Math 503 Homework 10

Due Fri. Nov. 30

1. Let H_1, H_2 be subgroups of a finite group G. Show that

$$[H_1 \colon H_1 \cap H_2] \leq [G : H_2].$$

2. (a) Prove that A_n has no subgroup of index 2.

Hint. Apply 1. to every subgoup $H_1 \subset A_n$ with $|H_1| = 3$.

(b) Deduce from (a) and 1. that if $H_2 \subset S_n$ has index 2, then $H_2 = A_n$.

3. Let H be a subgroup of a group G. Show that

$$N(H) := \{\, a \in G \mid aH = Ha \,\}$$

is a subgroup of G containing H.

4. Let n be a positive integer, and let $n\mathbb{Z}$ denote the subgroup of $\mathbb{Z}$ consisting of the multiples of n. Explain why $a \equiv b \pmod{n}$ means that a and b lie in the same coset of $n\mathbb{Z}$.

5. Let G be a cyclic group of order n. Show that for each divisor d of n, G has precisely one subgoup of order d, namely the set of all $g \in G$ such that $g^d = 1$.

Read §**50** in Clark, then do the following two problems:

6. **64**$\boldsymbol{\alpha}$.

7. **64**$\boldsymbol{\beta}$.

8. Establish an isomorphism between the group of automorphisms of a cyclic group of order n and the multiplicative group $(\mathbb{Z}/n)^*$ of units in $\mathbb{Z}/n$.

9. Prove that every non-cyclic group of order 4 is isomorphic to the one in Clark, **26**$\boldsymbol{\iota}$.