, get functorial

$$\iota_{\mathfrak{U}}\colon \mathbf{R}\Gamma \to \mathbf{1}$$

Local cohomology and derived tensor product

$\underline{\underline{\otimes}} = \underline{\underline{\otimes}}_S$ denotes left-derived tensor product.

Proposition

There is a natural $\mathbf{D}(S)$*-map*

$$\mathbf{R}\mathrm{Hom}^\bullet_S(G, E) \underline{\underline{\otimes}} \mathbf{R}\Gamma G \to \mathbf{R}\Gamma E \qquad \big(G, E \in \mathbf{D}(S)\big).$$

For $G = S$ this gives a functorial isomorphism

$$E \underline{\underline{\otimes}} \mathbf{R}\Gamma S \xrightarrow{\;\sim\;} \mathbf{R}\Gamma E.$$

Proof.

For the first assertion, can take G and E to be injective complexes, and drop the $\mathbf{R}$s to get a simple statement about ordinary complexes.

The second follows, by Neeman, from the (not quite trivial) fact that $\mathbf{R}\Gamma$ commutes with direct sums. Q.E.D.

"Trivial" Local Duality

$\varphi\colon R \to S$: ring homomorphism ($(S, \mathfrak{U})$ as before.)

$\varphi_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$: restriction-of-scalars functor.

$\mathbf{R}\mathrm{Hom}^\bullet_\varphi(E, F)\colon \mathbf{D}(S)^{\mathrm{op}} \times \mathbf{D}(R) \to \mathbf{D}(S)$: given by $\mathrm{Hom}^\bullet_R(E, I_F)$ for S-complexes E, R-complexes F and I_F an injective resolution of F.

$\exists$ natural identification

$$\varphi_*\mathbf{R}\mathrm{Hom}^\bullet_\varphi(E, F) = \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* E, F) \qquad \big(E \in \mathbf{D}(S),\ F \in \mathbf{D}(R)\big).$$

$\exists$ natural isomorphisms:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^\bullet_\varphi(\mathbf{R}\Gamma E, F) &\xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_\varphi(E \underline{\underline{\otimes}}_S \mathbf{R}\Gamma S, F) \\ &\xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_S(E, \mathbf{R}\mathrm{Hom}^\bullet_\varphi(\mathbf{R}\Gamma S, F)) =: \mathbf{R}\mathrm{Hom}^\bullet_S(E, \varphi^\# F)\end{aligned}$$

Apply the functor $H^0\varphi_*$ to get the local duality isomorphism

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathbf{R}\Gamma E, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(S)}(E, \varphi^\# F),$$

an adjunction between the functors $\varphi_*\mathbf{R}\Gamma$ and $\varphi^\#$.

"Nontrivial" Local Duality

"Nontrivial" versions of Duality convey more information about $\varphi^{\#}$.

Example

Suppose S J-adically topologized (J an S-ideal), module-finite over R.

$\hat{S} :=$ J-adic completion of S.

$F \in \mathbf{D}(R)$ such that $H^n(F)$ finitely generated $\forall n$ and 0 for $n \ll 0$.

Then:

$$\begin{aligned}\varphi^{\#}F := \mathbf{R}\mathrm{Hom}^{\bullet}_{\varphi}(\mathbf{R}\Gamma_J S, F) &\xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^{\bullet}_S\big(\mathbf{R}\Gamma_J S, \mathbf{R}\mathrm{Hom}^{\bullet}_{\varphi}(S, F)\big)\\ &\xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^{\bullet}_{\varphi}(S, F) \otimes_S \hat{S}\,.\end{aligned}$$

where the last isomorphism comes from Greenlees-May duality.

