<u>The 15 Puzzle</u>

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & x \end{pmatrix}$$

(x = blank)

Number the (movable) squares $1-15$, and the hole 16. Number the (stationary) spots where they initially reside in the same way.

To any arrangement, associate the permutation taking i to the number of the spot where the i^{th} square now sits.

<u>Example:</u> The arrangement

$$\begin{matrix} 2 & 4 & 5 & 7 \\ 3 & 12 & 15 & 14 \\ 13 & x & 10 & 9 \\ 11 & 1 & 6 & 8 \end{matrix}$$

gives the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 = x \\ 14 & 1 & 5 & 2 & 3 & 15 & 4 & 16 & 12 & 11 & 13 & 6 & 9 & 8 & 7 & 10 \end{pmatrix}$$

Every move clearly multiplies the permutation by the transposition (ij), where i is the spot which is blank before the move, and j is the spot which is blank after the move.

Moreover, if you associate to each spot a pair of coordinates (a, b) with a the row number and b the column number of the spot (both a and b between 1 and 4), then each move changes the sum of the coordinates of the blank spot by ± 1. (In the above example, the blank spot is at $(3, 2)$ and $3+2 = 5$; and it can be moved to $(3, 1), (3, 3), (2, 2), (2, 4)$, with respective sums $4, 6, 4, 6$. In general, a move changes either the row number or the column number by ± 1, and so changes the sum by the same amount).

Since the initial permutation is the identity, which is even, and the blank spot is at $(4, 4)$, and 4+4 is even, it follows that the <u>the parity of the permutation associated to any arrangement is the same as the parity of the sum of the coordinates of the blank spot.</u>

In particular, the arrangement

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 15 & 14 & x \end{pmatrix}$$

where the permutation is odd, but the coordinates of the blank spot are both even, cannot be achieved!

<u>Challenge:</u> Prove that exactly half (i.e. $\frac{16!}{2}$) of all permutations can be achieved.