MA 503 Homework 7

Due Wed. Oct. 31.

A *Pythagorean triple* is a triple (x, y, z) of positive integers such that $x^2 + y^2 = z^2$. So x, y and z are the sides of a right-angle triangle. Examples: $(3, 4, 5)$, $(9, 12, 15)$, $(5, 12, 13)$. The problem is to find a way of generating all such triples.

1. A Pythagorean triple (x, y, z) is *primitive* if the gcd of x and y is 1.

(a) Prove that if the Pythagorean triple (x, y, z) is primitive then the gcd's (y, z) and (x, z) are both 1.

(b) Prove that if (x, y, z) is a primitive Pythagorean triple then so is (y, x, z). (This is very easy.)

(c) Prove that if (x, y, z) is a primitive Pythagorean triple then exactly one of x and y is even.

Hint: Work mod 4 to see that x and y can't both be odd.

2. (a) Prove that if (x, y, z) is a Pythagorean triple then so is (dx, dy, dz) for any positive integer d.

(b) Prove that any Pythagorean triple has the form (dx, dy, dz) where (x, y, z) is a primitive Pythagorean triple.

3. Problem 2 makes it clear that to generate all Pythagorean triples you just need to know how to generate all primitive ones; and problem 1(b) shows it's enough to generate those primitive (x, y, z) for which y is even (since every other one is obtained from such a one by switching x and y—for example, you get (4,3,5) from (3,4,5) by doing that.

(a) Prove that if (x, y, z) is a primitive Pythagorean triple with y even then there are positive, relatively prime, integers $u > v$, just one of which is even, such that

$$x = u^2 - v^2$$
$$y = 2uv$$
$$z = u^2 + v^2.$$

Hint: Begin by showing that $(z + x)/2$ and $(z - x)/2$ are relatively prime, and that their product is a square.

(b) Show that conversely for any u, v as in (a), $(u^2 - v^2,\ 2uv,\ u^2 + v^2)$ is a primitive Pythagorean triple.

(c) Which values of u and v give you the triple (8,15,17)?

4. Find a formula that generates all triples $a < b < c$ of positive integers such that a^2, b^2, c^2 is an arithmetic progression, i.e., $b^2 - a^2 = c^2 - b^2$. (Example: $1^2, 5^2, 7^2$.)

Hint.

$$x^2 + z^2 = 2y^2 \iff \left(\frac{z-x}{2}\right)^2 + \left(\frac{z+x}{2}\right)^2 = y^2.$$

Remark. It holds, nontrivially, that the product of four distinct integers in arithmetic progression is never a square. In particular, four such numbers can't all be squares—a fact first stated by Fermat in the 1600s, and not proved by anyone else for over 100 years.

5. Do problem **120$\boldsymbol{\alpha}$** in Clark.

EXTRA CREDIT:

6. Read Theorem 114 in Clark, understand its proof, and do problem **114$\boldsymbol{\beta}$**.