Math 503 Homework 1

Due Wed. Aug. 29

1. Let X be a set. Let $\mathbb{Z}_2$ denote the integers (mod 2), consisting of two elements 0 and 1 with obvious rules for addition and multiplicaion. (See Clark, p. 10, **18**.) For any subset $A \subset X$, define the characteristic function $\chi_A \colon X \to \mathbb{Z}_2$ by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

(a) Prove that $A \mapsto \chi_A$ is a one-one correspondence (Clark, p. 7, **13**) between the set of subsets of X and the set of maps from X to $\mathbb{Z}_2$.

(b) The product fg of two functions f, g from X to $\mathbb{Z}_2$ is defined by $fg(x) = f(x)g(x)$, and similarly for the sum $f + g$. Prove:

(i) $\chi_{A \cap B} = \chi_A \chi_B$.

(ii) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$.

(c) Solve problems 8α and 8β on p. 4 in Clark, and the third part of problem 9α, p. 5, by using the fact that two subsets of X are the same if their characteristic functions are the same.

2. (a) Prove that if $m < n$ are natural numbers then there is no one-one correspondence between the sets $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$. Deduce that the number of elements in a finite set (Clark, p. 8, **15**) is a well-determined number.

3. Prove that every subset of a finite set is finite.

4. (Base b representation.) Let b be a positive integer. Show that every positive integer can be represented in one and only one way as

$$r_k b^k + r_{k-1} b^{k-1} + \cdots + r_1 b + r_0$$

where each r_i is an integer satisfying $0 \le r_i < b$.

5. Observe that $1^3 = 1^2, \quad 1^3 + 2^3 = (1+2)^2, \quad 1^3 + 2^3 + 3^3 = (1+2+3)^2, \ \ldots$

Formulate a similar statement involving an arbitrary natural number n, and prove the statement (by induction).

Do the following problems in Clark:

6. 15$\boldsymbol{\beta}$.

7. 15$\boldsymbol{\gamma}$.

8. 15$\boldsymbol{\theta}$.

9. 20$\boldsymbol{\gamma}$.

OPTIONAL

10. a) Show that the equation $x^3 + x = 2$ has precisely one real root, and find it (by inspection).

b) What does Cardan's formula give for this root? (If you apply an ambiguous operator like $\sqrt[3]{}$, then you should specify which value you are referring to.)

c) Prove that $\sqrt[3]{1 \pm \frac{2}{3}\sqrt{\frac{7}{3}}}$ has the real value $\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{7}{3}}$.