Math 553 Homework 2

Due Wed. Sept. 5, 2012

1. Exhibit explicitly a Sylow 2-subgroup of the symmetric group S_4, and show that it is (up to isomorphism) the dihedral group D_8.
Hint: D_8 is the group of symmetries of a square (see DF, p. 24); consider the resulting action on the vertices of the square, numbered $1, 2, 3, 4$. Or, see DF, p. 121, `#5`.

2. Let G be a non-abelian group of order 8.

(a) Show that the center of G has order 2. Denote its generator (an element of order 2) by c.

(b) If G contains an element $z \neq c$ of order 2, then G is isomorphic to a subgroup of S_4, and thus—by (1)—to D_4. (Hint: Consider the action of G on the left cosets of the subgroup $\{1, z\}$.)

(c) If c is the only order 2 element in G, then G is generated by two elements $\mathbf{j}$ and $\mathbf{k}$ of order 4 such that $\mathbf{j}^2 = \mathbf{k}^2$ and $\mathbf{jk} = \mathbf{kj}^{-1}$. Hence G is isomorphic to the quaternion group, i.e., the multiplicative group of complex matrices generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad (i^2 = -1).$$

(d) Is the (order 8) group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad (a, b, c \in \mathbb{Z}/2\mathbb{Z})$$

"dihedral" (i.e., as in (b)) or "quaternionic" (i.e., as in (c))?

3. Let G be a non-abelian group of order p^3 (p an odd prime), and let C be its center.

(a) Show that G/C is isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$ where $\mathbf{Z}_p$ is a group of order p.

(b) Prove that the map $f\colon G \to G$ defined by $f(x) = x^p$ is a group homomorphism.
Hint: By (a), for any x, y in G, there is a $z \in C$ such that $yx = xyz$.

(c) Prove that $f(G) \subset C$, and deduce that G has at least $p^2 - 1$ elements of order p.

(d) Prove that G has subgroups H and K of orders p^2 and p respectively, with $H \cap K = \{e\}$.

4. DF, p. 122, `#13.` Give a complete solution, i.e., include solutions to the other three problems mentioned in this one.