Example (Ct'd)

For $S = R$, assumed local, $J :=$ maximal ideal $\mathfrak{m}$, and $\varphi = \mathrm{id}$, get

$$\mathrm{id}^{\#} F = F \otimes_R \hat{R}\,.$$

If D is a normalized dualizing complex then $\exists$ natural isomorphisms, with $E \in \mathbf{D}(R)$ and $\mathcal{I}$ an R-injective hull of $R/\mathfrak{m}$,

$$\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_{\mathfrak{m}} E, D) \cong \mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_{\mathfrak{m}} E, \mathbf{R}\Gamma_{\mathfrak{m}} D) \cong \mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_{\mathfrak{m}} E, \mathcal{I})$$

The above "trivial" isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_{\mathfrak{m}} E, D) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_R(E, \mathrm{id}^{\#} D)$$

gives then a nontrivial natural isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_{\mathfrak{m}} E, \mathcal{I}) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_R(E, D \otimes_R \hat{R}\,).$$

When $E \in \mathbf{D}_{\mathrm{c}}(R)$ has finitely-generated homology this is just classical local duality, modulo Matlis dualization.

Example (Concluded)

For a more familiar form, apply homology H^{-i} to get the duality isomorphism

$$\mathrm{Hom}_R(\mathsf{H}^i_{\mathfrak{m}}E,\mathcal{I}) \xrightarrow{\sim} \mathrm{Ext}_R^{-i}(E, F\otimes_R \hat{R}\,). \tag{1}$$

If R is Cohen-Macaulay, then there is a derived-category isomorphism $F \cong \omega[d]$ where $\omega := \mathsf{H}^{-d}F$ (a canonical module of R); and (1) becomes

$$\mathrm{Hom}_R(\mathsf{H}^i_{\mathfrak{m}}E,\mathcal{I}) \xrightarrow{\sim} \mathrm{Ext}_R^{d-i}(E, \hat{\omega}).$$

"Nontrivial" Local Duality (Ct'd)

Example

$S = R[[t_1, \ldots, t_m]]$, topologized by $J := (t_1, \ldots, t_m)S$; $\varphi\colon R \to S$ obvious.

$\Omega_{S/R}$: the universal finite relative differential module.

$d\colon S \to \Omega_{S/R}$: the universal R-derivation.

$\Omega_{S/R}$ is free over S, with basis $(dt_1, \ldots, dt_m)$.

$\Omega^m_{S/R} := \Lambda^m_S \Omega_{S/R}$, free with basis $dt_1 \wedge \cdots \wedge dt_m$.

Using the standard calculation of local homology via the Čech complex defined by $(t_1, \ldots, t_m)$, one finds that *if the homology modules of $F \in \mathbf{D}(R)$ are all finitely-generated then there is an isomorphism*

$$\varphi^{\#}F \cong F \otimes_R \Omega^m_{S/R}[m].$$

In fact, $\exists$ a canonical such isomorphism, depending only on the topology of the R-algebra S.

Let E be an S-module. Recall the trivial duality isomorphism

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathbf{R}\Gamma_J E, R) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(S)}(E, \varphi^{\#}R),$$

or equivalently, via the preceding $\varphi^{\#}R \cong \Omega^m_{S/R}[m]$,

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathbf{R}\Gamma_J E, R[-m]) \xrightarrow{\sim} \mathrm{Hom}_S(E, \Omega^m_{S/R}).$$

J being m-generated, the Čech calculation gives $H^i_J E = 0\ \forall i > m$, whence

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathbf{R}\Gamma_J E, R[-m]) \cong \mathrm{Hom}_R(\mathrm{H}^m_J E, R).$$

Thus we have a functorial isomorphism,

$$\mathrm{Hom}_R(\mathrm{H}^m_J E, R) \xrightarrow{\sim} \mathrm{Hom}_S(E, \Omega^m_{S/R}),$$

making the identity map of $\Omega^m_{S/R}$ correspond to a canonical residue map

$$\mathrm{res}_{S/R} \colon \mathrm{H}^m_J \Omega^m_{S/R} \to R.$$

And hence:

Theorem (Canonical local duality)

The functor $\mathrm{Hom}_R(\mathrm{H}^m_J E, R)$ *of* S*-modules* E *is represented by the pair* $(\hat{\Omega}^m, \mathrm{res}_{S/R})$.

Concrete residues

Write Ω^m for $\Omega^m_{S/R}$. The Čech complex is a $\varinjlim$ of Koszul complexes, hence $\exists$ iso

$$\mathrm{H}^m_J\Omega^m \cong \varinjlim_i \Omega^m/(t_1^i,\dots,t_m^i)\Omega^m$$

with maps in the direct system coming from multiplication by $t_1t_2\cdots t_m$.

