Ordering the natural numbers.

This will illustrate the power of the Dedkind-Peano axioms by showing how one introduces an ordering into $(\mathbb{N}, 1, \sigma)$. (The approach will not be the same as the one in Landau's book, which uses Exercise (2) below as a definition.)

Definition 0.2. $x \le y$ means that every stable subset of $\mathbb{N}$ that contains x also contains y.

$x < y$ means that $x \le y$ but $x \ne y$.

Remarks. (a) Let $x_{\ge}$ be the intersection of all stable subsets containing x. Then $x_{\ge} = \{\, y \mid x \le y \,\}$, and $x \le y \iff x_{\ge} \supseteq y_{\ge}$. (Visualize this on the number line.)

(b) For all $x \in \mathbb{N}$, we have $1 \le x$. (This just restates the induction axiom.)

(c) For all $x \in \mathbb{N}$, we have $x < \sigma(x)$. ($\le$ is trivial; and prove $\ne$ by induction.)

Proposition 0.3. *For all $x, y, z \in \mathbb{N}$, and with $x + 1 := \sigma(x)$:*

(i) *$x \le x$.*

(ii) *If $x \le y$ and $y \le z$ then $x \le z$.*

(iii) *If $x < y$ then $x + 1 \le y$.*

(iv) (Ordering is total.) *One of $x \le y$, $y < x$ always holds.*

(v) *If $x \le y$ and $y \le x$ then $x = y$.* (*Thus the conditions in* (iv) *can't both hold.*)

Proof. (i) and (ii) are trivial. Make the class come up with the proofs.

(iii) (Might still be left to the class.) If S is a stable subset containing $x+1$ then $S \cup \{x\}$ is a stable subset containing x, hence y; and since $y \ne x$ therefore $y \in S$.

(iv) Fix x. Let $Y = \{\, y \mid \text{either } x \le y \text{ or } y < x \,\}$. If $x \le y$ then $x \le y \le \sigma(y)$, so, by (ii), $x \le \sigma(y)$; and if $y < x$ then by (iii), $\sigma(y) \le x$. Thus Y is stable; and since, by (a) above, $1 \in Y$, therefore $Y = \mathbb{N}$.

(v) If $x < y$ then by (iii) and (b) above, $x+1 \le y \le x$, so by (ii), $(*):\ x+1 \le x$. This is impossible. To see why, consider the set $Z := \{\, z \mid z+1 \not\le z \,\}$. Since $\sigma(\mathbb{N})$ is a stable set containing $\sigma(1)$ but not 1, therefore $1 \in Z$; and if $z \in Z$ then $\sigma(z) \in Z$, because of the following Lemma. Thus $Z = \mathbb{N}$. □

Lemma 1. *For any $x, y \in \mathbb{N}$ we have $x \le y \iff x + 1 \le y + 1$.*

Proof. Note that $\sigma(\mathbb{N}) \cup \{1\}$ is a stable set containing 1, hence is all of $\mathbb{N}$. Thus every $x \ne 1$ in $\mathbb{N}$ is of the form $\sigma(z)$ for some z, unique since σ is injective.

Now suppose $x \le y$. If $T \subset N$ is stable and contains $x + 1$, then either $1 \in T$, in which case $T = \mathbb{N}$ and $y + 1 \in T$, or $1 \notin T$, in which case the preceding remark shows that $T = \sigma(S)$ where $S \subset \mathbb{N}$ is easily seen to be a stable subset containing x, hence y, and so T contains $\sigma(y) = y + 1$. Thus $x + 1 \le y + 1$.

Suppose, conversely, that $x + 1 \le y + 1$. Let $S \subset \mathbb{N}$ be a stable subset containing x. Then $\sigma(S) \subset \mathbb{N}$ is is easily seen to be a stable subset that contains $x + 1$, hence $y + 1$ $(= \sigma(y))$, whence $y \in S$. Thus $x \le y$. □

Exercises (optional). (Use only what's been shown today, or the properties of addition and multiplication mentioned in class.)

(1) Show that if $x < y + 1$ then $x \le y$. (Use (iii) above and Lemma 1.)

(2) Show that $x < y$ if and only if there is a z such that $x + z = y$.

Hint: First show that the set $\{\, z \mid x < x + z \,\}$ is stable, and contains 1. Then show that $\{\, y \mid \text{either } y \le x \text{ or } y = x + z \text{ for some } z \,\}$ is stable, and contains 1.

(3) Show that $[x < y] \implies [x + z < y + z \text{ and } x \cdot z < y \cdot z]$.