Think of an icosahedron as a figure I having 12 non-coplanar vertices on a sphere about the origin, such that for any two vertices V_1 and V_2 of I, there is a rotation of the sphere permuting all the vertices among themselves, and taking V_1 to V_2. I has 30 edges, all of the same length, joining each vertex to its 5 nearest neighbors; and 20 faces, each of which is an equilateral triangle. Look at the picture below of an octahedron O about I. ("About" means that each of the 12 edges of O contains exactly one vertex of I.) Each of the 6 line segments joining a vertex of O to the origin intersects one edge of the icosahedron, and no two of these edges have a point in common. Call these the "six edges of I associated to O."

(a) Using the edges associated to any octahedron about I, or otherwise, show that the five rotations through multiples of $2\pi/5$ around a line joining one fixed vertex of I to the origin take O into five different octahedra $O = O_1$, O_2, O_3, O_4, O_5 about the icosahedron.

(b) Show that any symmetry of $\mathbb{R}^3$ taking I into itself permutes the five octahedra. Deduce a homomorphism h from the symmetry group $S(I)$ of I into the symmetric group S_5.

(c) Choose a coordinate system so that the vertices of O are at the eight points which have two coordinates 0 and the remaining coordinate ± 1. Show that the symmetry σ of $\mathbb{R}^3$ taking (x, y, z) to $(-x, -y, -z)$ maps I into itself and lies in the kernel of h.

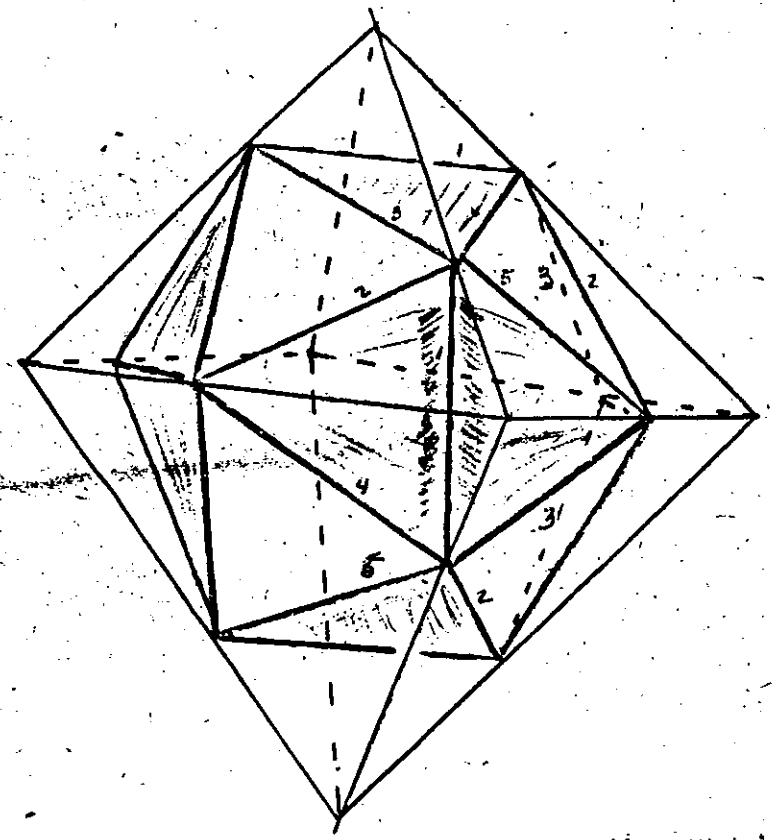
(d) Show that $S(I)$ has order 120. (Hint. How many symmetries leave a given vertex fixed?)

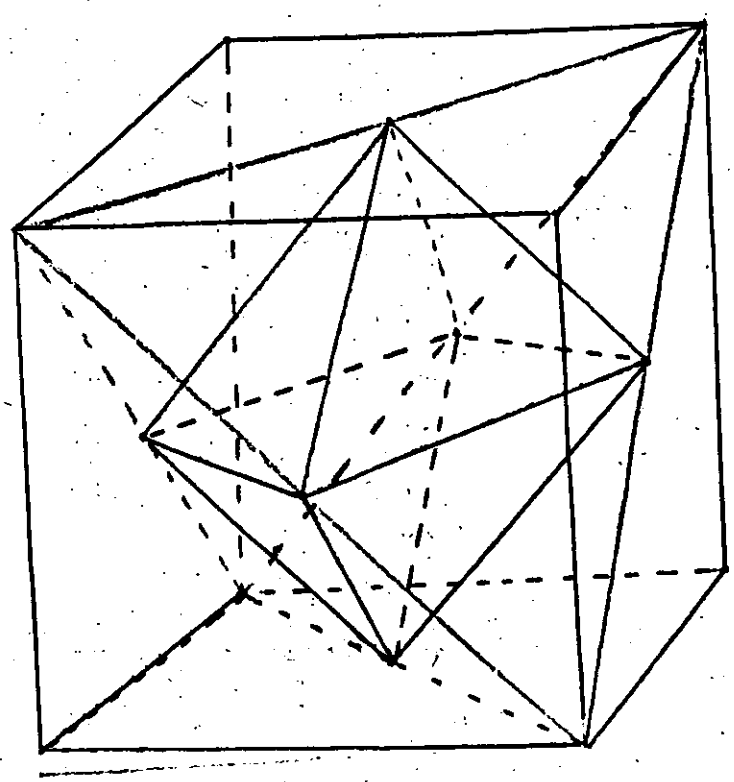
(e) Show that $S(I)$ contains 59 non-identity rotations. Of these, 6 are through an angle of $2\pi n/5$ for each $n = 1, 2, 3, 4$ (see above), 15 are through an angle π around a line joining the origin to the midpoint of an edge, and 20 are through an angle of $\pm 2\pi/3$ about a line joining the centroid of a face to the origin. Show that every such rotation produces an even, non-identity permutation of the octahedra.

(f) Deduce that the group of rotations of I (a subgroup of $S(I)$) is mapped *isomorphically* by h onto the alternating group A_5; that the kernel of h is $\{1, \sigma\}$; and that the image of h is A_5.

(g) Describe an isomorphism from $S(I)$ to $\mathbb{Z}_2 \times A_5$.

Octahedron (one of 5) about an icosahedron





Octahedron < Tetrahedron < Cube.