So can specify any element of $\mathrm{H}^m_J\hat\Omega^m$ by a symbol (non-unique) of the form

$$\begin{bmatrix} \nu \\ t_1^{n_1},\dots,t_m^{n_m}\end{bmatrix} := \kappa_{n_1,\dots,n_m}\pi_{n_1,\dots,n_m}\nu$$

for suitable $\nu\in\hat\Omega^m$ and positive integers $n_1,\dots,n_m$, with π and κ the natural maps

$$\pi_{n_1,\dots,n_m}\colon \hat\Omega^m \twoheadrightarrow \Omega^m/(t_1^{n_1},\dots,t_m^{n_m})\Omega^m,$$

$$\kappa_{n_1,\dots,n_m}\colon \Omega^m/(t_1^{n_1},\dots,t_m^{n_m})\Omega^m \to \mathrm{H}^m_J\hat\Omega^m.$$

Then,

$$\mathrm{res}_{S/R}\begin{bmatrix} \sum r_{i_1,\dots,i_m}t_1^{i_1}\cdots t_m^{i_m}\,dt_1\cdots dt_m \\ t_1^{n_1},\dots,t_m^{n_m}\end{bmatrix} = r_{n_1-1,\dots,n_m-1}.$$

Since it depends on choices, this formula is not a proper definition, but rather a consequence thereof.

3. Residue Theorem

The motivating result underlying this talk is the next theorem, stated here for smooth varieties, but extendable to singular varieties with Kunz's regular differential *m*-forms in place of the usual ones. (See Astérique 117, 1984.)

The theorem shows how differentials and residues give a canonical realization of, and compatibility between, local and global duality.

Theorem

(i) (Globalization of residues) *For each proper smooth m-dimensional variety V over a perfect field k, with ω_V the coherent sheaf of differential m-forms relative to k, there exists a unique k-linear map*

$$\int_V \colon \mathrm{H}^m(V, \omega_V) \to k$$

such that for each closed point $v \in V$, $\gamma_v \colon \mathrm{H}^m_v(\omega_{V,v}) \to \mathrm{H}^m(V, \omega_{V/k})$ being the map derived from the inclusion of the functor of sections supported at v into the functor of all sections, the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^m_v(\omega_{V,v}) \arrow[rr, "\gamma_v"] \arrow[dr, "\mathrm{res}_v"'] & & \mathrm{H}^m(V, \omega_V) \arrow[dl, "\int_V"] \\ & k & \end{tikzcd}$$

(ii) (Canonical global duality). *The pair $(\omega_V, \int_V)$ is dualizing, i.e., for each coherent $\mathcal{O}_V$-module E, the natural composition*

$$\mathrm{Hom}_{\mathcal{O}_V}(E, \omega_V) \to \mathrm{Hom}_k(\mathrm{H}^m(V,E), \mathrm{H}^m(V,\omega_V)) \xrightarrow{\text{via } \int_V} \mathrm{Hom}_k(\mathrm{H}^m(V,E), k)$$

is an isomorphism.

Remark. Part (i) can be reformulated as *the sum over all v of the residues of some global object is 0.* When $m = 1$, that object is any meromorphic differential.

Explanation of what follows

The proof of this Residue Theorem in Asterique 117 is quite roundabout, especially when V is not projective.
Likewise for a generalization to certain maps of noetherian schemes given by Hübl and Sastry in Amer. J. Math. 115 (1993).

My dream, largely but not entirely realized, is

(A): To generalize to more-or-less arbitrary proper maps of formal schemes.

(B): To find a direct definition of integrals, and an *a priori* connection between residues and integrals from which the generalized theorem can be deduced.

The strategy is based on a formalization of duality—outlined in the rest of this talk—which applies not only to complete local rings but also to their globalizations, i.e., formal schemes. It is this possibility of dealing simultaneously with local and global situations that makes it desirable to work with formal, rather than just ordinary, schemes.

