Math 553 Homework 11

Due Fri. Nov. 16, 2012

1. Let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree $n > 2$ such that $f(X) = f(-X)$. Prove that the galois group of f is not the symmetric group S_n.

2. Let K be the splitting field over $\mathbb{Q}$ of $X^4 - 2X^2 - 1$.

(a) Determine the galois group of $K/\mathbb{Q}$.

(b) Show that the only three subfields of K having degree 2 over $\mathbb{Q}$ are $\mathbb{Q}(\sqrt{-1}\,)$, $\mathbb{Q}(\sqrt{2}\,)$, and $\mathbb{Q}(\sqrt{-2}\,)$.

3. D&F, p. 582, #11.

4. Let k be a field of characteristic $\neq 2$, and let $f \in k[X]$ be an irreducible polynomial of degree 4. If r_1, r_2, r_3 and r_4 are the roots of f (in some splitting field), then the polynomial g whose roots are $r_1r_2+r_3r_4$, $r_1r_3+r_2r_4$ and $r_1r_4+r_2r_3$ is called the *resolvent cubic* of f.

(a) Show that the discriminant of f is the same as that of g.

(b) Let $G \subset S_4$ be the galois group of f, and let $V \lhd S_4$ be the unique normal subgroup of order 4. Prove that the fixed field T of $V \cap G$ is a splitting field of g.

(c) Let $t = [T : k]$ (see (b)). Prove that $G = S_4,\ A_4$ or V according as $t = 6,\ 3$, or 1. What are the possibilities for G when $t = 2$?

(d) Can the roots of $X^4 + X - 5 \in \mathbb{Q}[X]$ be constructed with ruler and compass? (Hint. Compute the resolvent cubic—see Homework 1.)

(e) Determine the galois group for the minimal polynomial over $\mathbb{Q}$ of each of

$$\sqrt{3+2\sqrt{2}}, \qquad \sqrt{7+2\sqrt{10}}, \qquad \sqrt{5+2\sqrt{5}}, \qquad \sqrt{5+2\sqrt{21}}.$$

(Cf. D&F, p. 618, #13.)