Math 503 Homework 3

Due Mon. Sept. 17

1. (a) Express the greatest common divisor $(X^3 + X^2,\ X^4 - 2X^2 - X)$ as a linear combination of these two polynomials.

(b) Find a polynomial $f(X)$ which satisfies both of the congruences

$$f(X) \equiv X^2 + X \pmod{X^3 + X^2},$$
$$f(X) \equiv X^3 - X \pmod{X^4 - 2X^2 - X}.$$

2. (a) Prove that every rational number can be represented in one and only one way as a fraction a/b where a and b are relatively prime integers and $b > 0$.

(b) Let $f(X) = X^n + a_1X^{n-1} + \cdots a_{n-1}X + a_n$ be a polynomial with integer coefficients. Let a/b be a rational root of $f(X)$, that is, $f(a/b) = 0$. Prove that a/b is an integer.

Hint: Use (a), and multiply $f(a/b)$ by b^n.

(c) Prove that $\sqrt[7]{123456789}$ is irrational.

3. Let R be a Euclidean domain.

(a) For any $a \neq 0$, b and c in R, prove that if a divides bc then $a/(a,b)$ divides c.

(b) Suppose that a, b, N, x_0 and y_0 in R satisfy $ax_0 + by_0 = N$. Show that the elements x and y in R satisfy $ax + by = N$ if and only if, for some $m \in R$,

$$x = x_0 + m\frac{b}{(a,b)}, \quad \text{and} \quad y = y_0 - m\frac{a}{(a,b)}.$$

4. Find all integer pairs (x, y) such that $85x + 145y = 505$.

5. Let a and b be two relatively prime positive integers. Prove that $ab - a - b$ is not of the form $ax + by$ with *nonnegative* x and y; but that all integers greater than $ab - a - b$ do have that form.

6. (a) Show that in the ring $\mathbf{Z}[\sqrt{-7}]$, $1 + \sqrt{-7}$ and $1 - \sqrt{-7}$ have greatest common divisor 1, that their product is a cube, but that neither has the form uv^3 with u a unit.

(b) Deduce from (a) that there cannot be a division algorithm in $\mathbf{Z}[\sqrt{-7}]$.

(c) Show that the ring $\mathbf{Z}[\sqrt{-2}]$ has a division algorithm. How many units are there in this ring?

(d) Find all positive integer pairs (x, y) such that $x^2 + 2 = y^3$.

Hints: Show that if $x^2 + 2 = y^3$ then x is odd, and that in $\mathbf{Z}[\sqrt{-2}]$ the elements $x + \sqrt{-2}$ and $x - \sqrt{-2}$ are relatively prime. (It may help to observe that any common divisor has to divide their difference.) Deduce that $x + \sqrt{-2} = (a + b\sqrt{-2})^3$ for suitable integers a, b, and analyze this equation.