That will be as far as the talk goes. But within this formal framework, one *can* concoct a map which in the local situation is the residue map and in the global situation is the integral; and from this the general theorem "should" result.

Monoidal categories

(Symmetric) monoidal category $(\mathbf{D}, \otimes, \mathcal{O})$:

- $\mathbf{D}$ a category,
- $\otimes\colon \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ a functor (product)
- $\mathcal{O}$ a $\otimes$-unit (up to isomorphism): $\forall E \in \mathbf{D}, \quad \mathcal{O} \otimes E \cong E \otimes \mathcal{O} \cong E.$

Product must be associative and commutative, up to isomorphism; and the associativity, commutativity, and unit isomorphisms must interact in natural ways.

Monoidal functor: $\xi_*\colon (\mathbf{D}_1, \otimes_1, \mathcal{O}_1) \to (\mathbf{D}_2, \otimes_2, \mathcal{O}_2)$ is a functor $\xi_*\colon \mathbf{D}_1 \to \mathbf{D}_2$ together with two maps (the first functorial)

$$\xi_* E \otimes_2 \xi_* F \to \xi_*(E \otimes_1 F), \qquad \mathcal{O}_2 \to \xi_* \mathcal{O}_1$$

compatible, in a natural sense, with the respective monoidal structures.

Closed categories

Closed category: Monoidal category together with a functor (internal hom)

$$[-,-]\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{D}$$

and a trifunctorial isomorphism (adjoint associativity)

$$\mathrm{Hom}_{\mathbf{D}}(E \otimes F, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(E, [F, G]).$$

Examples of closed categories

Example (Modules over rings.)

R a ring, $\mathbf{D}_R$ the category of R-modules, $\otimes$ the usual tensor product, $\mathcal{O} := R$, $[E, F] := \mathrm{Hom}_R(E, F)$.

For any ring-homomorphism $\xi\colon R \to S$, the restriction-of-scalars functor $\xi_*\colon \mathbf{D}_S \to \mathbf{D}_R$ is monoidal.

Example (Derived categories over ringed spaces.)

$(X, \mathcal{O}_X)$ a ringed space. i.e., X is a topological space with a sheaf $\mathcal{O}_X$ of commutative rings.

$\mathbf{D}_X$ the derived category of $\mathcal{O}_X$-modules, $\otimes$ the derived tensor product, $\mathcal{O} := \mathcal{O}_X$, $[E, F] := \mathbf{R}\mathcal{H}om_X(E, F)$.

For any ringed-space map $\xi\colon X \to Y$ (continuous map, plus $\mathcal{O}_X \to \xi_*\mathcal{O}_Y$), the derived direct-image functor $\mathbf{R}\xi_*\colon \mathbf{D}_S \to \mathbf{D}_R$ is monoidal.

Idempotent pairs in a monoidal category $\mathbf{D}$

An idempotent pair (A, α) in $\mathbf{D}$ consists of an object A and a map $\alpha\colon A \to \mathcal{O}$ such that the two composite maps

$$A \otimes A \xrightarrow{1\otimes\alpha} A \otimes \mathcal{O} \xrightarrow{\ \sim\ } A, \qquad A \otimes A \xrightarrow{\alpha\otimes 1} \mathcal{O} \otimes A \xrightarrow{\ \sim\ } A$$

are *equal isomorphisms*.

A map of idempotent pairs $\lambda\colon (B, \beta) \to (A, \alpha)$ is a $\mathbf{D}$-morphism $\lambda\colon B \to A$ making the following commute:

$$\begin{tikzcd} B \arrow[rr, "\lambda"] \arrow[dr, "\beta"'] & & A \arrow[dl, "\alpha"] \\ & \mathcal{O} & \end{tikzcd}$$

$B \preccurlyeq A$ means that for some β and α there exists a map—necessarily unique—of idempotent pairs $(B, \beta) \to (A, \alpha)$, a condition that is independent of the choice of β and α.

Examples of idempotent pairs

Example (Not quite trivial.)

$\mathbf{D}$ together with the identity map of $\mathcal{O}$ is an idempotent pair.

