Math 553 Homework 12

Due Mon. Nov. 26, 2012

1. Using D&F, p. 254, Prop. 11, show that a commutative ring that has no maximal ideals must be the trivial ring (i.e., it has just one element.)

2. Show that the intersection of all the prime ideals in a commutative ring R is the ideal

$$N := \{\, x \in R \mid x^n = 0 \text{ for some } n > 0 \,\}.$$

(This N is called the *nilradical* of R; and elements of N are called *nilpotent*.)

Hint: Start by showing that for any $x \in N$, the localization R_x has no prime ideals.

3. Show that the intersection of all the prime ideals containing a given ideal I in a commutative ring R is the ideal

$$\sqrt{I} := \{\, x \in R \mid x^n \in I \text{ for some } n > 0 \,\}.$$

(This ideal, called the *radical of* I, is denoted rad I in D&F.)

Hint: Consider R/I.

EXTRA CREDIT

4. Show that in Theorem 32 on page 700 of D&F, $\mathcal{I}(\mathcal{Z}(I))$ is the intersection of all the *maximal* ideals of $k[x_1, x_2, \ldots, x_n]$ containing I.