Math 553 Homework 1

Due Wed. Aug. 29, 2011

1. Dummit & Foote (DF), p. 106, #6 and p. 106, #7.

2. Let A, B, and C be subgroups of a group G with $A \triangleleft B$ and $C \triangleleft G$ (where "$\triangleleft$" denotes "normal subgroup of"). Prove that $CA \triangleleft CB$.

3. With the notation of DF, p. 103, Thm. 22:

(a) Show for $i, j > 0$ that $N_{i-1}(M_{j-1} \cap N_i) \triangleleft N_{i-1}(M_j \cap N_i)$.

(b) Use the Second Isomorphism Theorem (DF, p. 98) to establish isomorphisms

$$\frac{N_{i-1}(M_j \cap N_i)}{N_{i-1}(M_{j-1} \cap N_i)} \cong \frac{M_j \cap N_i}{(M_{j-1} \cap N_i)(M_j \cap N_{i-1})} \cong \frac{(M_j \cap N_i)M_{j-1}}{(M_j \cap N_{i-1})M_{j-1}}.$$

4. Prove **(2)** in DF, p. 103, Thm. 22 (using the preceding problem 3(b), or otherwise).

5. Apply DF, p. 103, Thm. 22 to a group of the form $\mathbb{Z}/n\mathbb{Z}$ to prove that unique factorization holds in $\mathbb{Z}$.

(OVER)

6. *Ferrari's method for solving degree-4 equations.* (~ 1550.)

Suppose x, c, d, e are complex numbers such that

$$x^4 + cx^2 + dx + e = 0. \tag{6.1}$$

a) Show that with the proper choice of $\sqrt{\ }$,

$$(x^2 + t/2)^2 - \left(x\sqrt{t-c} + \tfrac{1}{2}\sqrt{t^2 - 4e}\right)^2 = 0$$

if t satisfies the equation (called the "resolvent cubic")

$$(t - c)(t^2 - 4e) = d^2. \tag{6.2}$$

b) Deduce that (6.1) can be solved by radicals.

c) Show that if (6.2) has three distinct roots t_1, t_2, t_3 then (6.1) has four distinct roots, say r_1, r_2, r_3, r_4, where the labeling can be chosen so that

$$t_1 = r_1r_2 + r_3r_4, \quad t_2 = r_1r_3 + r_2r_4, \quad t_3 = r_1r_4 + r_2r_3.$$

Hint. Use the procedure from b). Actually it is not necessary to assume the roots distinct: you might try showing directly that if

$$X^4 + cX^2 + dX + e = (X - r_1)(X - r_2)(X - r_3)(X - r_4)$$

then

$$(T - c)(T^2 - 4e) - d^2 = (T - r_1r_2 - r_3r_4)(T - r_1r_3 - r_2r_4)(T - r_1r_4 - r_2r_3). \tag{6.3}$$

d) With notation as in c), show that

$$(r_1 + r_2)^2 = -t_2 - t_3 = t_1 - c = (r_3 + r_4)^2.$$

e) It follows from (6.2) or (6.3) that $(t_1 - c)(t_2 - c)(t_3 - c) = d^2$. So in the expression

$$r = \tfrac{1}{2}\left(\sqrt{t_1 - c} + \sqrt{t_2 - c} + \sqrt{t_3 - c}\right)$$

there are four combinations of choices of the $\sqrt{\ }$'s subject to the restriction that

$$\sqrt{t_1 - c}\sqrt{t_2 - c}\sqrt{t_3 - c} = -d.$$

Show that the resulting four values of r are the roots of (6.1).

(For which equation are the other four values of r the roots?)