Example

Let S be a ring with a topology $\mathfrak{U}$, and let $\iota_{\mathfrak{U}}(S)\colon \mathbf{R}\Gamma_{\mathfrak{U}}S \to S$ as before. The pair $\big(\mathbf{R}\Gamma_{\mathfrak{U}}S, \iota_{\mathfrak{U}}(S)\big)$ is idempotent. (Needs proof)

One shows that this process gives a bijection

$\{$topologies on $S\}\leftrightarrow\{$isomorphism classes of idempotent pairs in $\mathbf{D}(S)\}$.

If $\psi\colon S \to T$ is a ring-homomorphism, then (one checks) *the derived extension-of-scalars functor* $\psi*\colon \mathbf{D}(R) \to \mathbf{D}(S)$ *takes idempotent pairs in* $\mathbf{D}(R)$ *to idempotent pairs in* $\mathbf{D}(S)$. Moreover,

For topologies $\mathfrak{U}$ on S and $\mathfrak{V}$ on T, ψ continuous $\Leftrightarrow \mathbf{R}\Gamma_{\mathfrak{V}}T \preccurlyeq \psi^*\mathbf{R}\Gamma_{\mathfrak{U}}S$.

So, idempotents give a category-theoretic substitute for topologies.

Torsion objects

Let (A, α) be an idempotent pair in $\mathbf{D}$.
An (A, α)-torsion object $E \in \mathbf{D}$ is one such that the composition

$$E \otimes A \xrightarrow{1 \otimes \alpha} E \otimes \mathcal{O} \xrightarrow{\sim} E$$

is an isomorphism.

These are the objects of a full subcategory $\mathbf{D}_A \subset \mathbf{D}$, actually the essential image of the functor $- \otimes A$ (so $\mathbf{D}_A$ doesn't depend on the choice of α).

Example

For a topological ring $(S, \mathfrak{U})$, with $\mathbf{D} := \mathbf{D}(S)$ and $A := \mathbf{R}\Gamma_{\mathfrak{U}} S$, it turns out that the torsion objects are those S-complexes E such that each element in each $H^i(E)$ is annihilated by some open ideal.

5. Formal foundations—duality setups

Given a category $\mathcal{C}$, assign—

1. To each object $X \in \mathcal{C}$ a closed category $\mathbf{D}_X$, with unit object $\mathcal{O}_X$, together with a $\mathbf{D}_X$-idempotent pair (A_X, α_X) in $\mathbf{D}_X$.

Notation: $\mathbf{D}_{A_X} := (\mathbf{D}_X)_{A_X}$, see above.

2. To each $\mathcal{C}$-map $\psi\colon X \to Y$ a functor $\psi_*\colon \mathbf{D}_X \to \mathbf{D}_Y$ with monoidal structure given by

$$\mathbf{e} = \mathbf{e}_\psi\colon \psi_*E \otimes \psi_*E' \to \psi_*(E \otimes E') \qquad \bigl(E, E' \in \mathbf{D}_X\bigr)$$
$$\nu_\psi\colon \mathcal{O}_Y \to \psi_*\mathcal{O}_X,$$

such that:

- If $X = Y$ and ψ is the identity map then ψ_* is the identity functor.
- ψ_* has a left adjoint ψ^* such that the map $\mu_\psi\colon \psi^*\mathcal{O}_Y \to \mathcal{O}_X$ corresponding to ν_ψ is an isomorphism, and such that

- for all $F,\ G \in \mathbf{D}_Y$, the map corresponding to the natural composition

$$F \otimes G \to \psi_*\psi^*F \otimes \psi_*\psi^*G \xrightarrow{\mathbf{e}} \psi_*(\psi^*F \otimes \psi^*G)$$

  is an isomorphism

$$\mathbf{d} = \mathbf{d}_\psi \colon \psi^*(F \otimes G) \xrightarrow{\ \sim\ } \psi^*F \otimes \psi^*G.$$

- (Continuity) $\quad A_X \preccurlyeq \psi^*A_Y.$

- For all $E \in \mathbf{D}_X$ and $F \in \mathbf{D}_Y$ the composite projection map

$$\mathbf{p}_1 \colon \psi_*E \otimes F \xrightarrow{\text{natural}} \psi_*E \otimes \psi_*\psi^*F \xrightarrow{\mathbf{e}} \psi_*(E \otimes \psi^*F)$$

  is an *isomorphism*.

