Math 503 Homework 4

Due Fri. Sept. 28

1. Let F be a finite field of cardinality q, and let $0 \neq x \in F$. Given a positive integer n, let $d = (n, q-1)$.

(a) Show that if P is the product of the $q-1$ nonzero elements of F then $x^{q-1}P = P$; and deduce that $x^{q-1} = 1$.

(b) Show that every d-th power in F is an n-th power.

(c) Show that $x^n = 1 \iff x^d = 1$.

2. For any positive integer n, let $\tau(n)$ be the number of positive integers which divide n, and let $\sigma(n)$ be the sum of these divisors.

(a) Find formulas for $\tau(n)$ and $\sigma(n)$ in terms of the factorization $n = \prod p_i^{e_i}$ into prime powers.

(b) Show that for any positive integers m, n,

$$\sigma(m)\sigma(n) = \sigma\big((m,n)\big)\sigma\big([m,n]\big),$$
$$\tau(m)\tau(n) = \tau\big((m,n)\big)\tau\big([m,n]\big).$$

3. Set $j = \sqrt{-2}$. In the ring $\mathbb{Z}[j]$:

(a) Factor $14 - 7j$ and $5 + j$ into primes. Justify your answer (explain why the factors are primes).

(b) Find a linear combination of $14 - 7j$ and $5 + j$ which divides them both.

Do the following problems in Clark:

4. 87$\boldsymbol{\eta}$.

5. 89$\boldsymbol{\gamma}$.

Hint. Prove by induction—and use—that every positive integer n satisfies $n^p \equiv n \pmod{p}$.