- The functor $\psi_*(- \otimes A_X) \colon \mathbf{D}_X \to \mathbf{D}_Y$ has a right adjoint $\psi^{\#}$.

  So there is an **abstract local duality isomorphism**

  $\mathrm{Hom}_{\mathbf{D}_Y}(\psi_*(E\otimes A_X), F) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}_X}(E, \psi^{\#}F) \qquad (E \in \mathbf{D}_X,\ F \in \mathbf{D}_Y).$

3. (Monoidal pseudofunctoriality) To each pair of $\mathcal{C}$-maps $X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z$ an isomorphism of monoidal functors

$$(\varphi\psi)_* \xrightarrow{\sim} \varphi_*\psi_*,$$

associative (up to isomorphism) vis-à-vis $X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \xrightarrow{\chi} W$, and compatible in a natural way with the monoidal structure on the **D**s.

Local and Global examples

Example (Affine example)

$\mathcal{C}$:= the category *opposite to* that of topological rings.

$A_{(S,\mathfrak{U})} := \mathbf{D}(S)$.

ψ_* := restriction of scalars.

ψ^* := derived extension of scalars.

$\psi^\#$:= as in discussion of local duality.

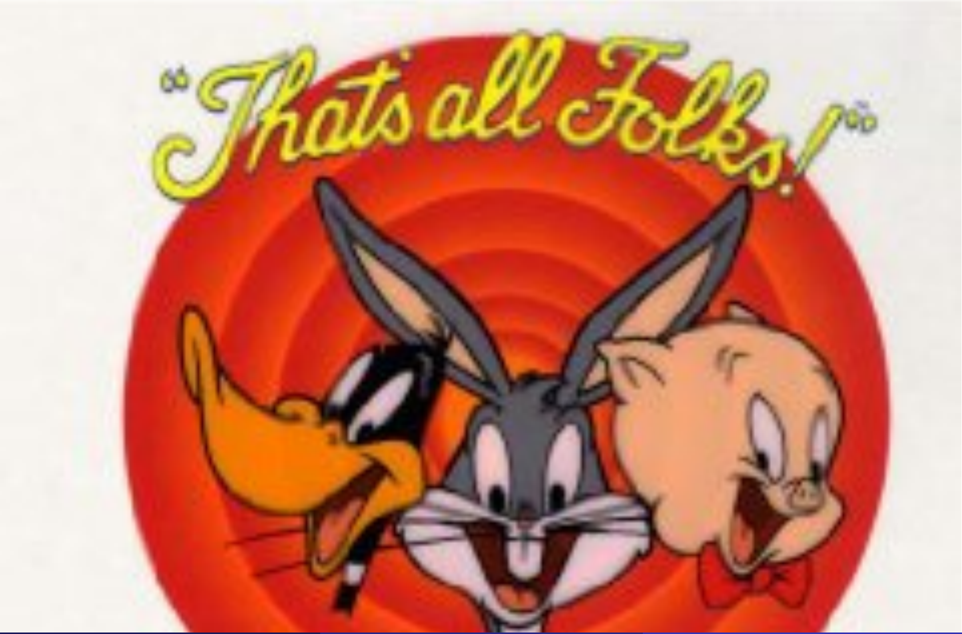
Example (Formal schemes)

The affine example can be generalized to formal schemes X, the topology $\mathfrak{U}$ being replaced by a set $Z \subset X$ closed under specialization, i.e., a union of closed subsets.
The existence of $\psi^\#$ is provided by a version of Grothendieck Duality which holds on formal schemes. [See Contemporary Math. 244 (1999).]

In conclusion

The “duality setup” formalism is very rich in consequences. It is the foundation on which duality theory, with “supports,” can be erected.

In particular, it does enable a common definition (not given in this talk) for local residues and global integrals, that presumably opens the way to a satisfying proof (yet to be completed) of a general Residue Theorem.


"That's all Folks!"